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ON THE RIESZ DIAMOND POTENTIAL AND ITS INVERSION PROBLEM

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Abstract: By the convolution of an adecuate function f with the Diamond Riesz kernel the Riesz Diamond potential of order (α, β) is introduced. We study some elementary properties and the inversion problem is solved by using hypersingular integral operators defining certain fractional derivative as the inverse operator.

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1. Introduction

This article deals with the Riesz Diamond operator or Riesz Diamond Potential defined on the basis of the Riesz Diamond kernel introduced by Kananthai (cf. [7]) who also introduced the Diamond differential operator (cf. [5]) $\lozenge$ defined as

$$\lozenge = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2, \tag{1.1}$$

where $p + q = n$, n the dimension of the space $\mathbb{R}^n$, which may be factorized as the "product" of the ultrahyperbolic differential operator and the Laplacian

$$\lozenge = \square\Delta\,, \tag{1.2}$$

where

$$\square = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}; \tag{1.3}$$

and

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$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\,. \tag{1.4}$$

On the basis of this "product" in finding the elementary solution he uses the convolution of functions that are elementary solutions of the operators $\square$ and Δ.

Let us consider the Riesz Diamond kernel $K_{\alpha,\beta}(x)$ introduced by Kananthai in [7] given by the convolution

$$K_{\alpha,\beta}(x) = R_\alpha^e * R_\beta^H\,, \tag{1.5}$$

where R_α^e is the elliptic Riesz kernel of order α

$$R_\alpha^e(x) = \frac{|x|^{\alpha-n}}{H_n(\alpha)}\,, \tag{1.6}$$

where $|x| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}}$, α is a complex number, n is the dimension of $\mathbb{R}^n$ and the constant $H_n(\alpha)$ is given by

$$H_n(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\alpha)}{\Gamma\left(\frac{n-\alpha}{2}\right)},$$

and $R_\beta^H(x)$ is the ultrahyperbolic Riesz kernel introduced by Nozaki [8, p. 72] given by

$$R_\beta^H(u) = \begin{cases} \frac{u^{\frac{\beta-n}{2}}}{K_n(\beta)} & \text{if } x \in K_+\,, \\ 0 & \text{if } x \notin K_+\,, \end{cases} \tag{1.7}$$

where $K_n(\beta)$ is the constant given by

$$K_n(\beta) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+\beta-n}{2}\right) \Gamma\left(\frac{1-\beta}{2}\right) \Gamma(\beta)}{\Gamma\left(\frac{2+\beta-p}{2}\right) \Gamma\left(\frac{p-\beta}{2}\right)}, \tag{1.8}$$

$\Gamma(z)$ is the Euler Gamma function, u is the quadratic form in n variables

$$u = u(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2\,, \tag{1.9}$$

where $p + q = n$, β a complex number and K_+ denotes the interior of the forward cone defined as $K_+ = \{x \in \mathbb{R} : x_1 > 0, u > 0\}$; $\overline{K}_+$ denotes its closure.

The support of the function $R_\alpha^H(u)$ is included in the cone $\overline{K}_+$.

It may be observed that the elliptic kernel R_α^e and the ultrahyperbolic kernel R_β^H which are ordinary functions if Re $(\alpha) \geq n$, and Re $(\beta) \geq n$, respectively, are distributions for Re $(\alpha) < n$, and Re $(\beta) < n$.

By putting $p = 1$ in (1.7) and (1.8), $R_\alpha^H(u)$ reduces to the hyperbolic Riesz kernel (cf. [9, p. 31])

$$M_\alpha(x) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)} & \text{if } x \in K_+\,, \\ 0 & \text{if } x \notin K_+\,, \end{cases} \tag{1.10}$$

where $u = x_1^2 - x_2^2 - \cdots - x_n^2$ and

$$H_n(\alpha) = 2^{\alpha-1}\pi^{\frac{n-2}{2}}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha-n+2}{2}\right),$$

here $\Gamma(z)$ is the Euler Gamma function.

2. Preliminaries

Definition 1. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the Euclidean n-dimensional space $\mathbb{R}^n$ and let $P = P(x)$ be the quadratic form in n variables given in (1.9) that we rewrite:

$$P = P(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2. \tag{2.1}$$

Gelfand and Shilov (cf. [4]) defined the $(P \pm i0)^\lambda$ generalized function as the limit:

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0}\left(P \pm i\varepsilon\,|x|^2\right)^\lambda, \tag{2.2}$$

where ε is a positive real number, λ a complex number and

$$|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2.$$

They are analytic function on λ everywhere except at $\lambda = -\frac{n}{2} - k$, $k = 0, 1, 2, \ldots$, where they have simple poles (cf. [4, p. 275]).

We will use the following expression of the $(P \pm i0)^\lambda$ distribution in terms of the P_+ and P_- generalized functions defined by:

$$P_+^\lambda = \begin{cases} P^\lambda & \text{if } P > 0\,, \\ 0 & \text{if } P \le 0\,, \end{cases} \tag{2.3}$$

$$P_-^\lambda = \begin{cases} 0 & \text{if } P \ge 0\,, \\ |P|^\lambda & \text{if } P < 0\,. \end{cases} \tag{2.4}$$

Then

$$(P \pm i0)^\lambda = P_+^\lambda + e^{\pm i\pi\lambda}P_-^\lambda\,. \tag{2.5}$$

Let us now consider the causal (anticausal) ultrahyperbolic kernel $H_\alpha(P \pm i0, n)$ given by

$$H_\alpha(P \pm i0, n) = C(\alpha, n) \cdot (P \pm i0)^{\frac{\alpha - n}{2}}, \tag{2.6}$$

where

$$C(\alpha, n) = \frac{e^{\pm i \frac{\pi}{2} q} \cdot \Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \cdot \Gamma\left(\frac{\alpha}{2}\right) \pi^{\frac{n}{2}}}. \tag{2.7}$$

This generalized function is the causal (anticausal) analogue of the elliptic kernel due to M. Riesz, and its Fourier transform is

$$\mathcal{F}[H_\alpha(P \pm i0, n)] = (Q \mp i0)^{\frac{\alpha}{2}}. \tag{2.8}$$

For $\alpha \in \mathbb{C}$ that Re $(\alpha) > 0$, and $f \in S$ we have the generalized causal Riesz potential of order α defined by the convolution

$$R^\alpha = H_\alpha(P + i0, n) * f. \tag{2.9}$$

The inversion of causal Riesz potentials was treated in a similar way which was considered for causal Bessel potentials by using hypersingular causals integral in differences (cf. [1]).

Definition 2. Let α be a real number, $\frac{n+\alpha}{2} \neq -\frac{n}{2} - k$, $k = 0, 1, \ldots$, $\alpha < l$, l non negative integer and f a function of the space S the Schwartz class of infinitely differentiable functions on $\mathbb{R}^n$ decreasing at infinity faster than $|x|^{-1}$ and let $(T_l^\alpha f)(x)$ be the operator defined by the formula

$$(T_l^\alpha f)(x) = \int_{\mathbb{R}^n} (P \pm i0)^{-\frac{n+\alpha}{2}} \left\{ \left(\Delta_t^l f \right)(x) \right\} dt, \tag{2.10}$$

$\alpha > 0$, $l > 0$, where

$$\left\{ \left(\Delta_t^l f \right)(x) \right\} = \sum_{k=0}^{l} \binom{l}{k} (-1)^k f(x - kt).$$

The operator $(T_l^\alpha f)(x)$ must be interpreted in the same sense that in the case of causal Bessel operator (cf. [3]).

The formula (2.10) defines an operator that was called "hypersingular causal integral in differences" by analogy with the integral defined by Samko (cf. [11, p. 1091, formula (1.3)]) used to invert elliptic Riesz potentials. Its Fourier transform is given by

$$\mathcal{F}[T_l^\alpha f](\xi) = d_{n,l}(\alpha)(Q - i0)^{\frac{\alpha}{2}} \mathcal{F}[f], \tag{2.11}$$

where $d_{n,l}(\alpha)$ is

$$d_{n,l}(\alpha) = \begin{cases} \dfrac{\pi^{\frac{n}{2}+1}\mathcal{A}_l(\alpha)\, e^{i\frac{\pi}{2}q}}{2^{\alpha}\Gamma\left(\frac{n+\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}+1\right)\sin\frac{\pi\alpha}{2}} & \text{for } \alpha \neq 2,4,6\,, \\ \dfrac{(-1)^{\frac{\alpha}{2}}\pi^{\frac{n}{2}}2^{1-\alpha}e^{i\frac{\pi}{2}q}}{\Gamma\left(\frac{\alpha}{2}+1\right)2^{\alpha}\Gamma\left(\frac{n+\alpha}{2}\right)}\dfrac{d}{d\alpha}\mathcal{A}_l(\alpha) & \text{for } \alpha = 2,4,6\,, \end{cases} \tag{2.12}$$

and

$$\mathcal{A}_l(\alpha) = \sum_{k=1}^{l}\binom{l}{k}(-1)^k k^{\alpha}\,.$$

Definition 3. The generalized causal Riesz derivative of order α of a function $f \in S$ is defined by

$$(\mathcal{D}^{\alpha} f)(x) = \frac{1}{d_{n,l}(\alpha)}(T_l^{\alpha} f)(x)\,, \tag{2.13}$$

where α is a real number, l non negative integer, $l > \alpha$ and $\alpha \neq 1,3,5,\ldots$, and $d_{n,l}(\alpha)$ is given by (2.12).

Then, from (2.11), we have

$$\mathcal{F}[D^{\alpha} f](\xi) = (Q - i0)^{\frac{\alpha}{2}}\mathcal{F}[f](\xi)\,. \tag{2.14}$$

This generalized causal Riesz derivatives inherits some properties from the Riesz derivatives. In fact, under the assumption for $\alpha, \beta \in \mathbb{R}^+$, $\varphi \in S$ and $\mathcal{D}^{\alpha}$ defined by (2.13) (cf. [2]); we have the semigroup property:

$$\mathcal{D}^{\alpha+\beta}\varphi = \mathcal{D}^{\alpha}\mathcal{D}^{\beta}\varphi. \tag{2.15}$$

Another important property of the generalized Riesz derivative is the following relation with the ultrahyperbolic differential operator given by the next

Lemma 2.1. *Let f be a function that belongs to S, L^k ultrahyperbolic differential operator iterated k-times, and $\mathcal{D}^{\alpha}$ the generalized causal Riesz derivative of order α, $\alpha \in \mathbb{R}$, $\alpha \geq 2k$, $k = 1,2,\ldots$.*

Then is valid

$$\mathcal{D}^{\alpha} f = \mathcal{D}^{\alpha-2k}\left\{L^k f\right\} = L^k\left\{\mathcal{D}^{\alpha-2k} f\right\}. \tag{2.16}$$

The proof of Lemma 2.1 appears in [1].

For $\alpha = 2k$, $k = 1,2,\ldots$ we have

$$\mathcal{D}^{2k} f = L^k f\,. \tag{2.17}$$

Moreover, we can obsserve that the inverse causal Riesz operator $(R^{\alpha})^{-1}$ and then the causal Riesz derivative of order α are, formally, a fractional power of the ultrahyperbolic differential operator

$$(R^{\alpha})^{-1} = \mathcal{D}^{\alpha} = \Box^{\frac{\alpha}{2}} . \tag{2.18}$$

This result coincide with the one due to Samko (cf. [10, p. 555]) who introduces a fractional power of the D'Alembertian as

$$(-\Box)^{\lambda} \varphi = \mathcal{F}^{-1} \left((P \mp i0)^{\lambda} \mathcal{F} [\varphi] \right) . \tag{2.19}$$

We may observe that if in (2.17) $q = 0$ is considered we obtain

$$\mathcal{D}^{2l} = \Delta^{l} f , \tag{2.20}$$

where Δ^l is the Laplacian iterated l times, and the left hand member is the integral introduced by Samko (cf. [11]) to invert elliptic Riesz potential.

Definition 4. Let $K_{\alpha,\beta}$ be the Riesz Diamond kernel of order (α, β), α and β complex numbers, and let f be a functions belongs to $\mathbb{C}_0^{\infty}(K_+)$. The Riesz Diamond potential of order (α, β) of the function f, denoted by $R^{(\alpha,\beta)} f$ is, by definition, the convolution

$$R^{(\alpha,\beta)} f = K_{\alpha,\beta} * f \tag{2.21}$$

for $\operatorname{Re}\alpha > 0$, and $\operatorname{Re}\beta > 0$.

Remark. Taking into accout than when $\beta = 0$, $K_{\alpha,\beta} = K_{\alpha,0}$ reduces to $R^e_{\alpha} * R^H_0 = R^e_{\alpha} * \delta = R^e_{\alpha}$, the elliptic Riesz potential and that when $\alpha = 0$, $K_{\alpha,\beta} = K_{0,\beta}$ reduces to $R^e_0 * R^H_{\beta} = \delta * R^H_{\beta} = R^H_{\beta}$ the ultrahyperbolic Riesz potentials, the operator defined in (2.21) is a generalization of the Riesz potentials. From here its name.

Definition 5. Let f be a function belongs to $\mathbb{C}_0^{\infty}(K_+)$, let α and β be complex numbers that Re $(\alpha) > 0$, Re $(\beta) > 0$.

We define the following operator

$$\left(T_l^{\alpha,\beta} \right) f(x) = \left(T_l^{\alpha} * T_l^{\beta} \right) f(x) . \tag{2.22}$$

Taking into account that (T_l^{α}) is a temperate distribution and $f \in \mathbb{C}_0^{\infty}(K_+)$, (2.22) exists and is a temperate distribution. Then its Fourier transform is

$$\mathcal{F} \left[\left(T_l^{\alpha} * T_l^{\beta} \right) f(x) \right] = d_{n,l(\alpha)} \cdot d_{n,l(\beta)} (Q - i0)^{\frac{\alpha}{2}} (Q - i0)^{\frac{\beta}{2}} \mathcal{F}[f] . \tag{2.23}$$

Definition 6. By putting $q = 0$ in T_l^{β} and in $(Q - i0)^{\frac{\beta}{2}}$ in (2.23) we define the generalized Riesz Diamond derivative of order (α, β) of a function $f \in \mathbb{C}_0^{\infty}(K_+)$ by

$$\left(\mathcal{D}^{\alpha,\beta} f \right)(x) = \frac{1}{d_{n,l}(\alpha)} \cdot \frac{1}{d_{n,l}(\beta)} \left(T_l^{\alpha} * T_l^{\beta} \right) f(x) , \tag{2.24}$$

and then

$$\mathcal{F}\left[\mathcal{D}^{(\alpha,\beta)}f\right](\xi) = (Q-i0)^{\frac{\alpha}{2}}\left(|Q|^2\right)^{\frac{\beta}{2}}\mathcal{F}[f](\xi)\ . \tag{2.25}$$

3. Some Properties of the Generalized Riesz Diamond Derivative

Firstly we will prove that the generalized Riesz Diamond derivative is a left inverse operator of the Riesz Diamond potential of the same order.

Theorem 7. *Let* $\varphi \in \mathbb{C}_0^\infty(K_+)$, α, β *complex numbers that* $\mathrm{Re}\,(\alpha) > 0$, $\mathrm{Re}\,(\beta) > 0$, *and* $\mathcal{D}^{(\alpha,\beta)}\varphi = f$. *Then* $\varphi = K^{(\alpha,\beta)}f$.

Proof. Given $f = \mathcal{D}^{(\alpha,\beta)}\varphi$, applying Fourier transform we get

$$\mathcal{F}[f] = \mathcal{F}\left[\mathcal{D}^{(\alpha,\beta)}\varphi\right]\ . \tag{3.1}$$

From (2.25) it results

$$\mathcal{F}[f] = (Q-i0)^{\frac{\alpha}{2}}\left(|Q|^2\right)^{\frac{\beta}{2}}\mathcal{F}[\varphi]\ , \tag{3.2}$$

or equivalently

$$\mathcal{F}[\varphi] = (Q-i0)^{-\frac{\alpha}{2}}\left(|Q|^2\right)^{-\frac{\beta}{2}}\mathcal{F}[f]\ . \tag{3.3}$$

It is well known (cf. [12, p. 268]) that the Fourier transform maps the convolution to the pointwise product, then the right hand member of (3.3) may be written as $\mathcal{F}[K_{\alpha,\beta} * f]$ and then we have

$$\mathcal{F}[\varphi] = \mathcal{F}[K_{\alpha,\beta} * f]\ . \tag{3.4}$$

By the uniqueness of the Fourier transform, it results

$$\varphi = K_{\alpha,\beta} * f = R^{(\alpha,\beta)}f \tag{3.5}$$

and the theorem follows. □

The last expression may be written as the following assertion: if $\mathcal{D}^{(\alpha,\beta)}\varphi = f$ then $\varphi = R^{(\alpha,\beta)}f$. Thus

$$\mathcal{D}^{(\alpha,\beta)}R^{(\alpha,\beta)}f = f\ . \tag{3.6}$$

This property is analogue to those valids for elliptic Riesz potentials (cf. [11, p. 1099]), and the ones due to us for the causal Riesz potentials (cf. [1]).

Lemma 3.1. *Let* (α, β) *and* (α', β') *be pairs of complex numbers satisfying* $\mathrm{Re}\,(\alpha) > 0$, $\mathrm{Re}\,(\beta) > 0$, $\mathrm{Re}\,(\alpha') > 0$ *and* $\mathrm{Re}\,(\beta') > 0$.

Then

$$\mathcal{D}^{(\alpha,\beta)+(\alpha',\beta')}f = \mathcal{D}^{(\alpha,\beta)}\mathcal{D}^{(\alpha',\beta')}f\ . \tag{3.7}$$

Proof. Let us now

$$\begin{aligned}\mathcal{F}\left[\mathcal{D}^{(\alpha,\beta)+(\alpha',\beta')}f\right] &= (Q-i0)^{\frac{\alpha+\alpha'}{2}}\left(|Q|^2\right)^{\frac{\beta+\beta'}{2}}\mathcal{F}[f] \\ &= (Q-i0)^{\frac{\alpha}{2}}(Q-i0)^{\frac{\alpha'}{2}}\left(|Q|^2\right)^{\frac{\beta}{2}}\left(|Q|^2\right)^{\frac{\beta'}{2}}\mathcal{F}[f] \\ &= (Q-i0)^{\frac{\alpha}{2}}\left(|Q|^2\right)^{\frac{\beta}{2}}\mathcal{F}\left[\mathcal{D}^{(\alpha',\beta')}f\right] \\ &= \mathcal{F}\left[\mathcal{D}^{(\alpha,\beta)}\mathcal{D}^{(\alpha',\beta')}f\right].\end{aligned}$$

Then, by virtue of the uniqueness of the Fourier transform we get

$$\mathcal{D}^{(\alpha,\beta)+(\alpha',\beta')}f = \mathcal{D}^{(\alpha,\beta)}\mathcal{D}^{(\alpha',\beta')}f. \qquad \square \tag{3.8}$$

Theorem 8. *Let f be a function $\in \mathbb{C}_0^\infty(K_+)$, and let $\Diamond^k$ be the Diamond differential operator iterated k times and $\mathcal{D}^{(\alpha,\beta)}$ the generalized Riesz Diamond derivative of order (α,β), $\operatorname{Re}(\alpha) > 2k$, $\operatorname{Re}(\beta) > 2k$, $k = 1, 2, \ldots$.*

Then

$$\mathcal{D}^{(\alpha,\beta)}f = \mathcal{D}^{(\alpha-2k,\beta-2k)}\left\{\Diamond^k f\right\} = \Diamond^k\left\{\mathcal{D}^{(\alpha-2k,\beta-2k)}f\right\}. \tag{3.9}$$

Proof. Let f be a function that verifies $f = R^{(\alpha,\beta)}\varphi$, for some $\varphi \in \mathbb{C}_0^\infty(K_+)$. Thus, we have

$$\mathcal{D}^{(\alpha-2k,\beta-2k)}R^{(\alpha,\beta)}.\varphi = \varphi \tag{3.10}$$

By substitution we get

$$\mathcal{D}^{(\alpha-2k,\beta-2k)}\left\{\Diamond^k R^{(\alpha,\beta)}\varphi\right\} = \varphi. \tag{3.11}$$

Putting $\alpha = 2k$, $\beta = 2k$ in (3.11) it results

$$\Diamond^k\left\{R^{(2k,2k)}\varphi\right\} = \varphi. \tag{3.12}$$

Then

$$\begin{aligned}\mathcal{D}^{(\alpha,\beta)}f &= \mathcal{D}^{(\alpha,\beta)}R^{(\alpha,\beta)}\varphi = \varphi = \Diamond^k\left\{R^{(2k,2k)}\varphi\right\} \\ &= \Diamond^k\mathcal{D}^{(\alpha-2k,\beta-2k)}R^{(\alpha-2k,\beta-2k)}\varphi \\ &= \Diamond^k\mathcal{D}^{(\alpha-2k,\beta-2k)}R^{(\alpha,\beta)}\varphi \\ &= \Diamond^k\mathcal{D}^{(\alpha-2k,\beta-2k)}f.\end{aligned} \tag{3.13}$$

From (3.11) and (3.13)

$$\mathcal{D}^{(\alpha,\beta)}f = \mathcal{D}^{(\alpha-2k,\beta-2k)}\Diamond^k f = \Diamond^k\left\{\mathcal{D}^{(\alpha-2k,\beta-2k)}f\right\} \tag{3.14}$$

and the theorem is proved. $\square$

The formulae (3.14) is analogue to the formula due to Samko (cf. [11, p. 1094]) for Riesz potential and to the one due to Marcel Riesz (cf. [9, p. 32]) valid for the Riemann-Liouville integral in the space with Lorentzian metric.

Corollary 9. *If we consider $\alpha = 2k$, and $\beta = 2k$ then*

$$\mathcal{D}^{(2k,2k)} f = \Diamond^k f \,. \tag{3.15}$$

Proof. From (3.14) when $\alpha = 2k$ and $\beta = 2k$ is considered, it results. □

The last formulae expresses that the inverse Riesz Diamond operator of order $(2k, 2k)$, $k = 1, 2, \ldots$, is given by the Riesz Diamond differential operator iterated k times.

Corollary 10. *If q, the number of negative terms of the quadratic form $\mathcal{P} = P(x)$ given by (2.1), is equal to zero, then*

$$\mathcal{D}^{(2k,2k)} f = \left(\Delta^2\right)^k f \,. \tag{3.16}$$

where Δ is the Laplacian.

Proof. If in (2.1) we consider the case $q = 0$, then the ultrahyperbolic Riesz kernel given by (1.7) reduces to the elliptic Riesz kernel (1.6). By similar reasons (2.25) reduces to the product of $|Q|^{\frac{\alpha}{2}} |Q|^{\frac{\beta}{2}} \mathcal{F}[f](\xi)$, that when $\alpha = \beta = 2k$ results

$$\mathcal{F}\left[\mathcal{D}^{(2k,2k)} f\right] \xi = |Q|^{2k} \mathcal{F}[f](\xi) = \left(\xi_1^2 + \cdots + \xi_n^2\right)^{2k} \mathcal{F}[f](\xi) \tag{3.17}$$

and by the uniqueness of the Fourier transform, the lemma follows. □

Corollary 11. *If $\alpha = 2k$, $\beta = 0$ and $q = 0$ we have*

$$\mathcal{D}^{(2k,0)} f = \Delta^k f \,. \quad \square \tag{3.18}$$

This formulae coincides with the one valid for elliptic Riesz potentials due to Samko (cf. [11, p. 1094]), and it is the same like that in (2.20).

Lemma 3.2. *Let us now $f = K_{\alpha,\beta}(x)$. Then*

$$\mathcal{D}^{(\alpha,\beta)} K_{\alpha,\beta}(x) = \delta \,. \tag{3.19}$$

Proof. Applying Fourier transform we have

$$\begin{aligned} \mathcal{F}\left[\mathcal{D}^{(\alpha,\beta)} K_{\alpha,\beta}(x)\right] &= (Q - i0)^{\frac{\alpha}{2}} \left(|Q|^2\right)^{\frac{\beta}{2}} \mathcal{F}[K_{\alpha,\beta}(x)](\xi) \\ &= (Q - i0)^{\frac{\alpha}{2}} \left(|Q|^2\right)^{\frac{\beta}{2}} (Q - i0)^{-\frac{\alpha}{2}} \left(|Q|^2\right)^{-\frac{\beta}{2}} = 1 \end{aligned}$$

Then, by the properties of the Fourier transforn we have

$$\mathcal{D}^{(\alpha,\beta)} K_{\alpha,\beta}(x) = \delta.$$

□

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