International Journal of Pure and Applied Mathematics
Volume 52 No. 2 2009, 239-245

SPANNED SHEAVES ON REDUCIBLE CURVES WITH DEGREES COMING FROM THEIR NON LOCAL FREENESS

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let X be a reduced projective curve. Here we study spanned torsion free sheaves on X with pure rank r and low degree (the degree coming only from the singularities of the sheaves). These sheaves come from the disconnecting points of X.

AMS Subject Classification: 14H51, 14H10, 14H20
Key Words: torsion free sheaf, spanned sheaf, reducible curve

1. Introduction

Let X be a reduced projective curve. For the elementary properties of depth 1 coherent sheaves on reduced curves, see [4], parts VII and VIII. We say that a depth 1 sheaf F on X has pure rank r if its restriction to X_{reg} is a pure rank 1 vector bundle. Let F be sheaf on X with depth 1 and pure rank r. Set $\mathrm{Sing}(F) := \{P \in X : F$ is not locally free at $P\}$. Hence $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(X)$. The degree $\deg(F)$ of F may be defined by the Riemann-Roch formula $\chi(F) = \deg(F) + r \cdot \chi(\mathcal{O}_X)$. Fix any $P \in X$. Let F_P denote the germ of F at P. Since F has pure rank r, the $\mathcal{O}_{X,P}$-module F_P has pure rank r. Since the locall ring $\mathcal{O}_{X,P}$ is reduced and 1-dimensional, there is a free $\mathcal{O}_{X,P}$ module M with rank r contained in F_P and such that F_P/M has finite length. Let $\ell(F,P)$ denote the minimum of all such lengths for all poissible M as above. Thus $\ell(F,P)$ is a non-negative integer and $\ell(F,P) = 0$ if and only if F is locally free at P. Hence $\ell(F) := \sum_{P\in X} \ell(F,P)$ is a non-negative integer and $\ell(F) = 0$ if and only if F is locally free. Now assume that P is either an ordinary node or

Received: March 8, 2009

an ordinary cusp of X. Let $m_{X,P}$ denote the maximal ideal of the local ring $\mathcal{O}_{X,P}$. By the classification of torsion free modules over an A_n one-dimensional singularity (see [4], pp. 164–166, for ordinary nodes, [3] for both cases), there is an integer x such that $0 \le x \le r$ and $F_P \cong m_{X,P}^x \oplus \mathcal{O}_{X,P}^{\oplus(r-x)}$. A local calculation gives $\ell(F,P) = x$. We say that F is *full* if there are a partial normalization $u : Y \to X$ and a locally free sheaf G on Y such that $F \cong u_*(G)$. Even if X is connected we do not require in the definition of full sheaf that Y is connected. If the triple (Y,u,G) is as above, then the Leray spectral sequence of u gives $h^i(X,F) = h^i(Y,G)$ for $i = 0,1$. Hence Riemann-Roch on Y and on X gives $\deg(G) = \deg(F) + r(\chi(\mathcal{O}_Y) - \chi(\mathcal{O}_Y))$. The triple (Y,u,G) is uniquely determined by F, up to isomorphisms. For any quasi-projective curve $U \neq \emptyset$ let $\eta(U)$ denote the number of its connected components. Now assume that X is connected. Fix $P \in \mathrm{Sing}(X)$ and an integer $k \ge 2$. The singular point P is called a *disconnecting point with type* k of X if $X\backslash\{P\}$ has k connected components. The point P is a *disconnecting point* of X if there is an integer $k \ge 2$ such that P is a disconnecting point with type k of X. Let $\Sigma_k(X)$ denote the set of all disconnecting points of X with type k of X. We say that $P \in \Sigma_k(X)$ is a *transversal disconnecteing point of type* k of X and write $P \in \Sigma_k(X)'$ if $p_a(X) = \sum_{i=1}^{k} p_a(C_i)$, where $C_1,\dots,C_k$ denote the closure in X of the connected components of $X\backslash\{P\}$ in X; equivalently the Zariski tangent space T_PX of X at P is the direct sum of the Zariski tangent spaces T_PC_i, $1 \le i \le k$. Set $\Sigma_+(X) = \cup_{k\ge2}\Sigma_k(X)$ and $\Sigma_+(X)' := \cup_{k\ge2}\Sigma_k(X)'$. Notice that the both unions are disjoint unions. If every singular point of X lying on at least two different irreducible components is an ordinary node of X, then $\Sigma_k(X) = \emptyset$ for all $k \ge 3$.

Theorem 1. *Let X be a connected projective curve. Assume that each point of X lying on at least two irreducible components of X is an ordinary node of X. Fix an integer $r \ge 1$. For all integers d,m,k let $\mathcal{A}(r,d,m,k)$ denote the set of all spanned depth 1 sheaves F on X such that $\deg(F) = d$, $\ell(F) = k$ and $\sharp(Sing(F)) = m$.*

(a) If $F \in \mathcal{A}(r,d,d,1)$ for some $1 \le d \le r$, then the point $Sing(F)$ is a disconnecting node of X. If $\Sigma_2(X) = \emptyset$, then $\mathcal{A}(r,d,d,1) = \emptyset$ for all $1 \le d \le r$.

(b) There is a bijection $\gamma : \Sigma_2(X) \to \mathcal{A}(r,r,r,1)$. For every $P \in \Sigma_2(X)$ the sheaf $\gamma(P)$ is the unique $F \in \mathcal{A}(r,r,r,1)$ such that $Sing(F) = \{P\}$. Every $F \in \mathcal{A}(r,r,r,1)$ is full and satisfies $h^0(X,F) = 2r$.

(c) Assume $\Sigma_2(X) \neq \emptyset$ and fix $P \in \Sigma_2(X)$. For any integer d such that $1 \le d < r$ the set $A(P,r,d)$ of all $F \in \mathcal{A}(r,d,d,1)$ such that $Sing(F) = \{P\}$ is non-empty and parametrized by the a non-empty open subset G_1 of the integral variety G_2 of all surjective $\mathbb{K}$-linear map $\mathbb{K}^{2r} \to \mathbb{K}^{r-d}$.

Take $P \in \Sigma_2(X)$ and let $u : Y \to X$ be the partial normalization of X in which we normalize only the point P. We will see during the proof of Theorem 1 that $u_*(\mathcal{O}_Y^{\oplus r})$ is the sheaf $\gamma(P)$ whose existence is claimed in part (b) of Theorem 1.

Part (b) of Theorem 1 extends [1], Theorem 1.

Proposition 1. *Let X be a connected projective curve. Assume that each point of X lying on at least two irreducible components of X is an ordinary node of X. Fix positive integers r and s. Let $B(X,s)$ be the set of all $S \subset X$ such that $\sharp(S) = s$ and $\eta(X\backslash S) = s+1$. There is a bijection $\gamma_s : B(X,s) \to \mathcal{A}(r,sr,sr,s)$ and $h^0(X,F) = r(s+1)$ for every $F \in \mathcal{A}(r,sr,sr,s)$.*

Fix $S \in B(X,s)$. Notice that $S \subseteq \Sigma_2(X)$. Let $u : Y \to X$ be the partial normalization of X in which we normalize only the points of S. We get $\gamma_s(S) \cong u_*(\mathcal{O}_Y)^{\oplus r}$.

Proposition 2. *Let X be a reduced and connected projective curve. Fix an integer $r \geq 1$ and any $S \subseteq \cup_{k\geq 2}\Sigma_k(X)$. Let $C_1,\dots,C_x$ denote the closures in X of the connected components of $X\backslash S$. Set $Y := \sqcup_{i=1}^{x} C_i$ and let $u : Y \to X$ be the natural map which is the identity on each connected component of Y. Set $F := u_*(\mathcal{O}_Y^{\oplus r})$. F is a full depth 1 sheaf on X with pure rank r, $h^0(X,F) = rx$ and degree $r(\chi(\mathcal{O}_Y) - \chi(\mathcal{O}_X))$. If $S \subseteq \Sigma_+(X)'$, then F is spanned.*

Theorem 2. *Fix a reduced projective curve, an integer $r \geq 1$ and a set $S \subseteq Sing(X)$, $S \neq \emptyset$. Let $A(r,S)$ be the set of all depth 1 sheaves F on X with pure rank r, $\deg(F) = \ell(F)$, $Sing(F) = S$, and F spanned outside finitely many points of X. Set $A(r,S,d) := \{F \in A(r,S) : \deg(F) = d\}$. Fix $P \in S$ and set $S' := S\backslash\{P\}$. Let $A_1,\dots,A_c$, be the closures in X of the connected components of $X\backslash\{P\}$.*

(a) Take any $F \in A(r,S)$. Then there is $G \in A(1,S)$ such that $F \cong G^{\oplus r}$. Hence there is a bijection $\alpha_{r,S} : A(1,S) \to A(r,S)$ and $\alpha_{r,S}(G)$ is spanned (or spanned at a fixed $Q \in X$) if and only if G has the same property.

(b) If $P \notin \Sigma_+(X)$, then $A(r,\{P\}) = \emptyset$.

(c) Assume $P \in \Sigma_c(X)$, $c \geq 2$, and that P is a seminormal point of X with multiplicity m.Then $m \geq c$, $A(1,\{P\})(d) = \emptyset$ if $d \geq c$. For every integer x such that $1 \leq x \leq c-1$ there are element of $A(1,\{P\})(x)$ spanned at P and the set of these sheaves is finite and completely described in step (ii) of the proof below. Each element of $A(1,\{P\})(x)$ spanned at P is spanned.

Warning concerning part (c): X may have automorphisms preserving P and these automorphisms may “identify” different elements of $A(1,\{P\})(x)$.

2. The Proofs

Remark 1. Let $u : Y \to X$ be a partial normalization of X, i.e. let Y be a reduced projective curve (not necessarly connected) and u a finite and surjective morphism such that there is an open and dense subset U of Y with $u|U : U \to u(U)$ an isomorphism. Let F be depth 1 sheaf on X with pure rank r. Set $G := u^*(F)/\mathrm{Tors}(u^*(F))$. The coherent sheaf G is a depth 1 sheaf on Y with pure rank r and there is an injective map $j_{F,u} : F \to u_*(F)$ with cokernel with finite support. Applying Riemann-Roch on Y and on X we get $\deg(G) = \deg(F) + \chi(\mathcal{O}_X) - \chi(\mathcal{O}_Y)$. Now assume that F is spanned. Since the tensor product is a right exact functor, $u^*(F)$ is spanned. Hence G is spanned. Since F and G are spanned, the natural map $H^0(X, F) \to H^0(Y, G)$ is injective.

We need the following well-known lemma whose proof is left to the reader, because we are unable to do better than copy [2], Proposition 3.2.4.

Lemma 1. *Let $u : Y \to X$ be a partial normalization of X. Let F be a depth 1 sheaf on X with pure rank r. Fix $S \subseteq Sing(F)$ such that $u|Y\backslash u^{-1}(S) \to Y\backslash S$ is an isomorphism. Set $G := u^*(F)/Tors(u^*(F))$. Then:*

(a) The sheaf G has depth 1, pure rank r and

$$\deg(F) - \sum_{P \in S} \ell(F, P) \leq \deg(G) \leq \deg(F) - \sharp(S).$$

(b) If G is locally free at each point of $u^{-1}(S)$, then $\deg(G) = \deg(F) - \sum_{P\in S} \ell(F, P)$ and $G \cong u^*(u_*(G))/Tors(u^*(u_*(G)))$.

(c) If Y is smooth at each point of $u^{-1}(S)$, then $\deg(G) = \deg(F) - \sum_{P\in S} \ell(F, P)$ and $G \cong u^*(u_*(G))/Tors(u^*(u_*(G)))$.

Proof of Theorem 1. Fix any integer d such that $1 \leq d \leq r$ and any $F \in \mathcal{A}(r, d, d, 1)$. Set $\{P\} := \mathrm{Sing}(F)$. Let $u : Y \to X$ be the partial normalization of X in which we normalize only the point P. Set $G := u^*(F)/\mathrm{Tors}(u^*(F))$. Lemma 1 gives $\deg(G) = 0$. Since G is a spanned vector bundle with pure rank r, we get $G \cong \mathcal{O}_Y^{\oplus r}$. Hence $h^0(Y, G) = r\beta$, where β is the number of the connected components of Y. Since any disconnecting point of X is an ordinary node, $\beta \in \{1, 2\}$ and $\beta = 2$ if and only if P is an ordinary node and a disconnecting point of X. Since $h^0(X, F) \leq h^0(X, G)$ (Remark 1), F is spanned and $\deg(F) > 0$, we have $h^0(X, F) > r$. Hence $\beta = 2$, i.e. P is a disconnecting node of P. Conversely, for any $P \in \Sigma_+(X)$ set $\gamma(P) := u_*(\mathcal{O}_Y^{\oplus r})$, where $u : Y \to X$ be the partial normalization of X in which we normalize only the point P. If $d = r$ we easily get that F is full and obtain the bijection γ claimed in part (b), the second sentence of part (b) and the sentence after the statement of Theorem 1. Now assume $1 \le d < r$. There is an inclusion $j_F : F \hookrightarrow u_*(\mathcal{O}_Y^{\oplus r}) = \gamma(P)$. Set $T_F = \mathrm{Coker}(j_F)$. Hence we have an exact sequence of sheaves

$$0 \to F \to \gamma(P) \to T_F \to 0. \tag{1}$$

The sheaf T_F is a skyscraper sheaf supported by P and with length $\deg(\gamma(P)) - \deg(F) = r - d$. Since $\gamma(P)$ is full, its fiber $\gamma(P)|\{P\}$ is a $2r$ -dimensional $\mathbb{K}$-vector space. Since tensor product is a right exact functor, restricting (1) to $\{P\}$ we get a surjection $\mathbb{K}^{2r} \to \mathbb{K}^{r-d}$. Conversely, any surjection $v' : \mathbb{K}^{2r} \to \mathbb{K}^{r-d}$ induces a surjection $v : \gamma(P) \to \mathbb{K}_P^{r-d}$, where $\mathbb{K}_P^{r-d}$ is the only $\mathcal{O}_X$ killed by the ideal sheaf of P and with length $r-d$. The sheaf $\mathrm{Ker}(v)$ has depth 1, pure rank r, degree d and it is locally free outside P. The defining short exact sequence of $\mathrm{Ker}(v)$ gives $h^0(X, \mathrm{Ker}(v)) \ge 2r - d$. In the same way we see that any proper subsheaf F' of F with F/F' supported by P satisfies $h^0(X, F') = r + \deg(F') \le r + d - 1$. Hence F is spanned at P. Since $\gamma(u)$ is spanned and v is induced by a surjection v' from the fiber of $\gamma(P)$ over P, we have the reverse inequality. Hence $h^0(X, \mathrm{Ker}(v)) = 2r - d$. Since $\ell(\gamma(u), P) = r$, and the target of v is a skyscraper sheaf with length $r-d$, $\ell(F, P) \ge r - (r-d) = d$. Identify the germ $\gamma(P)_P$ with $m_{X,P}^{\oplus r}$, where $m_{X,P}$ is the maximal ideal of $\mathcal{O}_{X,P}$. By the classification of depth 1 modules with pure rank r over an ordinary node ([4], pp. 164–166) we see that $\ell(\mathrm{Ker}(v), P) = d$ if and only if the evaluation map $\mathcal{O}_{X,P} \otimes \mathrm{Ker}(v') \to m_{X,P}^{\oplus r}$ has rank r outside P. This is an open condition, which is generically satisfied. Hence we get the non-empty open subset G_2 of G_1. To conclude it is sufficient to prove that $\mathrm{Ker}(v)$ is spanned outside P. Let $X' \subseteq X$ be the union of the two irreducible components of X passing through P. Let E be the subsheaf of F spanned by $H^0(X, F)$. We have $h^0(X, E) = h^0(X, F')$. Since Y has two connected components, each of them containing one of the point P, we get that the restriction map $H^0(X, \gamma(u)) \to H^0(X', \gamma(u)|X')$ and $H^0(X, F) \to H^0(X', F|X')$ are bijective. Since F is spanned at P, E has pure rank r in a neighborhood of P and $\ell(E, P) = \ell(F, P)$. Hence applying the previous proof to the pair $(X', (E|X')/\mathrm{Tors}(E|X'))$, we get $h^0(X', (E|X')/\mathrm{Tors}(E|X')) = r + \deg(E) < h^0(X, F)$, contradiction. □

Proof of Proposition 1. Adapt the proof of Theorem 1. □

Proof of Proposition 2. At this point only the last assertion need to be proven. Since $F = u_*(\mathcal{O}_Y)^{\oplus r}$, it is sufficient to do the case $r = 1$. Fix $P \in S$ and call $A_1, \dots, A_k$ the closures in X of the connected components of $X \backslash \{P\}$. Write $C := \cup_{i=2}^k A_i$. Let $v : A_i \sqcup C \to X$ denote the ovious surjective map. Notice that u factors through v, say $u = v \circ m$. By induction on the number of components we may assume that $M := m_*(\mathcal{O}_Y)$ is spanned. Since $P \in \Sigma_k(X)'$, we have a Mayer-Vietoris exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_{A_1} \oplus \mathcal{O}_C \to \mathcal{O}_P \to 0 \tag{2}$$

in which P has its reduced structure. Notices that (assuming $r = 1$) $\ell(M, P) = k - 1$ and $\ell(F, P) = k$. By tensoring (2) with F we get an exact sequence, because F has no torsion. Hence from the spannedness of M and of $\mathcal{O}_{A_1}$ we get the spannedness of F. □

Proof of Theorem 2. Fix $F \in A(r, S)$. By the definition of full sheaf there are a partial normalization $u : Y \to X$ and a locally free sheaf M on Y with pure rank r such that $u_*(M) \cong F$. We have $M \cong u^*(F)/\mathrm{Tors}(u^*(F))$ (part (c) of Lemma 1). Since F is spanned outside finitely many points of X, as in Remark 1 we get that $u^*(F)$ is spanned outside finitely many point of Y. Hence M is spanned outside finitely many point of Y. Since $\deg(M) = \deg(F) - \ell(F) = 0$ (part (b) of Lemma 1), $M \cong \mathcal{O}_Y^{\oplus r}$. Hence $F \cong G^{\oplus r}$ with $G \cong u_*(\mathcal{O}_Y)$. Since $\mathrm{Sing}(G) = \mathrm{Sing}(F)$ and F and G are spanned at the same points of X, part (a) follows. Hence it is sufficient to prove all the other statements of Theorem 2 in the case $r = 1$.

(i) Assume $A(1, S) \neq \emptyset$ and fix $F \in A(1, S)$. Since F is full, there are a partial normalization $u : Y \to X$ and a locally free sheaf M on Y with pure rank r such that $u_*(M) \cong F$. We just proved that $M \cong \mathcal{O}_Y^{\oplus r}$. Hence $h^0(X, F) = h^0(Y, M) = \eta(Y)$. Set $k := \eta(Y)$. Since $S \neq \emptyset$, we get $k \geq 2$, proving part (b). Let $D_1, \dots, D_k$ be the connected components of Y. Set $C_i := u(D_i)$. Set $\{Q_1, \dots, Q_x\} := u^{-1}(P)$ (as sets) with $Q_i \neq Q_j$ for all $i \neq j$. Assume that P is a seminormal point of X with multiplicity m. Hence each Q_i is a seminormal point of Y and, calling m_i its multiplicity, $m_1 + \cdots + m_x = m$. Since the m branches of X at P are linearly independent and $F \cong u_*(\mathcal{O}_Y)$, the fiber $F|\{P\}$ of F at P is a $\mathbb{K}$-vector space of dimension x. Hence the natural isomorphism $H^0(X, F) \to H^0(Y, \mathcal{O}_Y)$ shows that F is spanned at P if and only if no connected component D_i contains two of the points $Q_1, \dots, Q_x$. Now (as in part (c)) we add the assumption $S = \{P\}$. We have $\deg(F) = x$, because $p_a(Y) - p_a(Y) = x$. Let $w : W \to X$ be the partial normalzation of X in which we normalize only the point P. Since $S = \{P\}$ and we are looking at full sheaves with singular set $\{P\}$, u factors through P. Hence $k \leq c$. Notice that $P \in C_i$ for all i. Hence $k \leq x$ and $k = x$ if and only if F is spanned at P.

(ii) Conversely, every element of $A(1, \{P\})(x)$ is obtained in the following way. Label the m branches of X at P (or, equivalently, label the m points $w^{-1}(P)$) with the integers $\{1, \dots, m\}$ and call $(S_1, \dots, S_c)$ the partition of this set obtained taking in each subset the points of $w^{-1}(P)$ lying in the same connected component of W. Take a partition of the set $\{1, \dots, c\}$ into x disjoint non-empty subsets $O_1, \dots, O_s$). Let Y be obtained from W gluing together the counterimages of P belonging to the closure of the same connected component of $W \backslash w^{-1}(P)$ and then gluing together (again to seminormal points) the equivalent classes lying in connected components labelled with an element of $\{1, \dots, c\}$ in the same partition. We saw that each F arising in this way is spanned at P, $\deg(F) = \ell(F) = x$, $\mathrm{Sing}(F) = \{P\}$, and $h^0(X, F) = x + 1$. Since Q is a Cartier divisor of X and F has pure rank 1, we get an integer $t > 0$ such that $F(-tQ)$ is generically spanned. Since $F(-tQ)$ is full and associated to $(Y, u, M(-t^{-1}(Q))$, $\ell(F(-tQ)) = \ell(F) = \deg(F) > \deg(F(-tQ))$, part (c) of Lemma 1 gives a contradiction. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, Brill-Noether theory of rank r sheaves on stable curves: an extremal case, *International Journal of Pure and Applied Mathematics*, **50**, No. 4 (2009), 591-594.

[2] Ph.R. Cook, *Local and Global Aspects of the Moduli Theory of Singular Curves*, Ph.D. Thesis, Liverpol University (1993).

[3] G.-M. Greuel, H. Knörrer, Einfache Kurvensingularitäten und torsionfreie Moduln, *Math. Ann.*, **270** (1985), 417-425.

[4] C. Seshadri, Fibrés vectoriels sur les courbes algébriques, *Astérisque*, **96** (1982).