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ON THE GONALITY OF NODAL REDUCIBLE PLANE CURVES WITH SMOOTH IRREDUCIBLE COMPONENTS

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Abstract: Let $X \subset \mathbb{P}^2$ be a nodal curve of degree $d \geq 4$ whose irreducible components are smooth. Here we prove that X has "gonality" $d-1$ and classify all generically spanned coherent sheaves F on X with depth 1, pure rank 1, $\deg(F) = d-1$ and $h^0(X, F) \geq 2$.

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1. Introduction

Fix integers $s \geq 2$ and $d_i \geq 1$, $1 \leq i \leq s$, such that $d := d_1 + \cdots + d_s \geq 4$. Let $W(d_1, \ldots, d_s)$ denote the set of all pairs (X, τ) such that X is a nodal plane curve with s irreducible components, all of them smooth, equipped with an ordering $X_1, \ldots, X_s$ of its irreducible components such that $\deg(X_i) = d_i$ for all i. The map $(X, \tau) \mapsto X$ induces a natural morphism β from $W(d_1, \ldots, d_s)$ to the projective space of all degree d plane curves. Set $V(d_1, \ldots, d_s) := \mathrm{Im}(\beta)$. The sets $W(d_1, \ldots, d_s)$ and $V(d_1, \ldots, d_s)$ are integral quasi-projective varieties of dimension $\sum_{i=1}^{s}(d_i^2 + 3d_i)/2$. The morphism β is finite-to-one. β is injective if and only if $d_i \neq d_j$ for all $i \neq j$. Since $d \geq 4$, every element of $V(d_1, \ldots, d_s)$ is a stable curve. Fix any $(X, \tau) \in W(d_1, \ldots, d_s)$, say $X = X_1 \cup \cdots \cup X_s$ and set $S := \{i \in \{1, \ldots, s\} : d_i = 1\}$. For any $L \in \mathrm{Pic}(X)$ the s-ple of integers $(\deg(L|X_1), \ldots, \deg(L|X_s))$ is called the multidegree of L. For any $P \in X_{reg}$ set $L_P := \mathcal{O}_X(1)(-P) \in \mathrm{Pic}(X)$. If $P \in X_i$, then L_P has multidegree $(u_1, \ldots, u_s)$ where $u_i = d_i - 1$ and $u_j = d_j$ for all $i \neq j$. Since $h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1-d)) = 0$, $\mathcal{O}_X(1)$ is a spanned line bundle on X such that $h^0(X, \mathcal{O}_X(1)) = 3$. Since $\mathcal{I}_{P,\mathbb{P}^2}(1)$

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is spanned, for every $P \in X_{reg}$ the line bundle L_P is a spanned line bundle such that $h^0(X, L_P) = 2$. The latter equality implies that $L_P \neq L_Q$ for all $P, Q \in X_{reg}$ such that $P \neq Q$: L_P and L_Q induces different morphisms to $\mathbb{P}^1$ and these morphisms are induced by a complete linear system. For any coherent sheaf F on X set $\mathrm{Sing}(F) := \{P \in X : F$ is not locally free at $P\}$. If F has depth 1, then $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(X)$. For any $P \in \mathrm{Sing}(X)$ set $A_P := \mathcal{I}_P(1)$. A_P is a coherent sheaf on X with depth 1 and pure rank 1. A_P is spanned, $\deg(A_P) = d - 1$ and $h^0(X, A_P) = 2$. If $P, Q \in \mathrm{Sing}(X)$ and $P \neq Q$, then A_P and A_Q are not isomorphic, because they have different non-locally free loci.

Example 1. Assume $S \neq \emptyset$ and fix $i \in S$. We have $\sharp(X_i \cap \overline{X \backslash X_i}) = d - 1$. Since $X_i \cong \mathbb{P}^1$, there is a degree $d - 1$ morphism $v : X_i \to \mathbb{P}^1$ such that $v(X_i \cap \overline{X \backslash X_i})$ is a unique point, Q, of $\mathbb{P}^1$. The set of all such morphisms v is parametrized by a non-empty open subset of a $(d-1)$-dimensional projective space. Since each point of $X_i \cap \overline{X \backslash X_i}$ is an ordinary node of X and $\mathbb{P}^1$ is a smooth curve, a well-known property of certain singularities (singularities with a module ([4], §6) or universal singularities ([3], end of p. 387)) gives that the morphism v induces a unique morphism $u : X \to \mathbb{P}^1$ such that $u|X_i = v$ and $v(X_j) = Q$ for every $j \neq i$. Notice that u determines v and that v determines u. Set $L(i)_v := u^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Hence $L(i)_v$ is a degree $d - 1$ spanned line bundle on X.

(a) We claim that $h^0(X, L(i)_v) = 2$. Assume $h^0(X, L(i)_v) \geq 3$. Fix any $j \in \{1, \ldots, s\} \backslash \{i\}$ and a general $P \in X_j$. Since $h^0(X, L(i)_v) \geq 3$, $h^0(X, L(i)_v(-P)) \geq 2$. Since $L(i)_v|X_h$ is trivial for all $h \neq i$ and $X_h \cap X_j \neq \emptyset$, any $s \in H^0(X, L(i)_v(-P))$ vanishes at each point of $\overline{X \backslash X_i}$. Hence it vanishes at $d - 1$ points of X_i, too. Since $\deg(L(i)_v|X_i) = d - 1$, we get $h^0(X, L(i)_v(-P)) \leq 1$, contradiction.

(b) Now assume $\sharp(S) \geq 2$ and fix $i, j \in S$ such that $i \neq j$. Since $h^0(X, L(i)_v) = 2$, it is obvious that $L(i)_v$ and $L(j)_{v'}$ are not isomorphic.

Theorem 1. *Fix any $X \in V(d_1, \ldots, d_s)$.*

(i) $d - 1$ is the minimal positive integer t such that there is a degree t spanned line bundle on X.

(ii) Every degree $d - 1$ spanned line bundle on X is isomorphic either to a unique L_P, $P \in X_{reg}$, or to a unique line bundle $L(i)_v$ described in Example 1. The latter posibility does not occur if $d_j \geq 2$ for every $j \in \{1, \ldots, s\}$.

Example 2. Assume $S \neq \emptyset$ and fix $i \in S$. Let $u_i : X(i) \to X$ be the partial normalization of X in which we normalize only the $d - 1$ points $X_i \cap \mathrm{Sing}(X)$. Since $d_i = 1$ and $d \geq 2$, $X(i)$ has two connected components. Hence $A(i) := u_{i*}(\mathcal{O}_{X(i)})$ is a coherent sheaf on X with depth 1, pure rank 1,

$\mathrm{Sing}(A(i)) = X_i \cap \mathrm{Sing}(X)$, $\deg(A(i)) = d-1$, and $h^0(X, A(i)) = 2$. Since $X \backslash T$ is connected for every proper subset T of $X_i \cap \mathrm{Sing}(X)$, $A(i)$ is spanned. If $i, j \in S$ and $i \neq j$, then $A(i)$ and $A(j)$ are not isomorphic, because they have diffent non-locally free loci.

Example 3. Assume $S \neq \emptyset$ and fix $i \in S$. Fix an integer a such that $1 \leq a \leq d-2$ and $A \subset X_i \cap \overline{X \backslash X_i}$ such that $\sharp(A) = a$. Let $u : C \to X$ denote the partial normalization of X in which we only normalize the points of A. Let $E \subset C$ be the strict transform of X_i in C. Set $W := E \cap \overline{C \backslash E}$. We have $\sharp(W) = d-1-a$. Since $E \cong \mathbb{P}^1$, there is a degree $d-1-a$ morphism $v : E \to \mathbb{P}^1$ such that $v(W)$ is a unique point, Q, of $\mathbb{P}^1$. The set of all such morphisms v is parametrized by a non-empty open subset of a $(d-1-a)$-dimensional projective space. Since each point of W is an ordinary node of C and $\mathbb{P}^1$ is a smooth curve, a well-known property of certain singularities (singularities with a module ([4], §6) or universal singularities ([3], end of p. 387)) the morphism v induces a unique morphism $\eta : C \to \mathbb{P}^1$ such that $\eta | X_i = v$ and $\eta(X_j) = Q$ for every $j \neq i$. Set $L(i, a, A) := \eta^*(\mathcal{O}_{\mathbb{P}^1}(1))$. $L(i, a, A)$ is a degree $d-1-a$ spanned line bundle on C. Set $F(i, a, A) := u_*(L(i, a, A))$. We have $A = \mathrm{Sing}(F(i, a, A))$ and hence the isomorphism class of $F(i, a, A)$ determines uniquely A and hence a. Remark 2 below gives $\deg(F(i, a, A)) = a + \deg(L(i, a, A)) = d-1$. Notice that $L(i, a, A)$ uniquely determines E and hence the integer $i \in S$. We will prove in Remark 2 that $L(i, a, A)$ is uniquely determined by $F(i, a, A)$, because $L(i, a, A) \cong F(i, a, A)''$. We have $h^0(X, F(i, a, A)) = h^0(C, L(i, a, A)) \geq 2$. A byproduct of parts (i) of Theorems 1 and 2 below gives $h^0(X, F(i, a, A)) = 2$ and the spannedness of $F(i, a, A)$.

Theorem 2. *Fix any $X \in V(d_1, \ldots, d_s)$.*

(i) $d-1$ is the minimal positive integer t such that there is a degree t spanned coherent sheaf on X with depth 1, pure rank 1 and not locally free.

(ii) Every degree $d-1$ spanned coherent sheaf on X with depth 1, pure rank 1 and not locally free is isomorphic either to a unique A_P, $P \in Sing(X)$ or $S := \{i \in \{1, \ldots, s\} : d_i = 1\} \neq \emptyset$ and there is a unique $i \in S$ such that $F \cong A(i)$ or $S \neq \emptyset$ and there are a unique $i \in S$, a unique integer a, and a unique $A \subset X_i \cap \overline{X \backslash X_i}$ such that $1 \leq a \leq d-2$, $\sharp(A) = a$ and $F \cong F(i, a, A)$.

Remark 1. Assume the statements of Theorems 1 and 2. We get the non-existence of a coherent sheaf F on X with depth 1, pure rank 1, $h^0(X, F) \geq 2$, and F spanned by its global sections outside finitely many points of X, because the subsheaf G of X spanned by $H^0(X, F)$ has pure rank 1. Fix $i \neq \{1, \ldots, s\}$ and $P \in X_{reg} \cap X_i$. Let $w_i : X_i \hookrightarrow X$ denote the inclusion. The sheaf $w_{i*}(\mathcal{O}_{X_i}(1)(-P))$ is spanned, but it has not pure rank. If $d_i \geq 2$, then this sheaf is different from $w_{i*}(\mathcal{O}_{X_i})$.

Question 1. Theorems 1 and 2 show that the set of all coherent sheaves F on X with depth 1, pure rank 1, $\deg(F) = d-1$ and $h^0(X,F) \geq 2$ is very different if $d_i = 1$ for at least one index i or not. Now assume $d_j \geq 2$ for all j and the existence of $i \in \{1,\dots,s\}$ such that $d_i = 2$. Since $\sharp(X_i \cap \overline{X \setminus X_i}) = 2d-4$, it is easy to extend Examples 1, 2, 3 and get examples of degree $2d-4$ spanned coherent sheaves on X with depth 1 and pure rank 1. Is it possible (at least under the assumption "$d_j \geq 2$ for all j") to extend the classification up to the degree $2d-4$?

We work over an algebraically closed base field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

2. The Proofs

Remark 2. Let X be a nodal curve and F a coherent sheaf on X with depth 1 and pure rank 1. Let $u_F : X_F \to X$ denote the partial normalization of X in which we normalize only the points of $\mathrm{Sing}(F)$. Set $F' := u_F^*(F)$ and $F'' := F'/\mathrm{Tors}(F')$. F'' is a line bundle on X_F. The classification of all modules with depth 1 and pure rank 1 over a nodal curve singularity ([5], pp. 164–166) gives that the germ of F at P is isomorphic to the maximal ideal of the local ring $\mathcal{O}_{X,P}$. Hence $\deg(F'') = \deg(F) - \sharp(\mathrm{Sing}(F))$. Since F has no torsion, the natural maps $H^0(X,F) \to H^0(X_F,F') \to H^0(X_F,F'')$ are injective. Hence $h^0(X_F,F'') \geq h^0(X,F)$. Now assume that F is spanned. Since the tensor product is a right exact functor, F' is spanned. Hence F'' is spanned. There is a natural map $i_F : F \to u_{F*}(F')$. Since F has no torsion, i_F induces a map $j_F : F \to u_{F*}(F'')$. The maps i_F and j_F are isomorphisms outside $\mathrm{Sing}(F)$. Since F has no torsion, i_F and j_F are injective. Since $u_{F*}(F'')$ has depth 1, pure rank 1 and $\deg(u_{F*}(F'')) = \deg(F'') + \sharp(\mathrm{Sing}(F)) = \deg(F)$, j_F is an isomorphism. Hence $h^0(X,F) = h^0(X,F'')$.

Proof of Theorem 1. Let $X_1,\dots,X_s$ be an ordering of the irreducible components of X such that $\deg(X_i) = d_i$ for all i. Let L be a spanned line bundle of minimal positive degree t. Obviously, $t \leq d-1$. Set $b_i := \deg(L|X_i)$. Since L is spanned, $b_i \geq 0$ and $b_i = 0$ if and only if $L|X_i \cong \mathcal{O}_{X_i}$. Set $M := \{i \in S : b_i > 0\}$. We have $\sum_{i=1}^s b_i = \sum_{i \in M} b_i = t$. Let Z be the zero-locus of a general section σ of L. Let $h_L : X \to \mathbb{P}^r$, $r := h^0(X,L) - 1$, be the morphism induced by the complete linear system $|L|$. If $h^0(X,L) = 2$, then set $a_L := h_L$. If $h^0(X,L) \geq 3$ let $a_L : X \to \mathbb{P}^1$ be the composition of h_L with a general linear projection of $\mathbb{P}^r$ onto $\mathbb{P}^1$.

(a) Since $a_L : X \to \mathbb{P}^1$ is a dominant morphism, $h_L^{-1}(P)$ is finite for a general $P \in \mathbb{P}^1$. Hence a general zero-locus Z is always zero-dimensional.

(b) By step (a) we may assume that Z contains no irreducible component of X. Hence Z is a length t zero-dimensional subscheme of X. Since L is a spanned line bundle and σ is general, $Z \subset X_{reg}$. Hence $L' \cong \mathcal{O}_X(Z)$. Since char$(\mathbb{K}) = 0$, Z is reduced, say $Z = \{P_1, \dots, P_t\}$ with $P_i \neq P_j$ for all $i \neq j$. The spannedness of L implies $h^0(X, L(-P_i)) = h^0(X, L) - 1$ for all i. Riemann-Roch and Serre duality give $h^0(X, \omega_X(-Z)) = h^0(X, \omega_X(-Z + P_i))$ for all i. The adjunction formula gives $\omega_X \cong \mathcal{O}_X(d-3)$. Since $h^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) = 0$ for $i = 0, 1$, the restriction map $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3)) \to H^0(X, \mathcal{O}_X(d-3))$ is bijective. Hence $h^0(\mathbb{P}^2, \mathcal{I}_{Z \setminus \{P_i\}}(d-3)) = h^0(\mathbb{P}^2, \mathcal{I}_Z(d-3))$ for every $P_i \in Z$, i.e. the set $Z \subset \mathbb{P}^2$ satisfies the Cayley-Bacharach condition with respect to the line bundle $\mathcal{O}_{\mathbb{P}^2}(d)$. The case $k = 1$ of [2] gives $t = d - 1$ and that the linear span D_σ of Z is a line.

(c) Here we assume that D_σ is an irreducible component of X, say $D_\sigma = X_i$. Since h_L is a morphism, we get $b_j = 0$ for all $j \neq i$. Since $t = d - 1$, we get $b_i = d - 1$ and that h_L contracts to a point all curves X_j with $j \neq i$. Since any two curves in a plane meet, we get $h_L(X_j) = h_L(X_h)$ for all $j, h \in \{1, \dots, s\} \setminus \{i\}$. Hence $h_L(\overline{X \setminus X_i})$ is a point. Thus $L \cong L(i)_v$ for some v.

(d) Here we assume that D_σ is not an irreducible component of X. Since $Z \subseteq D_\sigma \cap X_{reg}$ there is a unique $P \in X_{reg}$ such that $D \cap X = \{P\} \cup Z$, with the convention that if $P \in Z$, then the reduction of the right hand side is Z and in the right hand side the point P appears with multiplicity 2. Since $\mathcal{O}_X \cong \mathcal{O}_X(1)$, L is a line bundle and P is a Cartier divisor of X, $L \cong \mathcal{O}_X(1)(-P)$. □

Proof of Theorem 2. Let F be generically spanned coherent sheaf on X with depth 1, pure rank 1, not locally free and with minimal degree t. Obviously, $t \leq d-1$. The minimality of t and the particular case $t' \leq t - 1 \leq d - 2$ of Theorem 1 give that F is spanned. Hence the line bundle F'' on X_F is spanned (Remark 2). Set $a := \sharp(\mathrm{Sing}(F))$. By assumption $a > 0$. Remark 2 gives $F \cong u_{F*}(F'')$, $\deg(F'') = t - a$ and that the natural map $H^0(X, F) \to H^0(X_F, F'')$ is an isomorphism. Let $h_{F''} : X_F \to \mathbb{P}^1$ be the morphism induced by F''. Since F'' is spanned, we have $a \leq t$. First assume $a = t$. Since F'' is spanned and with degre zero, $F'' \cong \mathcal{O}_{X_F}$. Since $h^0(X_F, F'') = 2$ (Remark 2), X_F has two connected components. Call U and $\overline{X \setminus U}$ the two subcurves of X whose strict transform in X_F give the two connected components of X_F. Without loosing regerality we may assume $\deg(U) \leq \deg(\overline{X \setminus U})$. Set $k := \deg(U)$. Since $t = a$, X_F is not connected and U intersects transversally $\overline{X \setminus U}$, we get $t = a = k(d-k)$. Hence $a = t = d - 1$ and U is a line. We also get $\mathrm{Sing}(F) = U \cap \overline{X \setminus U}$. Since U is an irreducible component of X, Example 2 arises if and only if $t = a$. Hence from now on we may assume $1 \leq a \leq t-1$. Since $\sharp(T \cap \overline{X\backslash T}) \geq d-1$ for every proper subcurve T of X and $a \leq t-1 \leq d-2$, X_F is connected. We identify each X_i with the unique irreducible component of X_F mapped isomorphically onto X_i by u_F. Let $h_{F''} : X_F \to \mathbb{P}^1$ be the morphism induced by the complete linear system $|F''|$.

(a) Let Z be the zero-locus of a general section σ of F''. Since $h_{F''} : X_F \to \mathbb{P}^1$ is a dominant morphism, $h_L^{-1}(P)$ is finite for a general $P \in \mathbb{P}^1$. Hence a general zero-locus Z is always zero-dimensional. Thus $\deg(Z) = t-a$.

(b) By step (a) we may assume that Z contains no irreducible component of X_F. Hence Z is a length $t-a$ zero-dimensional subscheme of X_F. Since F'' is a spanned line bundle and σ is general, $Z \cap u_F^{-1}(\mathrm{Sing}(X)) = \emptyset$. Hence $F'' \cong \mathcal{O}_{X_F}(Z)$ and u_F maps isomorphically Z onto a subscheme B' of X_{reg} and hence of $\mathbb{P}^2$. Since $\mathrm{char}(\mathbb{K}) = 0$, Z is reduced, say $Z = \{P_1, \dots, P_{t-a}\}$ with $P_i \neq P_j$ for all $i \neq j$. Hence B' is reduced, $\sharp(B') = t-a$ and $B' = \{Q_1, \dots, Q_{t-a}\}$, where $Q_i := u_F(P_i)$ for all i. The spannedness of F'' implies $h^0(X_F, F''(-P_i)) = h^0(X_F, F'') - 1$ for all i. Riemann-Roch and Serre duality give $h^0(X_F, \omega_{X_F}(-Z)) = h^0(X_F, \omega_{X_F}(-Z+P_i))$ for all i. The adjunction formula gives $\omega_X \cong \mathcal{O}_X(d-3)$. Since each point of $\mathrm{Sing}(F)$ is an ordinary node of X, the theory of nodal plane curves gives an isomorphism $\beta : H^0(\mathbb{P}^2, \mathcal{I}_{\mathrm{Sing}(F)}(d-3)) \to H^0(X_F, \omega_{X_F})$. Hence there is an isomorphism $\beta_\sigma : H^0(\mathbb{P}^2, \mathcal{I}_{\mathrm{Sing}(F) \cup B'}(d-3)) \to H^0(X_F, \mathcal{I}_Z \otimes \omega_{X_F})$. Since $Z \neq \emptyset$ and $h^0(X_F, \omega_{X_F}(-Z)) = h^0(X_F, \omega_{X_F}(-Z+P_1))$, we get that $\mathrm{Sing}(F) \cup B'$ does not impose independent conditions to the linear system of all plane curves of degree $d-3$. We have $\sharp(\mathrm{Sing}(F) \cup B') = t \leq d-1$. It is well-known that this implies $t = d-1$ and that $\mathrm{Sing}(F) \cup B'$ is contained in a line D_σ (e.g. apply a particular case of [1], Lemma 4.1, or apply [2], Lemma 1.2, with $A := \emptyset$, $t := 0$, $n := d$, $B := \mathrm{Sing}(F) \cup B'$ and $b := t$). Since obviously $t \geq 2$, the line D_σ is uniquely determined by Z and $\mathrm{Sing}(F)$. Notice that we also proved that $\mathrm{Sing}(F)$ is contained in D_σ. If $a \geq 2$, then D_σ is uniquely determined by $\mathrm{Sing}(F)$. Hence if $a \geq 2$ varying σ and hence Z we get that D_σ is an irreducible component, say X_h, of X and that $b_i = 0$ for all $i \neq h$. Since $A \subset X_h$, $\overline{X_F\backslash X_h}$ is connected. Since $h_{F''}$ contracts the strict transform of each X_j, $j \neq i$, to a point Q_j, we get $Q_i = Q_j$ for all $i, j \in \{1, \dots, \}\backslash\{h\}$. Hence $Q_i = Q_j$ for all $i, j \in \{1, \dots, \}\backslash\{h\}$, i.e. $h_{F''}(\overline{X_F\backslash X_h})$ is a point. Hence $F'' = h_{F''}^*(\mathcal{O}_{\mathbb{P}^1}(1)) \cong L(h, a, A)$. Since $F \cong u_{F*}(F'')$ (Remark 2) we get $F \cong F(h, a, A)$. The same proof works if $a = 1$ and D_σ is an irreducible component of X, because in this case D_σ does not depend upon the choice of a general section σ. Now assume $a = 1$, say $\mathrm{Sing}(F) = \{P\}$, and that D_σ is not an irreducible component of X. Since $t = d-1$, $\sharp(Z) = t-1$, and $P \notin Z$, we get that $D_\sigma \cap X$ is the union of P

(counted with multiplicity 2) and Z. Since $D_\sigma \cap X$ is finite, while Z moves for different general sections σ, we get $F \cong A_P$. □

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