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BRILL-NOETHER THEORY ON SINGULAR CURVES (EXAMPLES WITH NON-EXISTENCE)

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Abstract: Here we discuss the existence of stable curves X with the Brill-Nother theory of a general curve of its genus, but whose gonality is not computed by line bundles on X. We classify all the examples in genus 4. We do not have new examples for genus ≥ 5.

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For all integers g, r, d set $\rho(g,r,d) := g-(r+1)(g+r-d) = (r+1)d-rg-r(r+1)$ (the Brill-Noether number for curves with genus g). Fix integers $g > q \geq 0$ such that $g \geq 2$. Let C be a general curve of genus q and X a general curve with arithmetic genus g and with $g-q$ singular points, each of them being either an ordinary node or an ordinary cusp. Fix positive integers g, r, d such that $\rho(g,r,d)$. An easy consequence of the proof of Gieseker-Petri Theorem is the existence of a degree spalled line bundle L on X such that $\deg(L) = d$ and $h^0(X,L) = \max\{r+1, d+1-g\}$. However, if we drop the generality assumptions, we could need non-locally free sheaves (see [2], Example 2.2). Our aim is to find an example in which X has the Brill-Noether theory of a general curve (but only in genus 4). In genus 4 we give a complete classification when X is Gorenstein and ω_X very ample. By [1], Theorem 1.2, or [5]; Theorem 3.6, this classification covers all non-hyperelliptic stable curves of genus 4 without a disconnecting node. If there is a disconnecting node, then there is a degree

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1 spanned sheaf with pure rank 1 (not locally free, but contained in a degree 2 lline bundle). We find only one case (part (b) of Example 2) with infinitely many spanned degree 3 line bundles.

Example 1. Let X be an integral and nodal projective curve of genus 4. Assume that X is not hyperelliptic. Since ω_X is very ample ([1], Theorem 1.2, or [5], Theorem 3.6) we will indentify X with its canonical model $X \subset \mathbb{P}^3$. Hence $\deg(X) = 6$, $\omega_X \cong \mathcal{O}_X(1)$ and X is the complete intersection of a quadric surface T and a cubic surface. If T is smooth, then X has two distinct spanned line bundles on X of degree d. The study of curves of type $(3,3)$ on a smooth quadric surface shows that the converse holds. Here we assume that T is singular. Since X is integral and non-degenerate, T is a quadric cone. Call O the vertex of T. If $O \notin X$, then X has a unique degree 3 spanned line bundle, even if X is singular. Hence from now on we assume $O \in X$. Hence X is singular and $O \in \mathrm{Sing}(X)$ ([8], Example V.5.6). Let $u : W \to T$ be the blowing-up of O. Hence W is isomorphic to the Hirzebruch surface F_2 and we will take as a basis of $\mathrm{Pic}(W)$ the curve $h := u^{-1}(O)$ and a fiber f of the ruling of W. We have $h^2 = -2$, $h \cdot f = 1$ and $f^2 = 0$. Let $Y \subset W$ denote the strict transform of X in W. $u|Y$ is a birational morphism $u_1 : Y \to X$. Since $O \in \mathrm{Sing}(X)$ and X is integral, the linear projection from O onto a general linear section D of the cone T (i.e. onto a smooth plane conic) shows that one of the following cases occurs:

(i) X has multiplicity 2 at O and the linear projection of X from O induces a degree 2 rational map onto D. In this case Y is an integral member of $|2h+6f|$.

(ii) X has multiplicity 4 at O and the linear projection of X from O induces a birational map onto D

Let $v : C \to X$ denote the partial normalization of X in which we normalize the point O. There is a morphism $w : C \to Y$ such that $v = w \circ u_1$.

(a) Here we study case (i), i.e. we assume $Y \in |2h + 6f|$. Since Y is the strict transform of X, not its total transform, h is not a component of Y. Since $h \cdot (h + 2f) = 0$, we get $h \cap Y = \emptyset$. Hence $C = Y$ and $v = u_1$. Since $\omega_W \cong \mathcal{O}_W(-2h - 4f)$, the adjunction formula gives $\omega_Y \cong \mathcal{O}_Y(2f)$. Hence $p_a(Y) = 3$. Since $Y \in |2h + 6f|$ the ruling of W induces a spanned $R \in \mathrm{Pic}(Y)$ such that $\deg(R) = 2$.

Claim. $h^0(Y, R) = 2$.

Proof of Claim. Since $h^0(W, \mathcal{O}_W(f)) = 2$ and $h^1(W, \mathcal{O}_W(-2h - 5f)) = h^1(W, \mathcal{O}_W(f)) = 0$ (Serre duality and the cohomology of W), the Claim is true. Set $M := u_{1*}(R)$. We have $h^i(X, M) = h^i(Y, R)$, $i = 1, 2$. Hence the Claim gives $h^0(X,M) = 2$. Since ω_X is very ample, X has no degree 1 or degree 2 spanned torsion free sheaf M' with rank 1 and $h^0(X,M') \geq 2$. Hence M is spanned. Here we check that there is no other degree 3 torsion free sheaf N on X with rank 1 and $h^0(X,N) \geq 2$. Assume the existence of such a sheaf N. Since X has no degree 1 or degree 2 spanned torsion free sheaf M' with rank 1 and $h^0(X,M') \geq 2$, N is spanned and $h^0(X,N) = 2$. By the geometric form of Riemann-Roch the zero-locus of any non-zero section of N is contained in a line. By Bezout theorem each such line is contained in T. Since X contains no line of T, we get that N is induced by the pencil of lines of T through O, i.e. $N \cong M$. We also notice that rank 1 torsion free sheaf F on X with $h^0(X,F) \geq 3$ has degree at least 5, because X is not hyperelliptic and Clifford's Theorem is true for integral curves ([7], Theorem A of Appendix). If $\deg(F) = 5$, then $h^0(X,F) = 3$ and there is $P \in X$ such that $F \cong \mathcal{I}_P \otimes \omega_X$.

(b) Here we study case (ii), i.e. we assume $Y \in |h + 6f|$. Hence $Y \cong \mathbb{P}^1$. Since X has multiplicity 4 at O, X has neither an ordinary node nor an ordinary cusp at O. We want to prove the non-existence of $L \in \mathrm{Pic}(X)$ such that $\deg(L) = 3$ and $h^0(X,L) \geq 2$. Assume the existence of $L \in \mathrm{Pic}(X)$ such that $\deg(L) = 3$ and $h^0(X,L) \geq 2$. Since ω_X is very ample, L is spanned and $h^0(X,L) = 2$. Hence $|L|$ induces a degree 3 morphism $h_L : X \to \mathbb{P}^1$. The geometric form of Riemann-Roch shows that any fiber of h_L spans a line. Bezout theorem shows that any such line is contained in T. Since $O \in X$, we see the non-existence of h_L and hence the non-existence of L. As in (b) we also get the uniqueness of the rank 1 torsion free sheaf with degree 3 and at least two linearly independent sections.

Example 2. Let X be a reduced, Gorenstein and connected genus 4 curve for which ω_C is very ample. See [1], Theorem 1.2, or [5], Theorem 3.6, for the description of the stable curves for which ω_C is very ample. We identify X with its canonical model. Hence X is contained in the intersection of a quadric surface T and a cubic surface T' not containing T. As in the case "X integral" considered in Example 1 the quadric T is smooth if and only if X has 2 spanned degree 3 line bundles, while the curve X has a spanned degree 3 line bundle if T is a quadric cone with vertex O and $O \notin X$.

(a) Here we assume that T is a quadric cone with vertex O. Again, we find $O \in \mathrm{Sing}(X)$ and that one of the following cases occurs:

(i) X has multiplicity 2 at O and the linear projection of X from O induces a degree 2 rational map onto D. In this case $Y \in |2h + 6f|$.

(ii) X has multiplicity 4 at O.

As in Example 1 let $Y \subset W$ be the strict transform of X. The map $u|Y$ is a birational morphism $u_1 : Y \to X$. Let $v : C \to X$ denote the partial normalization of X in which we normalize the point O. There is a morphism $w : C \to Y$ such that $v = w \circ u_1$.

(a1) Here we study case (i), i.e. we assume $Y \in |2h + 6f|$. Since Y is the strict transform of X, not its total transform, h is not a component of Y. Since $h \cdot (h + 2f) = 0$, we get $h \cap Y = \emptyset$. Hence $C = Y$ and $v = u_1$. Since $\omega_W \cong \mathcal{O}_W(-2h - 4f)$, the adjunction formula gives $\omega_Y \cong \mathcal{O}_Y(2f)$. Hence $p_a(Y) = 3$. The geometric form of Riemann-Roch gives the non-existence of a degree 3 line bundle on X with at least 2 linearly independent sections.

(a2) Here we study case (ii), i.e. we assume $Y \in |h + 6f|$. Since h is not an irreducible component of Y, the reduced curve Y is a union of e distinct fibers of the ruling of W (for some integer e such that $0 \le e \le 4$, and an integral $Y' \in |h + (6 - e)f|$. Hence $Y' \cong \mathbb{P}^1$. Since X is assumed to be reducible, $e > 0$. Hence X is a union of e lines R_i, $1 \le i \le e$ such that $O \in X_i \subset T$ for all i, and an integral rational curve $X_0 \subset T$ of degree $6 - e$. Assume the existence of $L \in \mathrm{Pic}(X)$ such that $\deg(L) = 3$, L is spanned and $h^0(X, L) \ge 2$. Set $d_i := \deg(L|X_i)$, $0 \le j \le e$. Hence $d_j \ge 0$ for all $j \in \{0, \dots, e\}$ and $\sum_{j=0}^{e} d_j = 3$. We get a morphism $h_L : X \to \mathbb{P}^r$, $r := h^0(X, L) - 1$, which contracts a component X_j if and only if $d_j = 0$. Taking a general linear projection $\mathbb{P}^r \to \mathbb{P}^1$ (if $r > 1$, we again obtain a morphism $h'_L : X \to \mathbb{P}^1$ such that $\deg(h'_L|X_j) = d_j$ for all j. Take a general $A \in \mathbb{P}^1$. Then ${h'_L}^{-1})(A)$ is a length 3 zero-dimensional scheme. The geometric form of Riemann-Roch gives that ${h'_L}^{-1}(A)$ is contained in a line. Hence L is the non-locally free spanned sheaf induced by the ruling of W, contradiction.

(b) Here we assume that T is not integral. Since X is reduced and non-degenerate, T is a union of two distinct planes H_1 and H_2. Set $X_i := (X_i \cap H_i)_{red}$ and $d_i := \deg(X_i)$, $i = 1, 2$. Hence each X_i is a plane curve of degree d_i and $d_1 + d_2 \ge 6$ with strict inequality if and only if the line $H_1 \cap H_2$ is an irreducible component of X. There is an injective map $j_{i!} : \omega_{X_i} \to \omega_X|X_i \cong \mathcal{O}_{X_i}$. Hence $d_1 \le 3$, $d_2 \le 3$. Hence $d_1 = d_2 = 3$, X does not contain the line $H_1 \cap H_2$, and X is the complete intersection of T and a cubic surface T'. Assume the existence of $L \in \mathrm{Pic}(X)$ such that $\deg(L) = 3$, L is spanned and $h^0(X, L) \ge 2$. Let $h'_L : X \to \mathbb{P}^1$ be an associated morphism. Take a general $A \in \mathbb{P}^1$. Then ${h'_L}^{-1})(A)$ is a length 3 zero-dimensional scheme. The geometric form of Riemann-Roch gives that ${h'_L}^{-1}(A)$ is contained in a line. Hence there is $i \in \{1, 2\}$ such that $L|X_i \cong \mathcal{O}_{X_i}(1)$ and $L|X_{2-i} \cong \mathcal{O}_{X_{2-i}}$. Obviously, this morphism sends the scheme $X_1 \cap X_2$ into a point. Hence it is obtained in the following way. Fix any

$Q \in H_1 \cap H_2$ such that $Q \notin X$. $H'_L|X_i$ is the linear projection from Q, while $h'_L(X_{2-i})$ is a point. Conversersely, any such Q and the choice of the index $i \in \{1, 2\}$ gives the morphism. Hence X has 2 one-dimensional families of such spanned line bundles. In the boundary (i.e. if $Q \in X \cap H_1 \cap H_2$) we also find non-locally free degree 3 spanned sheaves with pure rank 1.

Remark 1. Take X as in Examples 1 or 2 (case T an integral quadric cone) with X nodal and stable. Let F be the only spanned torsion free sheaf on X such that $\deg(F) = 3$ and $h^0(X, F) \geq 3$. Let M be the quasi-stable curve with $m : M \to X$ as its stable reduction and a unique exceptional component E of M with $m(E) = O$. Let $L\mathrm{Pic}(M)$ be the only line bundle on M obtained from F as in [3]. The general theory of [3] (the properness of the relative compactified Picard functor), semicontinuity and the fact that a general smooth curve of genus 4 is trigonal give that L is balanced in the sense of [11], Definition 1.1, or [4], 2.1.1.

Question 1. Are there an integer $g \geq 5$ and $X \in \overline{\mathcal{M}}_g$ with the following properties:

(i) if $k \leq \lfloor (g + 1)/2 \rfloor$, then there is no degree k balanced line bundle L;

(i)' there is a torsion free sheaf F with degree d and $h^0(X, F) \geq r + 1$ (resp. a quasistable model X' of X and a balanced $L \in \mathrm{Pic}^d(X')$) if and

(ii) there is no degree $\lfloor (g + 3)/2 \rfloor$ balanced line bundle L on X with $h^0(X, L) \geq 2$.

Instead of (i) we may require the non-existence of depth 1 sheaves F on X with pure rank 1, degree k and $h^0(X, F) \geq 2$.

By [3] if X is as in Question 1, then there is a quasistable model $m : X' \to X$ such that $X' \neq X$ and X has a degree $\lfloor (g + 3)/2 \rfloor$ balanced line bundle L such that $h^0(X', L) = 2$. Conditions (i) and (ii) are satisfied in [2], Example 2.2, (at least if in the examples z is very near to q) which is even integral, but (i)' fails for these examples.

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