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POSITIVE REGULAR SOLUTIONS TO A SINGULAR INTEGRAL EQUATION

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Abstract: Let n be a positive integer and let α satisfy $0 < \alpha < n$. Consider a positive regular solution $u(x)$ to the integral equation

$$u(x) = \int_{R^n} |x-y|^{\alpha-n} u(y)^{(n+\alpha)/(n-\alpha)} dy.$$

We use the method of moving planes to prove that for every direction $u(x)$ is symmetric about a plane perpendicular to the direction and monotone on the two sides of the plane. It follows that $u(x)$ is radially symmetric about a point and is a strictly decreasing function of the radius. It then follows that $u(x)$ is a constant multiple of a function of form

$$\left(\frac{t}{t^2 + |x-x_0|^2}\right)^{(n-\alpha)/2},$$

where $t > 0, x_0 \in R^n$. Our work here adds to and modifies our previous works on the same problem.

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1. Introduction

Consider the singular integral equation on the Euclidean space R^n

$$u(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} u(y)^{\frac{n+\alpha}{n-\alpha}} dy \,, \tag{1.1}$$

where α satisfies $0 < \alpha < n$. This equation arises in the context of the Hardy-Littlewood-Sobolev (HLS) inequalities (cf. [19] and [20]) as the Euler-Lagrangian equation for functions that attain an optimizing constant. In the case of $n \geq 3$ and $\alpha = 2$, the equation (1.1) is closely associated with the well known semilinear differential equation $-\Delta u = u^{(n+2)/(n-2)}$. And in the case of $n \geq 5$ and α being an even integer larger than two but less than n, the equation is associated with a higher order differential equation.

It is well known that (1.1) is invariant of certain scaling, translation, and inversion transforms. Particularly, if $u(x)$ is a solution of (1.1) then all the following functions also satisfy (1.1): 1) $s^{(n-\alpha)/2}u(sx)$ for any positive constant s; 2) $u(x - x_1)$ for any point x_1 in R^n; and 3) $|x|^{\alpha-n}u(x/|x|^2)$.

As pointed out in [19], (1.1) has many singular solutions in addition to many regular solutions. For example, a constant multiple of $|x|^{(\alpha-n)/2}$ is a singular solution. Consequently, all the functions resulting from a translation transform compounded with an inversion transform of this singular solution are also singular solutions, generally having two singular points. Apparently, solutions with two singular points cannot be radially symmetric about any point.

With regular solutions, however, we know them all. In three previous papers [6], [7], [8] we have studied several problems related to positive regular solutions of (1.1). In particular, we have proved that if a positive solution is locally $L^{2n/(n-\alpha)}$ integrable, then it is radially symmetric and monotone and is a constant multiple of a function of form

$$u_{t,x_0}(x) := \left(\frac{t}{t^2 + |x - x_0|^2}\right)^{(n-\alpha)/2}, \tag{1.2}$$

where $t > 0$ and $x_0 \in R^n$. It is elementary to verify that

$$\begin{aligned} s^{(n-\alpha)/2}u_{t,x_0}(sx) &= u_{t/s,\, x_0/s}(x) \qquad (s > 0)\,, \\ u_{t,x_0}(x - x_1) &= u_{t,\, x_0+x_1}(x) \qquad (x_1 \in R^n)\,, \end{aligned}$$

$$|x|^{\alpha-n}u_{t,x_0}(x/|x|^2) \quad = \quad u_{t/(t^2+|x_0|^2),\, x_0/(t^2+|x_0|^2)}(x).$$

In other words, a function of form (1.2) is still of form (1.2) after a scaling, translation, or inversion transform.

In this paper we present alternative proofs on the radial symmetry and monotonicity of positive regular solutions of (1.1) using the method of moving planes. In comparison with the proof in our previous paper [6], the proofs here are at a more elementary level. We also have alternative proofs on the part of Lieb's work in [19] and [20] that determines the form of the solutions from the radial symmetry and monotonicity. In addition, we will have a regularity result for (1.1) that modifies our previous proof.

Our proofs are first directed at positive solutions of (1.1) satisfying the assumption that

$$\begin{cases} u(x) \text{ is continuous everywhere and} \\ u_\infty := \lim_{x\to\infty} |x|^{n-\alpha}u(x) > 0. \end{cases} \tag{1.3}$$

This assumption is really without much loss of generality. In fact, a solution $u(x)$ that is in $L^{2n/(n-\alpha)}(R^n)$, which is natural in the context of the HLS inequalities, must satisfy (1.3). This regularity result was obtained by Y.-Y. Li in [17]. Here we will modify our proof in [7] on the same regularity.

We also have in mind continuous entire solutions of (1.1) without any assumption on the rate of decay at infinity. By performing an inversion transform on such a solution we obtain a positive solution $v(x)$ that satisfies (1.3) but with a possible singularity at the origin. However, the possible singularity will not cause any difficulty. Eventually the possible singularity will be shown to be removable. Thus we can show that any continuous entire solution is a constant multiple of a function of form (1.2) as well. We summarize our results as follows.

Theorem 1.1. *Suppose $u(x)$ satisfies (1.3) and is a solution of (1.1). Then $u(x)$ is a constant multiple of a function of form (1.2).*

Theorem 1.2. *Suppose $u(x)$ is a continuous entire solution of (1.1). Then $u(x)$ satisfies (1.3) and thus is a constant multiple of a function of form (1.2).*

Theorem 1.3. *Suppose $u(x)$ is in $L^{2n/(n-\alpha)}(R^n)$ and satisfies (1.1). Then $u(x)$ satisfies (1.3) and thus is a constant multiple of a function of form (1.2).*

Let λ be a real number and let the moving plane be $x_1 = \lambda$. We use Σ_λ to denote the region to the right of the moving plane; that is,

$$\Sigma_\lambda := \{x = (x_1, \dots, x_n) \mid x_1 \geq \lambda\}.$$

Define

$$x^\lambda := (x_1 - 2\lambda, x_2, \dots, x_n), \quad u_\lambda(x) := u(x^\lambda).$$

Our plan is as follows. In Section 2 we state or prove two preliminary lemmas. In Section 3 we prove that there exists a negative number $-N$ such that $u(x) \geq u_\lambda(x)$ on Σ_λ for all $\lambda \leq -N$. In Section 4 we prove that if for a negative λ_0

$$u(x) \geq u_{\lambda_0}(x) \quad \text{but} \quad u(x) \not\equiv u_{\lambda_0}(x)$$

on Σ_{λ_0}, then there exists an $\epsilon > 0$ such that $u(x) \geq u_\lambda(x)$ on Σ_λ for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$. In Section 5 we discuss the radial symmetry and monotonicity of $u(x)$. We also study continuous entire solutions without a decay assumption at infinity. In Section 6 we come to know the form of the solution $u(x)$. Our discussion includes an exposition on the relevant work of Lieb. In Section 7 we give a slight modification on our earlier proof on that a solution $u(x)$ of (1.1) in $L^{2n/(n-\alpha)}(R^n)$ must satisfy (1.3). In Section 8 we make a few further remarks.

2. Two Preliminary Lemmas

We will use c for a general positive constant that depends on n, α, and the solution $u(x)$ itself. This c is usually different when it appears in different places.

Lemma 2.1. *We have*

$$u(x) - u_\lambda(x) = \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}}\right)\left(u(y)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}}\right) dy. \quad (2.1)$$

Remark. This lemma was in our previous paper [6]. We split the integrals for $u(x)$ and $u_\lambda(x)$ on R^n into integrals on the left and right of the plane $x_1 = \lambda$. The four terms are then factored into the product.

In the next lemma, B_R is the ball centered at the origin and with radius R, and B_R^C is the exterior region of the ball.

Lemma 2.2. *We have*

$$\int_{B_R^C} \frac{1}{(1+|x|)^{2\alpha}} \frac{1}{|x-y|^{n-\alpha}} dx \leq \frac{c}{R^\alpha} \quad \textit{for all } y \in B_R^C. \quad (2.2)$$

Proof. We decompose the set B_R^C into three parts D_1, D_2, D_3 with

$$\begin{aligned} D_1 &= \{x \mid x \in B_R^C, |x-y| \leq \frac{1}{2}|y|\}, \\ D_2 &= \{x \mid x \in B_R^C, |x-y| > \frac{1}{2}|y|, |x-y| > |x|\}, \end{aligned}$$

$$D_3 \quad = \quad \{x \mid x \in B_R^C, |x-y| > \frac{1}{2}|y|, |x-y| \le |x|\}.$$

On D_1, $|x| \ge |y| - |x-y| \ge \frac{1}{2}|y|$ and thus

$$\begin{aligned}
&\int_{D_1} \frac{1}{(1+|x|)^{2\alpha}} \frac{1}{|x-y|^{n-\alpha}} dx \\
&\le \frac{1}{(1+\frac{1}{2}|y|)^{2\alpha}} \int_{\{x \mid |x-y| \le \frac{1}{2}|y|\}} \frac{1}{|x-y|^{n-\alpha}} dx \\
&\le \frac{c}{|y|^{2\alpha}} |y|^{\alpha} = \frac{c}{|y|^{\alpha}} \le \frac{c}{R^{\alpha}}.
\end{aligned}$$

On D_2, $|x-y| > |x|$ and thus

$$\begin{aligned}
&\int_{D_2} \frac{1}{(1+|x|)^{2\alpha}} \frac{1}{|x-y|^{n-\alpha}} dx \\
&\le \int_{B_R^C} \frac{1}{(1+|x|)^{2\alpha}} \frac{1}{|x|^{n-\alpha}} dx \\
&\le \int_{B_R^C} \frac{1}{|x|^{n+\alpha}} dx \le \frac{c}{R^{\alpha}}.
\end{aligned}$$

On D_3, $|x| \ge |x-y|$ and thus

$$\begin{aligned}
&\int_{D_3} \frac{1}{(1+|x|)^{2\alpha}} \frac{1}{|x-y|^{n-\alpha}} dx \\
&\le \int_{D_3} \frac{1}{(1+|x-y|)^{2\alpha}} \frac{1}{|x-y|^{n-\alpha}} dx \\
&\le \int_{D_3} \frac{1}{|x-y|^{n+\alpha}} dx \\
&\le \int_{\{x \mid |x-y| \ge \frac{1}{2}|y|\}} \frac{1}{|x-y|^{n+\alpha}} dx \\
&\le \frac{c}{(\frac{1}{2}|y|)^{\alpha}} \le \frac{c}{R^{\alpha}}.
\end{aligned}$$

The lemma is now proved. □

3. Reflection about the Planes far Away

In this section we prove Theorem 3.1 that ensures that $u(x) \ge u_\lambda(x)$ on Σ_λ for all $\lambda \le -N$ where N is a large positive number. We provide two proofs.

Theorem 3.1. *There exists a number R_0 depending on n, α, and the* *function $u(x)$ itself such that if*

$$u(x) \geq u_\lambda(x) \ \text{ on } \ \Sigma_\lambda \cap B_{R_0} \ \text{ for a } \ \lambda \leq 0,$$

then $u(x) \geq u_\lambda(x)$ on all Σ_λ.

Remark. Here and later in Theorem 4.1 we restrict our attention to λ satisfying $\lambda \leq 0$ because we will make use of the following inequality

$$\frac{1}{|y^\lambda|} \leq \frac{1}{|y|} \ \text{ for } \ y \in \Sigma_\lambda.$$

It follows that for $y \in \Sigma_\lambda$ with $\lambda \leq 0$,

$$\frac{u(y)^{\frac{n+\alpha}{n-\alpha}} - u(y^\lambda)^{\frac{n+\alpha}{n-\alpha}}}{u(y) - u(y^\lambda)} \leq \frac{c}{(1+|y|)^{2\alpha}}. \tag{3.1}$$

Indeed, by the mean value theorem, the quotient on the left equals

$$\frac{n+\alpha}{n-\alpha}(\xi u(y) + (1-\xi)u(y^\lambda))^{\frac{2\alpha}{n-\alpha}}$$

for some $\xi \in (0,1)$ and is bounded above by

$$\frac{n+\alpha}{n-\alpha}(\frac{\xi c}{(1+|y|)^{n-\alpha}} + \frac{(1-\xi)c}{(1+|y^\lambda|)^{n-\alpha}})^{\frac{2\alpha}{n-\alpha}}.$$

for a constant c by the assumption (1.3).

We will also use the inequality

$$\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \geq 0 \quad \text{for all } x, y \text{ in } \ \Sigma_\lambda.$$

This inequality, however, is true no matter whether λ is positive or not.

From the assumption (1.3) we know that there exists an N such that $u(x) \geq u_\lambda(x)$ on $\Sigma_\lambda \cap B_{R_0}$ for all $\lambda \leq -N$. It follows from Theorem 3.1 that $u(x) \geq u_\lambda(x)$ on all Σ_λ for all $\lambda \leq -N$.

First Proof of Theorem 3.1. Let R_0 be a number to be determined later and assume $u(x) \geq u_\lambda(x)$ on $\Sigma_\lambda \cap B_{R_0}$. Define

$$\begin{aligned} \Sigma_\lambda^- &:= \{x \mid x \in \Sigma_\lambda, u(x) - u_\lambda(x) < 0\}, \\ \Sigma_\lambda^+ &:= \{x \mid x \in \Sigma_\lambda, u(x) - u_\lambda(x) \geq 0\}. \end{aligned} \tag{3.2}$$

By the assumption $\Sigma_\lambda^- \subset B_{R_0}^C$. It follows from (2.1) that for x in Σ_λ,

$$\begin{aligned} & u(x) - u_\lambda(x) \\ \geq & \int_{\Sigma_\lambda^-} (\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}})(u(y)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}})dy \\ \geq & \int_{\Sigma_\lambda^-} \frac{1}{|x-y|^{n-\alpha}}(u(y)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}})dy \end{aligned}$$

$$
\begin{aligned}
&= \int_{\Sigma_\lambda^-} \frac{1}{|x-y|^{n-\alpha}} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}}}{u(y)-u_\lambda(y)} (u(y)-u_\lambda(y))dy \\
&\geq \int_{\Sigma_\lambda^-} \frac{1}{|x-y|^{n-\alpha}} \frac{c}{(1+|y|)^{2\alpha}} (u(y)-u_\lambda(y))dy.
\end{aligned}
$$

In the last step we used (3.1).

Now define

$$
\mu = \int_{\Sigma_\lambda^-} \frac{1}{(1+|x|)^{2\alpha}} (u(x)-u_\lambda(x))dx.
$$

Note that the integral makes sense because of the decay assumption in (1.3). Then by the preceding inequality,

$$
\begin{aligned}
\mu &\geq \int_{\Sigma_\lambda^-} \Big(\int_{\Sigma_\lambda^-} \frac{1}{(1+|x|)^{2\alpha}} \frac{1}{|x-y|^{n-\alpha}} dx \Big) \frac{c}{(1+|y|)^{2\alpha}} (u(y)-u_\lambda(y))dy \\
&\geq \int_{\Sigma_\lambda^-} \Big(\int_{B_{R_0}^C} \frac{1}{(1+|x|)^{2\alpha}} \frac{1}{|x-y|^{n-\alpha}} dx \Big) \frac{c}{(1+|y|)^{2\alpha}} (u(y)-u_\lambda(y))dy \\
&\geq \frac{c}{R_0^\alpha} \int_{\Sigma_\lambda^-} \frac{1}{(1+|y|)^{2\alpha}} (u(y)-u_\lambda(y))dy \quad \text{(using Lemma 2.2)} \\
&= \frac{c}{R_0^\alpha} \mu.
\end{aligned}
$$

Now let us choose $R_0 = (2c)^{1/\alpha}$. Then we have $\mu \geq \frac{1}{2}\mu$, which implies $\mu = 0$. Thus Σ_λ^- must be an empty set. □

Second Proof of Theorem 3.1. Let R_0 be a number to be determined later and assume that $u(x) \geq u_\lambda(x)$ on $\Sigma_\lambda \cap B_{R_0}$ but the inequality is not true on all Σ_λ. By the assumption (1.3), the minimum

$$
-m = \min_{x \in \Sigma_\lambda^-} |x|^{n-\alpha}(u(x)-u_\lambda(x)) \quad (m > 0)
$$

is attained at a point $x_0 \in \Sigma_\lambda^- \subset \Sigma_\lambda \cap B_{R_0}^C$. Thus

$$
u(x) - u_\lambda(x) \geq -\frac{m}{|x|^{n-\alpha}} \quad \text{on } \Sigma_\lambda^- \tag{3.3}
$$

and

$$
u(x_0) - u_\lambda(x_0) = -\frac{m}{|x_0|^{n-\alpha}}.
$$

By (2.1),

$$
\begin{aligned}
-\frac{m}{|x_0|^{n-\alpha}} &= u(x_0) - u_\lambda(x_0) \\
&\geq \int_{\Sigma_\lambda^-} \Big(\frac{1}{|x_0-y|^{n-\alpha}} - \frac{1}{|x_0^\lambda - y|^{n-\alpha}} \Big) (u(y)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}})dy
\end{aligned}
$$

$$
\begin{aligned}
&\geq \int_{\Sigma_\lambda^-} \frac{1}{|x_0-y|^{n-\alpha}} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}}}{u(y)-u_\lambda(y)} (u(y)-u_\lambda(y))dy \\
&\geq \int_{\Sigma_\lambda^-} \frac{1}{|x_0-y|^{n-\alpha}} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}}}{u(y)-u_\lambda(y)} (-\frac{m}{|y|^{n-\alpha}})dy \\
&\geq \int_{\Sigma_\lambda^-} \frac{1}{|x_0-y|^{n-\alpha}} \frac{c}{(1+|y|)^{2\alpha}} (-\frac{m}{|y|^{n-\alpha}})dy \quad \text{(using (3.1))}.
\end{aligned}
$$

We are led to

$$
\begin{aligned}
\frac{1}{|x_0|^{n-\alpha}} &\leq c\int_{\Sigma_\lambda^-} \frac{1}{|x_0-y|^{n-\alpha}} \frac{1}{|y|^{n+\alpha}}dy \\
&\leq c\int_{B_{R_0}^C} \frac{1}{|x_0-y|^{n-\alpha}} \frac{1}{|y|^{n+\alpha}}dy.
\end{aligned}
$$

Next we show that we can choose R_0 so that the preceding inequality is impossible to hold. Indeed, this inequality further leads to

$$
\begin{aligned}
1 &\leq c\int_{B_{R_0}^C} \frac{1}{|\frac{x_0}{|x_0|} - \frac{y}{|x_0|}|^{n-\alpha}} \frac{1}{|y|^{n+\alpha}}dy \\
&= c\int_{B_{R_0/|x_0|}^C} \frac{1}{|z_0-z|^{n-\alpha}} \frac{1}{|x_0|^{n+\alpha}|z|^{n+\alpha}} |x_0|^n dz \\
&\quad \text{(where } z_0 = x_0/|x_0|, z = y/|x_0|) \\
&= \frac{c}{|x_0|^\alpha} \int_{B_{R_0/|x_0|}^C} \frac{1}{|z_0-z|^{n-\alpha}} \frac{1}{|z|^{n+\alpha}}dz \\
&= \frac{c}{|x_0|^\alpha} \Big(\int_{\{z \mid |z|\geq R_0/|x_0|, |z-z_0|\leq \frac{1}{2}\}} \frac{1}{|z_0-z|^{n-\alpha}} \frac{1}{|z|^{n+\alpha}}dz \\
&\quad + \int_{\{z \mid |z|\geq R_0/|x_0|, |z-z_0| > \frac{1}{2}\}} \frac{1}{|z_0-z|^{n-\alpha}} \frac{1}{|z|^{n+\alpha}}dz \Big) \\
&\leq \frac{c}{|x_0|^\alpha} \Big(\int_{\{z \mid |z-z_0|\leq \frac{1}{2}\}} \frac{1}{|z_0-z|^{n-\alpha}} 2^{n+\alpha}dz + \int_{\{z \mid |z|\geq R_0/|x_0|\}} 2^{n-\alpha} \frac{1}{|z|^{n+\alpha}}dz \Big) \\
&\leq \frac{c}{|x_0|^\alpha} \Big(1 + \frac{1}{(R_0/|x_0|)^\alpha}\Big) \\
&= \frac{c}{|x_0|^\alpha} + \frac{c}{R_0^\alpha} \leq \frac{c}{R_0^\alpha}.
\end{aligned}
$$

If we choose $R_0 = (2c)^{1/\alpha}$, then the inequality is impossible to hold. Thus for this choice of R_0 our assumption in the beginning of the proof would not be valid. □

4. The Moving Planes

Our next step is to prove Theorem 4.1 that ensures either we can move the planes $x_1 = \lambda$ all the way to $x_1 = 0$ such that $u(x) \geq u_\lambda(x)$ on Σ_λ or we can move the planes to some $x_1 = \lambda_0$ ($\lambda_0 < 0$) about which the solution is symmetric.

Theorem 4.1. *Suppose for some $\lambda_0 < 0$, $u(x) \geq u_{\lambda_0}(x)$ on Σ_{λ_0} but $u(x) \not\equiv u_{\lambda_0}(x)$. Then there exists an ϵ depending on n, α, and the solution $u(x)$ itself such that $u(x) \geq u_\lambda(x)$ on Σ_λ for all λ in $[\lambda_0, \lambda_0 + \epsilon)$.*

Proof. By (2.1) and the continuity of $u(x)$ we know $u(x) > u_{\lambda_0}(x)$ in the interior of Σ_{λ_0}. Thus for λ close to λ_0, $u(x) < u_\lambda(x)$ on Σ_λ is possible only if x is large or is close to the plane $x_1 = \lambda$. Define $\Sigma_\lambda^-, \Sigma_\lambda^+$ as in (3.2). By (2.1), we know for $x \in \Sigma_\lambda^-$,

$$
\begin{aligned}
&u(x) - u_\lambda(x)\\
&\geq \int_{\Sigma_\lambda^-} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}}\right)\left(u(y)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}}\right)dy\\
&\geq \int_{\Sigma_\lambda^-} \frac{1}{|x-y|^{n-\alpha}} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}}}{u(y) - u_\lambda(y)}(u(y) - u_\lambda(y))dy\\
&\geq \int_{\Sigma_\lambda^-} \frac{1}{|x-y|^{n-\alpha}} \frac{c}{(1+|y|)^{2\alpha}}(u(y) - u_\lambda(y))dy.
\end{aligned}
$$

We are led to

$$
\begin{aligned}
&\int_{\Sigma_\lambda^-} \frac{1}{(1+|x|)^{2\alpha}}(u(x) - u_\lambda(x))dx \qquad (4.1)\\
\geq &\int_{\Sigma_\lambda^-} \left(\int_{\Sigma_\lambda^-} \frac{1}{(1+|x|)^{2\alpha}} \frac{c}{|x-y|^{n-\alpha}}dx\right)\frac{1}{(1+|y|)^{2\alpha}}(u(y) - u_\lambda(y))dy.
\end{aligned}
$$

Let δ, R_1 be positive numbers and let

$$D(\lambda, \delta, R_1) = \{x \mid x \in \Sigma_\lambda, \lambda < x_1 < \lambda + \delta, |x| \leq R_1\}.$$

We are to choose δ, R_1 such that

$$\int_{D(\lambda,\delta,R_1)\cup B_{R_1}^C} \frac{c}{(1+|x|)^{2\alpha}} \frac{1}{|x-y|^{n-\alpha}}dx \leq \frac{1}{2} \qquad (4.2)$$

for any $y \in D(\lambda, \delta, R_1) \cup B_{R_1}^C$. Here the constant c is the same one as in (4.1). Let us first assume that we can make such a choice. Then by the continuity of $u(x)$ there exists an ϵ such that for $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$,

$$\Sigma_\lambda^- \subset \text{closure of } D(\lambda, \delta, R_1) \cup B_{R_1^C}.$$

Define

$$\mu = \int_{\Sigma_\lambda^-} \frac{1}{(1+|x|)^{2\alpha}}(u(x) - u_\lambda(x))dx.$$

By (4.1) and (4.2),

$$\begin{aligned}\mu &\geq \int_{\Sigma_\lambda^-} (\int_{D(\lambda,\delta,R_1)\cup B_{R_1}^C} \frac{c}{(1+|x|)^{2\alpha}} \frac{1}{|x-y|^{n-\alpha}} dx) \\ &\quad \frac{1}{(1+|y|)^{2\alpha}}(u(y) - u_\lambda(y))dy \\ &\geq \frac{1}{2}\int_{\Sigma_\lambda^-} \frac{1}{(1+|y|)^{2\alpha}}(u(y) - u_\lambda(y))dy = \frac{1}{2}\mu,\end{aligned}$$

which implies $\mu = 0$. Therefore $u(x) \geq u_\lambda(x)$ on Σ_λ for $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$.

Now we verify (4.2). Recall that by a symmetrization argument (cf. Theorem 3.4 in [20] for example), for any Lebesgue measurable set E

$$\int_E \frac{1}{|x-y|^{n-\alpha}} dx \leq c(|E|)^{\alpha/n},$$

where $|E|$ is the Lebesgue measure of E. Thus

$$\begin{aligned}&\int_{D(\lambda,\delta,R_1)} \frac{c}{(1+|x|)^{2\alpha}} \frac{1}{|x-y|^{n-\alpha}} dx \\ &\leq c\int_{D(\lambda,\delta,R_1)} \frac{1}{|x-y|^{n-\alpha}} dx \\ &\leq c|D(\lambda,\delta,R_1)|^{\alpha/n} \leq c(R_1^{n-1}\delta)^{\alpha/n}\end{aligned}$$

for any $y \in R^n$. Next, let us consider the integral

$$\int_{B_{R_1}^C} \frac{c}{(1+|x|)^{2\alpha}} \frac{1}{|x-y|^{n-\alpha}} dx.$$

If $|y| \geq \frac{1}{2}R_1$, then by Lemma 2.2

$$\begin{aligned}&\int_{B_{R_1}^C} \frac{c}{(1+|x|)^{2\alpha}} \frac{1}{|x-y|^{n-\alpha}} dx \\ &\leq \int_{B_{R_1/2}^C} \frac{c}{(1+|x|)^{2\alpha}} \frac{1}{|x-y|^{n-\alpha}} dx \leq \frac{c}{(R_1/2)^\alpha}.\end{aligned}$$

If $|y| \leq \frac{1}{2}R_1$, then $1/|x-y|^{n-\alpha} \leq 1/(|x|/2)^{n-\alpha}$ and thus

$$\begin{aligned}&\int_{B_{R_1}^C} \frac{c}{(1+|x|)^{2\alpha}} \frac{1}{|x-y|^{n-\alpha}} dx \\ &\leq c\int_{B_{R_1}^C} \frac{1}{(1+|x|)^{2\alpha}} \frac{1}{|x|^{n-\alpha}} dx\end{aligned}$$

$$\leq \quad c\int_{B_{R_1}^C} \frac{1}{|x|^{n+\alpha}}dx \leq \frac{c}{R_1^\alpha}.$$

Combining these estimates, we know we can choose δ, R_1 so that (4.2) is valid. Theorem 4.1 is now proved. □

5. Radial Symmetry and Monotonicity

For a solution $u(x)$ of (1.1) satisfying (1.3), Theorems 3.1 and 4.1 imply that either for some $\lambda_0 < 0$, $u(x) \equiv u_{\lambda_0}(x)$ or $u(x) \geq u_0(x)$ on Σ_0. In the second case we consider the function

$$u(-x_1, x_2, \cdots, x_n),$$

which is also a solution of (1.1) satisfying (1.3). Applying the two theorems to the new function we know that $u(x)$ is symmetric about a plane $x_1 = \bar{\lambda}_0 \geq 0$. In any of the cases $u(x)$ is symmetric about a plane perpendicular to the x_1 axis and is monotone on the two sides of the plane. It follows that for every direction $u(x)$ is symmetric about a plane perpendicular to the direction and monotone on the two sides of the plane. We then choose n mutually perpendicular directions and determine n planes such that $u(x)$ is symmetric about each of the planes. It is obvious that at the intersection point of these planes $u(x)$ attains its unique absolute maximum. Then every symmetry plane of $u(x)$ must pass through this point. Therefore $u(x)$ is radially symmetric about the point and is a strictly decreasing function of the radius.

For a continuous entire solution $u(x)$ of (1.1) without the decay assumption in (1.3), by a translation we can assume $u(x)$ is not radially symmetric about the origin, for otherwise $u(x)$ would be a constant function. Set $v(x) = |x|^{\alpha-n}u(x/|x|^2)$. Then $v(x)$ satisfies (1.1), has the desired decay rate at infinity, and is continuous everywhere except possibly at the origin. Theorems 3.1 and 4.1 can be applied to $v(x)$ with only a few modifications. Indeed, in the estimate (3.1) for $v(y)$ we restrict $|y| \geq 1$. Accordingly, in choosing R_0 in Theorem 3.1 and R_1 in (4.2) we restrict to $R_0, R_1 \geq 1$. For Theorem 3.1, note that since $v(x)$ satisfies the integral equation (1.1), it is bounded from below by a strictly positive number in a punctured neighborhood of the origin. For Theorem 4.1, note that by (2.1) for $v(x)$, if $v(x) \geq v_{\lambda_0}(x)$ but $v(x) \not\equiv v_{\lambda_0}(x)$ on Σ_{λ_0} minus the origin for some $\lambda_0 < 0$, then $v(x) - v_{\lambda_0}(x)$ is bounded from below by a strictly positive number in a punctured neighborhood of the origin. Thus the possible singularity at the origin does not cause any difficulty for these two theorems.

Applying the modified Theorems 3.1 and 4.1 to $v(x)$ we conclude that for every direction $v(x)$ is symmetric about a plane perpendicular to the direction. Since $v(x)$ was made to be not symmetric about the origin, not all the symmetry planes pass through the origin. Hence the origin is a removable singularity. Accordingly, the entire solution $u(x)$ must satisfy the decay assumption in (1.3). That is, a continuous entire solution $u(x)$ of (1.1) must satisfy (1.3).

6. The Form of the Solutions

Let $u(x)$ be a solution of (1.1) satisfying (1.3). Its scaling, translation, and inversion transforms then are all solutions of (1.1) satisfying (1.3). Thus all these solutions are radially symmetric about a point and are strictly decreasing functions of the radius. It follows from these symmetry and monotonicity that the solution $u(x)$ must be a constant multiple of a function of form (1.2). Lieb made this observation in [19] and [20] by transforming the integral equation (1.1) on R^n to an integral equation on the unit sphere S^n in R^{n+1}. Although he only worked with the solutions that optimize a constant in the HLS inequalities, which he proved to be radially symmetric and monotone, his proof works for general functions whose scaling, translation, and inversion transforms are all radially symmetric about a point and are strictly decreasing functions of the radius. In [6] we had an alternative proof. Here we present another proof in the case of $n \geq 2$, which is relevant to a work of Ou in [22]. In the one dimensional case, this proof does not work; we instead present a proof transported from Lieb's proof. We will also give a brief exposition on Lieb's work.

Assume without loss of generality that

$$\begin{cases} u(x) \text{ is symmetric about the origin and} \\ u(0) = u_\infty = \lim_{x\to\infty} |x|^{n-\alpha} u(x), \end{cases} \tag{6.1}$$

for otherwise we make a translation transform and a scaling transform on $u(x)$. Let $e_1 = (1, 0, \ldots, 0)$ and set

$$v(x) = \frac{1}{|x|^{n-\alpha}} u\left(\frac{x}{|x|^2} - e_1\right).$$

Since $v(x)$ is an inversion transform of $u(x - e_1)$, it is radially symmetric about a point on the x_1 axis and is a strictly decreasing function of the radius. Furthermore,

$$v(e_1) = u(0) \text{ and } v(0) = u_\infty.$$

Thus $v(x)$ is radially symmetric about $e_1/2$. Now assume that the dimension $n \geq 2$. We recall that the inversion $x/|x|^2$ maps the plane (line) $x_1 = 1/2$ to the unit sphere (circle) centered at e_1; that is,

$$|\frac{x}{|x|^2} - e_1| = 1 \text{ if } x = (\frac{1}{2}, x_2, \ldots, x_n).$$

Thus for any $x = (\frac{1}{2}, x_2, \ldots, x_n)$,

$$\begin{aligned} v(\frac{1}{2}, x_2, \ldots, x_n) &= \frac{1}{(\frac{1}{4} + x_2^2 + \ldots + x_n^2)^{(n-\alpha)/2}} u(\frac{x}{|x|^2} - e_1) \\ &= \frac{1}{(\frac{1}{4} + x_2^2 + \ldots + x_n^2)^{(n-\alpha)/2}} c, \end{aligned}$$

where c is the value the solution u takes on the unit sphere centered at the origin. By the radial symmetry of $v(x)$ about $e_1/2$, for any x

$$v(x) = \frac{1}{(1/4 + |x - e_1/2|^2)^{(n-\alpha)/2}} c.$$

Thus $v(x)$ is a constant multiple of a function of form (1.2), with $t = 1/2$ and $x_0 = e_1/2$. It then follows that $u(x)$ itself is a constant multiple of a function of form (1.2).

The preceding proof does not work in the one dimensional case. The same argument gives only

$$u(x) = \frac{1}{|x|^{1-\alpha}} u(\frac{1}{x}) \tag{6.2}$$

after a long calculation. Instead, we consider the following function

$$v(x) = \frac{(1+y^2)^{(1-\alpha)/2}}{|1-xy|^{1-\alpha}} u(\frac{x+y}{1-xy}), \tag{6.3}$$

which comes from Lieb's proof. The function $v(x)$ can be expressed as

$$\frac{(1+\frac{1}{y^2})^{(1-\alpha)/2}}{|x - \frac{1}{y}|^{1-\alpha}} u(-\frac{1}{y} - \frac{1+\frac{1}{y^2}}{x - \frac{1}{y}}).$$

One can see that $v(x)$ is the result of a combination of scaling, translation, and inversion transforms of $u(x)$, and therefore is still a solution of (1.1).

With $y = 1$, we verify that $v(1) = v(-1)$ because of $u(0) = u_\infty$, and therefore $v(x)$ is symmetric about the origin. It follows that (6.2) is valid.

Once (6.2) is established we verify that for any y the function $v(x)$ in (6.3) satisfies $v(1) = v(-1)$ and is therefore symmetric about the origin. Next, in the expression

$$\frac{(1+y^2)^{(1-\alpha)/2}}{|1-xy|^{1-\alpha}} u(\frac{x+y}{1-xy}) = \frac{(1+y^2)^{(1-\alpha)/2}}{|1+xy|^{1-\alpha}} u(\frac{-x+y}{1+xy}),$$

which stands for $v(x) = v(-x)$, we substitute $x = \tan(-\theta/2 + \pi/4), y =$

$\tan(\theta/2)$. The result is

$$u(\tan(-\theta+\frac{\pi}{4})) = \frac{|\cos(-\theta+\frac{\pi}{4})|^{1-\alpha}}{|\cos(\frac{\pi}{4})|^{1-\alpha}}u(1).$$

Then we see that $u(x)$ is a constant multiple of $(1+|x|^2)^{-(1-\alpha)/2}$. The proof is now complete for the one dimensional case.

At this point we give an exposition on Lieb's proof. Lieb observed that the integral equation (1.1) can be transformed to an integral equation on the unit sphere S^n in R^{n+1} using the stereographic projection.

We first recall a few facts on the stereographic projection. Let

$$\sigma = (\sigma_1, \ldots, \sigma_n, \sigma_{n+1}), \quad \tau = (\tau_1, \ldots, \tau_n, \tau_{n+1})$$

be the images of x, y under the stereographic projection respectively. That is, the components of σ satisfy

$$\sigma_1 = \frac{2x_1}{1+|x|^2}, \ldots, \sigma_n = \frac{2x_n}{1+|x|^2}, \ \sigma_{n+1} = \frac{1-|x|^2}{1+|x|^2}$$

and τ has a similar expression in terms of y. It is elementary to verify that

$$|\sigma-\tau|^2 = \frac{4}{(1+|x|^2)(1+|y|^2)}|x-y|^2 \ \text{ and } \ dS_\tau = (\frac{2}{1+|y|^2})^n dy.$$

Here the distance between σ and τ is the distance between the two points in R^{n+1}, and dS_τ is the area element on the sphere S^n and dy the volume element in R^n. It would be helpful to recognize that the stereographic projection from R^n to S^n is induced by the inversion transform in R^{n+1} about the sphere centered at the north pole $(0, \ldots, 0, 1)$ and with the radius $\sqrt{2}$. Thus relations between the distances or the area elements on R^n and S^n can also be obtained using an elementary geometric argument for inversions (the same geometric argument implies the fact that the stereographic projection maps spheres (circles) and planes (lines) in R^n to spheres (circles) on S^n).

Set

$$U(\sigma) := (\frac{1+|x|^2}{2})^{(n-\alpha)/2}u(x). \tag{6.4}$$

Then $U(\sigma)$ satisfies the integral equation on the sphere

$$U(\sigma) = \int_{S^n} \frac{1}{|\sigma-\tau|^{n-\alpha}} U(\tau)^{\frac{n+\alpha}{n-\alpha}} dS_\tau. \tag{6.5}$$

This observation by Lieb equating (1.1) with (6.5) is very insightful. As an immediate result, it is easy to see that a constant function satisfies (6.5). It then follows that a constant multiple of a function of form (1.2) satisfies (1.1). Note that this is not a simple fact on its own. A much harder proof using the Fourier and the Laplace transforms can be found in [25, p. 131] or [19]. There one can also find an elementary expression for the normalizing constant in terms of the Gamma function.

Lieb then went on to show that if $u(x)$ satisfies (6.1) then $U(\sigma)$ given by (6.4) is a constant function. Note that already $U(\sigma)$ is symmetric about the x_{n+1} axis and has equal values at the north and south poles.

In the case of $n \geq 2$, he considered a new solution $V(\sigma)$ of (6.5) and its corresponding solution $v(x)$ of (1.1):

$$V(\sigma) = U(\sigma_1, \ldots, -\sigma_{n+1}, \sigma_n) = (\frac{1+|x|^2}{2})^{(n-\alpha)/2} v(x). \tag{6.6}$$

It is easy to verify that $v(x)$ satisfies $v(e_i) = v(-e_i),\ i = 1, \ldots, n$. Thus $v(x)$ is symmetric about the origin, and accordingly $V(\sigma)$ is symmetric about the x_{n+1} axis. It follows that $U(\sigma)$ is symmetric about both the x_{n+1} and the x_n axes. Thus $U(\sigma)$ must be a constant function.

In the case of $n = 1$, the same argument implies that $U(\sigma) = U(\sigma_1, \sigma_2)$ is symmetric about both the x_1 and the x_2 axes. To show that $U(\sigma)$ is symmetric about any line through the origin, Lieb considered

$$\begin{cases} V_\theta(\sigma) = U(\cos\theta\sigma_1 - \sin\theta\sigma_2, \sin\theta\sigma_1 + \cos\theta\sigma_2) \\ = (\frac{1+|x|^2}{2})^{(n-\alpha)/2} v_\theta(x). \end{cases} \tag{6.7}$$

Then $v_\theta(1) = v_\theta(-1)$ is verified. It follows that $v_\theta(x)$ is symmetric about the origin, $V_\theta(\sigma)$ is symmetric about the x_2 axes, and $U(\sigma)$ is symmetric about the line through the origin and along the direction $(\cos(\pi/2-\theta), \sin(\pi/2-\theta))$. Particularly, $U(\cos(\pi-2\theta), \sin(\pi-2\theta))$ equals $U(1,0)$. Since θ is arbitrary, $U(\sigma)$ must be a constant function.

Lieb's manipulation on functions on the spheres can of course be transported to manipulations on functions on the Euclidean space. In fact in the one dimensional case, we essentially carried out Lieb's proof earlier. The function $v_\theta(x)$ defined in (6.7) equals $v(x)$ in (6.3) with $y = \tan(\theta/2)$. For higher dimensions, Lieb's proof carried out on functions on the Euclidean spaces is more complicated than the proof we have presented here.

7. Regularity of Integrable Solutions

To make our presentation complete, we give a proof on that an $L^{2n/(n-\alpha)}$ integrable solution $u(x)$ of (1.1) must satisfy (1.3). This proof is a slight modification on our previous proof in [7].

Consider $U(\sigma)$ given by (6.4) and satisfying (6.5). Then $U(\sigma)$ is $L^{2n/(n-\alpha)}$ integrable on the sphere S^n. Define

$$U_M(\sigma) := \begin{cases} U(\sigma) & \text{if } U(\sigma) \geq M; \\ 0 & \text{if } U(\sigma) < M. \end{cases}$$

We write the equation (6.5) into

$$U_M(\sigma) = \int_{S^n} \frac{1}{|\sigma - \tau|^{n-\alpha}} U_M(\tau)^{2\alpha/(n-\alpha)} U_M(\tau) dS_\tau + G(\sigma)\,,$$

where $G(\sigma)$ is a bounded function (in comparison with our proof in [7], there is no difficulty with infinity here). Define the linear function operator

$$T(W)(\sigma) = \int_{S^n} \frac{1}{|\sigma - \tau|^{n-\alpha}} U_M(\tau)^{2\alpha/(n-\alpha)} W(\tau) dS_\tau.$$

Using the Hardy-Littlewood-Sobolev inequalities on spheres and the Hölder inequality we can show that T is a bounded linear map from $L^p(S^n)$ space to $L^p(S^n)$ space for any $p > n/(n-\alpha)$. (Note that the HLS inequalities on spheres should be slightly easier to prove than the HLS inequalities on Euclidean spaces because there is no infinity point on spheres. We followed through the proof in Stein [25] without encountering extra difficulty.)[1] In more detail, choose $q > 1$ by $1/p = 1/q - \alpha/n$, and let $r > 1$. Then there exists a constant $C_{n,\alpha,p}$ such that

$$\begin{aligned} ||T(W)||_{L^p(S^n)} &\leq C_{n,\alpha,p} ||U_M^{2\alpha/(n-\alpha)} W||_{L^q(S^n)} \\ &\leq C_{n,\alpha,p} (\int_{S^n} U_M(\sigma)^{\frac{2\alpha}{n-\alpha} qr} dS_\sigma)^{\frac{1}{qr}} (\int_{S^n} W(\sigma)^{q\frac{r}{r-1}} dS_\sigma)^{\frac{r-1}{qr}}. \end{aligned}$$

Next, choose r such that $rq = n/\alpha$. It follows that $r > 1$ and $qr/(r-1) = p$.

[1]Another approach to the HLS inequality on the unit sphere is to use the HSL inequality on R^n, and the argument goes as follows. Let

$$T_\alpha(f)(\sigma) = \int_{S^n} \frac{1}{|\sigma - \tau|^{n-\alpha}} f(\tau) dS_\tau.$$

Consider first that $f(\sigma)$ is supported on the spherical cap centered at the north pole and enclosed by the 45° parallel. Then on the southern hemisphere, $T_\alpha(f)(\sigma)$ is uniformly bounded by a constant multiple of the L^1 norm of $f(\sigma)$; on the northern hemisphere, it follows from the standard HLS inequality that the L^p norm of $T_\alpha(f)(\sigma)$ is bounded by a constant multiple of the L^q norm of $f(\sigma)$ with p, q satisfying $1 < q < n/\alpha$ and $1/p = 1/q - \alpha/n$. Thus it follows that

$$||T_\alpha(f)(\sigma)||_{L^p(S^n)} \leq C_{n,\alpha,p} ||f(\sigma)||_{L^q(S^n)}.$$

For a general function $f(\sigma)$ we can decompose it as a direct sum of a finite number of functions such that: 1) each of the functions is supported on a spherical cap with the size not larger than the spehrical cap enclosed by the 45° parallel and 2) at each point only one of the functions is not zero. The HSL inequality on the unit sphere then follows.

We have

$$||T(W)||_{L^p(S^n)} \leq C_{n,\alpha,p}||U_M||^{2\alpha/(n-\alpha)}_{L^{2n/(n-\alpha)}(S^n)}||W||_{L^p(S^n)}.$$

It is clear that we can choose M so large that T is a contraction mapping from $L^{p_i}(S^n)$ to $L^{p_i}(S^n)$ for

$$p_1 = \frac{2n}{(n-\alpha)} \text{ and } p_2 = \frac{2n}{\alpha}\frac{(n+\alpha)}{(n-\alpha)}.$$

Then we see that $U_M(\sigma)$ is the unique solution of the equation for $W(\sigma)$ in both $L^{p_i}(S^n) : W = T(W) + G$. It follows that $U_M(\sigma)$ as well as $U(\sigma)$ itself are in the more highly integrable space $L^{p_2}(S^n)$. It then follows that $U(\sigma)^{(n+\alpha)/(n-\alpha)}$ is in $L^{2n/\alpha}(S^n)$. By Hölder's inequality $U(\sigma)$ is bounded and thus continuous on S^n. It then follows from (6.4) that $u(x)$ satisfies (1.3).

8. Further Remarks

In [19] and [20] Lieb proved the existence of functions optimizing a constant in the HLS inequalities. He proved that these optimizing functions are positive, radially symmetric and monotone. In the special case of maximizing the ratio

$$\frac{\int_{R^n}\int_{R^n}|x-y|^{\alpha-n}f(x)f(y)dxdy}{||f||^2_{L^{2n/(n+\alpha)}}},$$

a maximizer $f(x)$ satisfies an integral equation. Set $u(x)$ as $f(x)^{(n-\alpha)/(n+\alpha)}$ multiplied by a normalizing constant. Then $u(x)$ satisfies (1.1). Lieb determined the form of these optimizing solutions and raised up the question whether the same result would hold for positive regular solutions of (1.1). We have solved this problem of Lieb in our previous papers and the current paper. Y.-Y. Li in [17] also solved the problem. He used a method of moving spheres to show that the regular solutions are radially symmetric and monotone. We refer to a recent work of Fengbo Hang [13] on Lieb's question on a more general system of integral equations.

In the early 1950's Alexanderov first used the method of moving planes to prove that embedded surfaces in R^n with a constant mean curvature must be spheres. The method was much developed in the works of Serrin [24], Gidas-Ni-Nirenberg [12], Caffarelli-Gidas-Spruck [3], Chen-Li [4], Berestyski-Nirenberg [1], and Chang-Yang [9] for elliptic partial differential equations. We also refer to the book of Franklin [11] for an exposition on the development of the method.

Our proofs on Theorems 3.1 and 4.1 were inspired by an idea in [1] and [21]. The use of a finite ball in the proofs is similar to what was used in [16],

[4], and [23].

The result for $-\Delta u = u^{(n+2)/(n-2)}$ on $R^n (n \geq 3)$ corresponding to Theorem 1.1 was obtained in [12]. The result for the same differential equation corresponding to Theorem 1.2 was first obtained in the last section of [3] and was later simplified in [4]. This well-known differential equation arises in the contexts of optimizing functions for certain inequalities and of the conformal geometry of spheres.

The regularity proof we had in Section 7 is similar to one of the several proofs on the regularity of the $L^{2n/(n-2)}$ integrable positive weak solutions of the differential equation $-\Delta u = u^{(n+2)/(n-2)}$. Y.-Y. Li's proof [15] is of the similar nature. We refer to the paper of Brezis [2] for this proof.

In our earlier paper [6] we proved the radial symmetry and monotonicity directly for $L^{2n/(n-\alpha)}(R^n)$ integrable solutions, and thus skipped the step of proving continuity from integrability of the solution. Roughly speaking, the integrability assumption serves as a decay assumption at infinity.

If $n \geq 5$ and α is an even integer larger than 2 but less than n, (1.1) is related to the higher order differential equation $(-\Delta)^{\alpha/2} u = u^{(n+\alpha)/(n-\alpha)}$. To see this connection we recall that the function operator of convolution with $|x|^{-(n-\alpha)}$ is proportional to the composition of $\alpha/2$ function operators of convolution with $|x|^{-(n-2)}$ (cf. [25]). See Wei-Xu [26] and our paper [6] on the higher order differential equation.

Also relevant are the studies on positive harmonic functions on the upper-half space satisfying a critical nonlinear boundary condition. See the works of Ou [23], Li-Zhu [18], Chipot-Shafrir-Fila [10]. These harmonic functions on the boundary plane satisfy an integral equation (1.1) in the case of $n \geq 2$ and $\alpha = 1$. The connection can be established by using Poisson's formula for harmonic functions on the upper-half space.

References

[1] H. Berestycki, L. Nirenberg, On the method of moving planes and the sliding method, *Bol. Soc. Brazil. Mat.*, **22**, No. 1 (1991), 1-37.

[2] H. Brézis, Uniform estimates for solutions of $-\Delta u = V(x)u^p$, In: *Partial Differential Equations and Related Subjects*, Volume **269** in *Pitman Research Notes in Mathematics* (Ed. M. Miranda), John Wiley and Sons, New York (1992).

[3] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, *Comm. Pure Appl. Math.*, **XLII** (1989), 271-297.

[4] W. Chen, C. Li, Classification of solutions of some nonlinear elliptic equations, *Duke Math. J.*, **63** (1991), 615-622.

[5] W. Chen, C. Li, A priori estimates for prescribing scalar curvature equations, *Annals of Math.*, **145** (1997), 547-564.

[6] W. Chen, C. Li, B. Ou, Classification of solutions for an integral equation, *Comm. on Pure and Appl. Math.*, To Appear.

[7] W. Chen, C. Li, B. Ou, Qualitative properties of solutions for an integral equation, *Discrete and Cont. Dynamical Systems*, **12**, No. 2 (2005), 347-354.

[8] W. Chen, C. Li, B. Ou, Classification of solutions for a system of integra equations, *Comm. in PDEs*, **30** (2005), 59-65.

[9] A. Chang, P. Yang, On uniqueness of an n-th order differential equation in conformal geometry, *Math. Res. Letters*, **4** (1997), 1-12.

[10] M. Chipot, I. Shafrir, M. Fila, On a solution to some elliptic equation with nonlinear Neumann boundary conditions, *Adv. Differential Equations*, **1**, No. 1 (1996), 91-110.

[11] L. Fraenkel, *An Introduction to Maximum Principles and Symmetry in Elliptic Problems*, Cambridge University Press, New York (2000).

[12] B. Gidas, W.-M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in R^n, In: *Mathematical Analysis and Applications*, Volume 7a of *Advances in Mathematics, Supplementary Studies,* Academic Press, New York (1981).

[13] F. Hang, On the integral systems related to Hardy-Littlewood-Sobolev inequality, *Math. Res. Lett.*, **14**, No. 3 (2007), 373-383.

[14] C. Jin, C. Li, Symmetry of solutions to some systems of integral equations, *Proc. Amer. Math. Soc.*, **134**, No. 6 (2006), 1661-1770.

[15] C. Li, Local asymptotic symmetry of singular solutions to nonlinear elliptic equations, *Invent. Math.*, **123** (1996), 221-231.

[16] Y. Li, W.-M. Ni, Radial symmetry of positive solutions of nonlinear elliptic equations in R^n, *Comm. in PDEs*, **18** (1993), 1043-1054.

[17] Y.-Y. Li, Remarks on some conformally invariant integral equations: the method of moving spheres, *J. of Euro. Math. Soc.*, **6** (2004), 153-180.

[18] Y.-Y. Li, M.-J. Zhu, Uniqueness theorems through the method of moving spheres, *Duke Math. J.*, **80**, No. 2 (1995), 383-417.

[19] E. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. of Math.*, **118** (1983), 349-374.

[20] E. Lieb, M. Loss, *Analysis*, Second Edition, American Mathematical Society, Rhode Island (2001).

[21] L. Nirenberg, On the maximum principle, *American Mathematical Society*, Rhode Island (1992), Videocassette.

[22] B. Ou, A Remark on a singular integral equation, *Houston J. of Math.*, **25**, No. 1 (1999), 181-184.

[23] B. Ou, Positive harmonic functions on the upper-half space satisfying a nonlinear boundary condition, *Diff. and Int. Equations*, **9**, No. 5 (1996), 1157-1164.

[24] J. Serrin, A symmetry problem in potential theory, *Arch. Rational Mech. Anal.*, **43** (1971), 304-318.

[25] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton (1970).

[26] J. Wei, X. Xu, Classification of solutions of higher order conformally invariant equations, *Math. Ann.* (1999), 207-228.