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ON THE ANALYSIS OF M/M/1 FEEDBACK QUEUE WITH CATASTROPHES USING CONTINUED FRACTIONS

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Abstract: A transient solution is obtained analytically using continued fractions for the system size in an M/M/1 feedback queue with the possibility of catastrophes at the service counter. Asymptotic behavior of the probability of the server being idle and mean system size are obtained. The steady state probability of the system size is also presented. Some key performance measures are computed. Busy period and its moments are also obtained. Further numerical illustrations are used to discuss the system performance measures using *MAPLE* software.

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1. Introduction

In the study of queueing systems, determination of transient solution is very much essential to analyze the behavior of the system. Transient analysis is very useful for all queueing models to obtain optimal solutions which pave way to control the system. Among several methods, continued fraction is one of the techniques that is used to obtain transient solution. Even in the case of an simple M/M/1 queue, analytical approach to obtain transient behavior

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is very difficult. In this regard, we have obtained the transient solution for an M/M/1 feedback queue which is subject to catastrophes by employing an effective continued fraction technique which is very simple.

In particular, queues with feedback occur in production systems subject to rework, computer networks, telecommunication systems, supermarkets, banking industries, hospital management, etc. Takacs [33], in his interesting paper has introduced a queue with feedback. Several authors have investigated queueing systems subject to feedback. To mention a few, Disney, Mc Nickle and Simon [11] have studied several random processes that occur in M/G/1 queues with instantaneous Bernoulli feedback. D'Avignon and Disney [10] have also considered the same queue with state dependent feedback mechanism.

Computer networks with a virus may be modeled by queueing networks with catastrophes. Studies have been made on stochastic models for the growth of population subject to catastrophes. These include works by Hanson and Tuckwell [15], Pakes et al [23], Murthy [22], Trajstman [34], Brockwell et al [3], [4], [5] and [6]. In recent years, queueing systems with catastrophes have been studied by Boucherie and Boxma [2], Jain and Sigman [16], Dudin and Nishimura [12], Gelenbe and Pujolle [14], Chao et al [8], Artalejo [1], Krishna Kumar and Arivudainambi [30], Krishna Kumar and Pavai Madheswari [31], Krishna Kumar, Krishnamoorthy, Pavai Madheswari and Sadiq Basha [32].

The transient analysis of queuing system demands methodologically simple and easily numerically implementable approach. In this content, we propose simple and mathematically elegant method of obtaining transient solution to M/M/1 feedback queue with catastrophes using continued fractions and also illustrate the numerical adaptability. A systematic study of the theory of continued fractions can be found in Jones and Thron [17]. Its application to the study of birth and death processes was initiated by Murphy and O'Donohoe [21]. Pearce [29], Conolly and Langaris [9], Flajolet and Guillemin [13], Parthasarathy and Lenin [24], [25], [26] and [28], Parthasarathy and Selvaraju [27] and Krishnakumar et al [32] have applied continued fraction technique to study the transient behavior of the stochastic systems.

The rest of the paper is organized as follows: In the next section, we describe the mathematical model and present the detailed analysis of the transient solution using continued fraction methodology. Section 3 provides corresponding steady state results and certain performance measures are given in Section 4. Section 5 investigates the busy period of our model. Finally, we give some numerical examples to discuss the system performance measures.

2. The M/M/1 Feedback Queue with Catastrophes

Consider an M/M/1 queue with instantaneous Bernoulli feedback which is subject to catastrophes. Customers arrive at the service station one by one according to a Poisson stream with arrival rate $\lambda(>0)$. There is a single server which provides service to all the arriving customers. Service times are independently and identically distributed exponential random variable with parameter μ. After the completion of each service, the customer can either join at the end of the queue with probability p or he can leave the system with probability q where $p+q=1$. The customer both newly arrived and those that are fed back are served in the order in which they join the tail of the original queue. We do not distinguish between the regular arrival and feedback arrival. The customers are served according to the first come, first served rule. Apart from arrival and service processes, the catastrophes also occur at the service facilities as a Poisson process with rate γ. Whenever a catastrophe occurs at the system, all the available customers are destroyed immediately, the server get inactivated momentarily and the server is ready for service only when a new arrival occurs.

Now we obtain a system of difference differential equations satisfied by the M/M/1 feedback queue which is subject to catastrophes and obtain the transient solution of the queueing system.

Theorem 1. *The system of Chapman-Kolmogorov forward differential difference equations to describe M/M/1 feedback queue with catastrophes is given by* (2.1) *and* (2.2) *with initial condition* (2.3) *and the transient solution is given by*

$$P_n(t) = n\left(\frac{\lambda}{\mu q}\right)^{\frac{n}{2}} \int_0^t P_0(u) e^{-(\lambda+\mu q+\gamma)(t-u)} \frac{I_n(2\sqrt{\lambda\mu q}(t-u))}{(t-u)}\, du, \ n = 1, 2, \cdots,$$

where

$$\begin{aligned} P_0(t) &= \sum_{n=0}^{\infty} \frac{(n+1)}{(\lambda\mu q)^{\frac{n+1}{2}}} (\mu q)^n e^{-(\lambda+\mu q+\gamma)t} \frac{I_{n+1}(2\sqrt{\lambda\mu q}\ \ t)}{t} \\ &\quad + \gamma \sum_{n=0}^{\infty} \frac{(n+1)}{(\lambda\mu q)^{\frac{n+1}{2}}} (\mu q)^n \int_0^t e^{-(\lambda+\mu q+\gamma)u} \frac{I_{n+1}(2\sqrt{\lambda\mu q}\ \ u)}{u}\, du. \end{aligned}$$

Proof. We model this queueing system as a continuous time Markov chain (CTMC). Let $\{X(t) : t \in \mathbb{R}^+\}$ be the number of customers in the system at time t. Let $P_n(t) = P(X(t) = n)$, $n = 0, 1, \ldots$ be the state probabilities that there are n customers in the system at time t. Based on the above assumptions and by the Chapman-Kolmogorov forward differential-difference equations, the state probabilities $P_n(t), n = 0, 1, \ldots$ for the queueing system under investigation are

$$P_0^{'}(t) = -\lambda P_0(t) + \mu q P_1(t) + \gamma(1 - P_0(t))\,, \tag{2.1}$$

$$P_n^{'}(t) = \lambda P_{n-1}(t) - (\lambda + \mu q + \gamma)P_n(t) + \mu q P_{n+1}(t), \quad n = 1, 2, \ldots, \tag{2.2}$$

where λ, μ, q and γ are described above.

Without loss of generality, assume that initially there is no customer in the system, i.e.

$$P_0(0) = 1. \tag{2.3}$$

In the sequel, for any function $f(\cdot)$, let $f^*(z)$ denote its Laplace transform. By taking Laplace transform, the above system of equations is transformed to the following system of simultaneous equations.

$$(z + \lambda + \gamma)P_0^*(z) = 1 + \frac{\gamma}{z} + \mu q P_1^*(z)\,, \tag{2.4}$$

$$(z + \lambda + \mu q + \gamma)P_n^*(z) = \lambda P_{n-1}^*(z) + \mu q P_{n+1}^*(z), \quad n = 1, 2, \ldots. \tag{2.5}$$

After some algebraic adjustments, equations (2.4) and (2.5) respectively reduce to

$$P_0^*(z) = \frac{1 + \frac{\gamma}{z}}{(z + \lambda + \gamma) - \frac{\mu q P_1^*(z)}{P_0^*(z)}}, \tag{2.6}$$

$$\frac{P_n^*(z)}{P_{n-1}^*(z)} = \frac{\lambda}{(z + \lambda + \mu q + \gamma) - \frac{\mu q P_{n+1}^*(z)}{P_n^*(z)}}, \quad n = 1, 2, \ldots. \tag{2.7}$$

Now using (2.7) iteratively in (2.6), we express ${P_0}^*(z)$ as a continued fraction

$$P_0^*(z) = \frac{1 + \frac{\gamma}{z}}{(z + \lambda + \gamma) - \frac{\lambda \mu q}{(z + \lambda + \mu q + \gamma) - \frac{\lambda \mu q}{(z + \lambda + \mu q + \gamma) - \cdots}}}. \tag{2.8}$$

The above equation is written as

$$P_0^*(z) = \frac{1 + \frac{\gamma}{z}}{(z + \lambda + \gamma) - \phi(z)}, \tag{2.9}$$

where

$$\phi(z) = \cfrac{\lambda\mu q}{(z+\lambda+\mu q+\gamma) - \cfrac{\lambda\mu q}{(z+\lambda+\mu q+\gamma)-\cdots}} \tag{2.10}$$
$$= \frac{\lambda\mu q}{(z+\lambda+\mu q+\gamma)-\phi(z)}\,.$$

It is clear that $\phi(z)$ satisfies the quadratic equation

$$\phi^2(z) - (z+\lambda+\mu q+\gamma)\phi(z) + \lambda\mu q = 0$$

the roots of which are

$$\alpha(z), \beta(z) = \frac{\omega \pm \sqrt{\omega^2 - 4\lambda\mu q}}{2},$$

where $\omega = z+\lambda+\mu q+\gamma$. It is seen that $\beta(z)$ is a unique real root within $[0,1)$ and $0 \leq z < 1$. Hence we consider only $\beta(z)$ for further discussion.

Substituting $\beta(z)$ for $\phi(z)$ in (2.9), we get after some algebraic calculations

$$P_0^*(z) = \frac{1+\frac{\gamma}{z}}{(z+\lambda+\gamma) - \left[\frac{\omega - \sqrt{\omega^2-4\lambda\mu q}}{2}\right]} \tag{2.11}$$

$$= \frac{\left[1+\frac{\gamma}{z}\right]\left[\frac{\omega - \sqrt{\omega^2-4\lambda\mu q}}{2\lambda\mu q}\right]}{1-\mu q\left[\frac{\omega - \sqrt{\omega^2-4\lambda\mu q}}{2\lambda\mu q}\right]} \tag{2.12}$$

$$= \sum_{n=0}^{\infty}(\mu q)^n\left(\frac{\omega - \sqrt{\omega^2-4\lambda\mu q}}{2\lambda\mu q}\right)^{n+1}$$
$$+\frac{\gamma}{z}\sum_{n=0}^{\infty}(\mu q)^n\left(\frac{\omega - \sqrt{\omega^2-4\lambda\mu q}}{2\lambda\mu q}\right)^{n+1}. \tag{2.13}$$

On inversion, (2.13) yields the following explicit expression for $P_0(t)$.

$$P_0(t) = \sum_{n=0}^{\infty}\frac{(n+1)}{(\lambda\mu q)^{\frac{n+1}{2}}}(\mu q)^n e^{-(\lambda+\mu q+\gamma)t}\frac{I_{n+1}(2\sqrt{\lambda\mu q}\;\; t)}{t}$$
$$+\gamma\sum_{n=0}^{\infty}\frac{(n+1)}{(\lambda\mu q)^{\frac{n+1}{2}}}(\mu q)^n\int_0^t e^{-(\lambda+\mu q+\gamma)u}\frac{I_{n+1}(2\sqrt{\lambda\mu q}\;\; u)}{u}\,du\,, \tag{2.14}$$

where $I_n(\cdot)$ is the modified Bessel function of the first kind of order n.

Now, we obtain the remaining transient probabilities in terms of $P_0(t)$. From equation (2.7), for $n = 1, 2, \cdots$ we get the continued fraction as

$$\frac{P_n^*(z)}{P_{n-1}^*(z)} = \cfrac{\lambda}{(z+\lambda+\mu q+\gamma) - \cfrac{\lambda\mu q}{(z+\lambda+\mu q+\gamma) - \cfrac{\lambda\mu q}{(z+\lambda+\mu q+\gamma) - \cdots}}}. \tag{2.15}$$

By similar arguments, as before, the above equation is written as,

$$\frac{P_n^*(z)}{P_{n-1}^*(z)} = \frac{\lambda}{\left[\frac{\omega + \sqrt{\omega^2 - 4\lambda\mu q}}{2}\right]}, \text{for } n = 1, 2, \ldots, \tag{2.16}$$

which yields after some calculations,

$$P_n^*(z) = \left(\frac{\omega - \sqrt{\omega^2 - 4\lambda\mu q}}{2\mu q}\right)^n P_0{}^*(z), \text{ for } n = 1, 2, \ldots \tag{2.17}$$

where $\omega = z + \lambda + \mu q + \gamma$.

On inversion, we get, for $n = 1, 2, \ldots$

$$P_n(t) = n\left(\frac{\lambda}{\mu q}\right)^{\frac{n}{2}} \int_0^t P_0(u) e^{-(\lambda+\mu q+\gamma)(t-u)} \frac{I_n(2\sqrt{\lambda\mu q}(t-u))}{(t-u)}\, du. \tag{2.18}$$

Thus equations (2.14) and (2.18) completely determine all the state dependent probabilities $P_n(t)$ $n = 0, 1, 2, \ldots$ of the system size which completes the proof of the theorem. □

Remark. When $\gamma = 0$ and $q = 1$, then equation (2.18) coincides exactly with Krishna Kumar et al [32] by taking $\nu = 0$ in that paper.

Theorem 2. *If $\gamma > 0$, then the asymptotic behavior of the probability of the server being idle is given by*

$$P_0(t) \to \gamma \sum_{k=0}^{\infty} \frac{1}{(\lambda+\gamma)^{k+1}} \left[\frac{(\lambda+\mu q+\gamma) - \sqrt{(\lambda+\mu q+\gamma)^2 - 4\lambda\mu q}}{2}\right]^k \quad \text{as } t \to \infty\,. \tag{2.19}$$

Proof. We have,

$$P_0^*(z) = \frac{1+\frac{\gamma}{z}}{(z+\lambda+\gamma) - \dfrac{\lambda\mu q}{\omega - \left[\frac{\omega - \sqrt{\omega^2 - 4\lambda\mu q}}{2}\right]}}, \tag{2.20}$$

where $\omega = z + \lambda + \mu q + \gamma$.

Now taking the limits as $z \to 0$, the above equation reduces to

$$\lim_{z\to 0} zP_0(z) = \frac{\gamma}{(\lambda+\gamma) - \dfrac{2\lambda\mu q}{(\lambda+\mu q+\gamma) + \sqrt{(\lambda+\mu q+\gamma)^2 - 4\lambda\mu q}}}. \tag{2.21}$$

Multiplying numerator and denominator of the right hand side of above equation by

$$\frac{(\lambda+\mu q+\gamma) - \sqrt{(\lambda+\mu q+\gamma)^2 - 4\lambda\mu q}}{4\lambda\mu q}$$

and after some simple algebraic simplification, equation (2.21) reduces to

$$P_0^*(z) \sim \frac{\gamma}{z}\sum_{k=0}^{\infty}\frac{1}{(\lambda+\gamma)^{k+1}}\left[\frac{(\lambda+\mu q+\gamma) - \sqrt{(\lambda+\mu q+\gamma)^2 - 4\lambda\mu q}}{2}\right]^k, \quad \text{as } z \to 0. \tag{2.22}$$

By using the Tauberian Theorem, the result (2.19) follows from (2.22). □

Theorem 3. *If $\gamma > 0$, then, the asymptotic behavior of the mean system size $m(t)$ is given by*

$$m(t) \to \frac{(\lambda-\mu q)}{\gamma} + \frac{2\mu q}{2(\lambda+\gamma) - \left[(\lambda+\mu q+\gamma) - \sqrt{(\lambda+\mu q+\gamma)^2 - 4\lambda\mu q}\right]} \quad as,\ t \to \infty. \tag{2.23}$$

Proof. Let us consider equations (2.1) and (2.2) with initial condition (2.3).

Let $P(z,t) = \sum_{n=0}^{\infty} P_n(t)z^n$. From the above assumptions, it is clear that the probability generating function $P(z,t)$ satisfies the partial differential equation

$$\frac{\partial P(z,t)}{\partial t} = \left[\lambda z + \frac{\mu q}{z} - (\lambda+\mu q+\gamma)\right]P(z,t) + \mu q\left[1 - \frac{1}{z}\right]P_0(t) + \gamma. \tag{2.24}$$

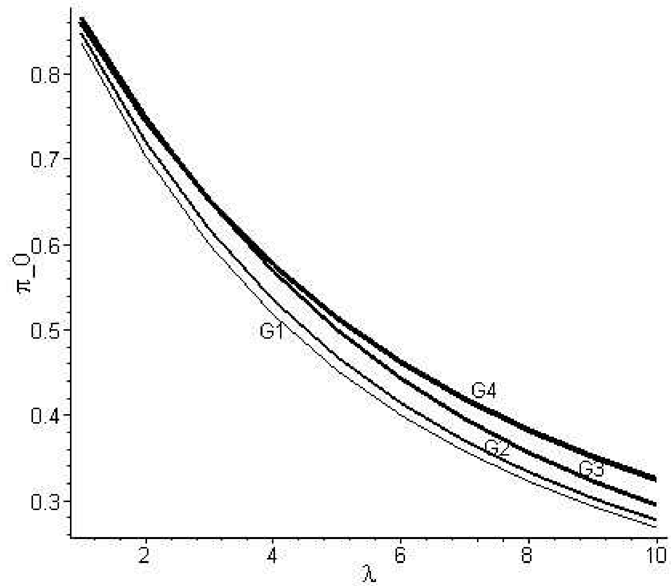

Figure 1: The graph between π_0 and λ.
G1: $\mu = 5, \gamma = 3, q = 0.5$ G2: $\mu = 5, \gamma = 3, q = 0.6$
G3: $\mu = 8, \gamma = 3, q = 0.5$ G4: $\mu = 5, \gamma = 4, q = 0.5$

The mean system size is $m(t) = \sum_{n=1}^{\infty} nP_n(t) = \frac{\partial P(z,t)}{\partial z}|_{z=1}$.

Differentiating (2.24) with respect to z at $z = 1$, we get

$$\frac{dm(t)}{dt} + \gamma m(t) = \lambda - \mu q[1 - P_0(t)]. \tag{2.25}$$

Solving the above differential equation for $m(t)$ with $m(0) = \sum_{n=1}^{\infty} nP_n(t) = 0$, we obtain,

$$m(t) = \frac{\lambda}{\gamma}(1 - e^{-\gamma t}) - \frac{\mu q}{\gamma}(1 - e^{-\gamma t}) + \mu q \int_0^t P_0(u) e^{-\gamma(t-u)} du. \tag{2.26}$$

If $m^*(z)$ is the Laplace transform of $m(t)$, then from above equation we get

$$m^*(z) = \frac{\lambda}{z(z+\gamma)} - \frac{\mu q}{z(z+\gamma)} + \frac{\mu q}{z+\gamma} P_0^*(z). \tag{2.27}$$

If $\gamma > 0$, then from (2.27) and (2.11) we obtain the required result (2.23). □

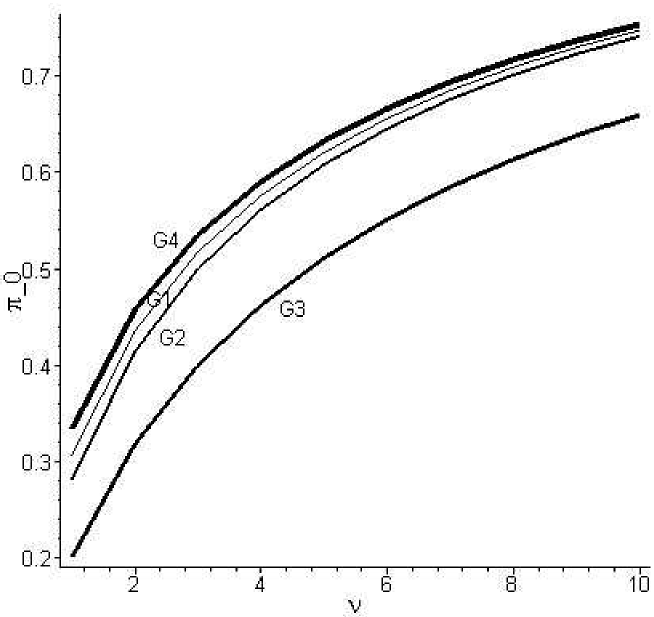

Figure 2: The graph between π_0 and γ.
G1: $\lambda = 4, \mu = 5, q = 0.5$ G2: $\lambda = 4, \mu = 5, q = 0.4$
G3: $\lambda = 6, \mu = 5, q = 0.5$ G4: $\lambda = 4, \mu = 6, q = 0.5$

3. Steady State Analysis

In this section, we shall derive the steady state probability distribution for our queueing model.

Theorem 4. *For $\gamma > 0$, the steady state distribution $\{\pi_n; n \geq 0\}$ of the M/M/1 feedback queue with catastrophe corresponds to*

$$\Pi_0 = (1-\rho), \tag{3.1}$$

$$\Pi_n = (1-\rho)\rho^n \; n = 1, 2, \ldots \tag{3.2}$$

where

$$\rho = \frac{(\lambda + \mu q + \gamma) - \sqrt{\lambda^2 + \mu^2 q^2 + \gamma^2 + 2\lambda\gamma + 2\mu q\gamma - 2\lambda\mu q}}{2\mu q}. \tag{3.3}$$

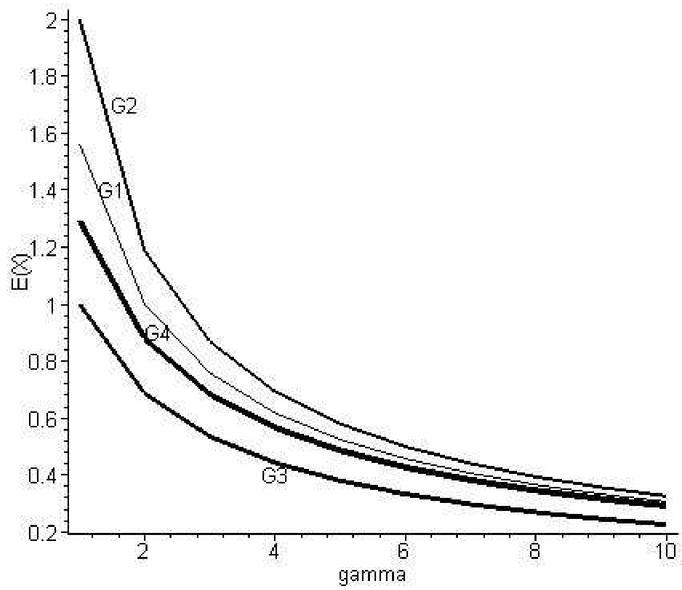

Figure 3: The graph between $E(X)$ and γ.
G1: $\lambda = 4, \mu = 5, q = 0.8$ G2: $\lambda = 4, \mu = 5, q = 0.6$
G3: $\lambda = 3, \mu = 5, q = 0.8$ G4: $\lambda = 4, \mu = 6, q = 0.8$

Proof.

$$P_0^*(z) = \frac{1 + \dfrac{\gamma}{z}}{(z + \lambda + \gamma) - \left[\dfrac{\omega - \sqrt{\omega^2 - 4\lambda\mu q}}{2}\right]}, \tag{3.4}$$

where $\omega = z + \lambda + \mu q + \gamma$.

Now multiplying (3.4) by z on both sides and taking the limit as $z \to 0$ we get,

$$\lim_{z\to 0} z P_0^*(z) = \Pi_0 = \frac{\gamma}{(\gamma + \lambda) - \left[\dfrac{(\lambda + \mu q + \gamma) - \sqrt{(\lambda + \mu q + \gamma)^2 - 4\lambda\mu q}}{2}\right]}.$$

After some algebraic manipulation, the above expression reduces to

$$\Pi_0 = \frac{-(\lambda + \gamma) + \mu q + \sqrt{(\lambda + \mu q + \gamma)^2 - 4\lambda\mu q}}{2\mu q},$$

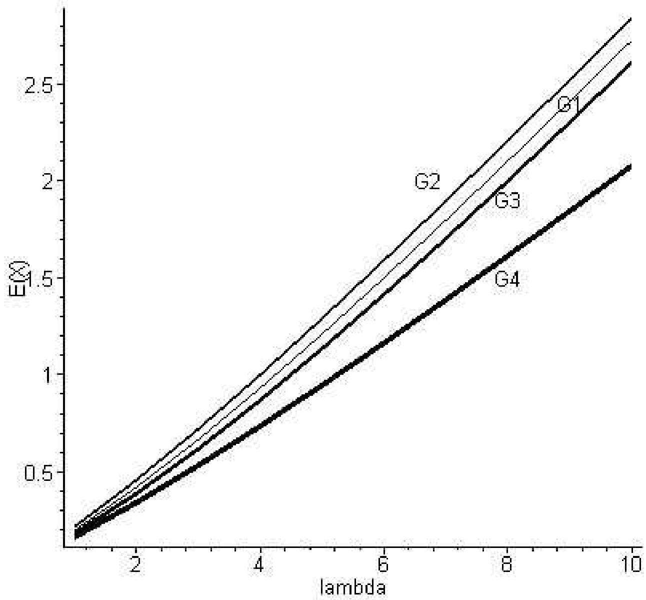

Figure 4: The graph between $E(X)$ and λ.

G1: $\mu = 5, \gamma = 3, q = 0.8$ G2: $\mu = 5, \gamma = 3, q = 0.4$

G3: $\mu = 6, \gamma = 3, q = 0.5$ G4: $\mu = 5, \gamma = 3, q = 0.5$

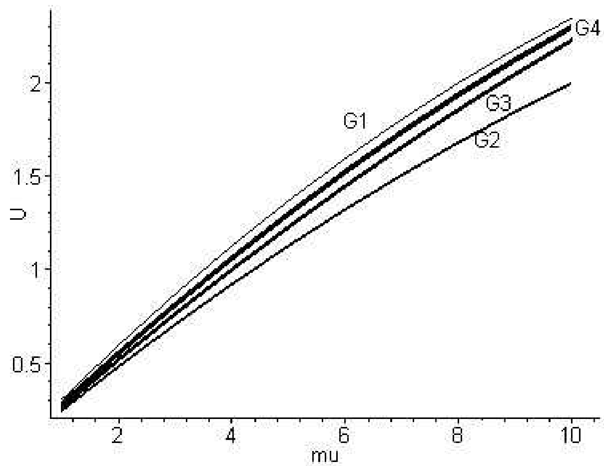

Figure 5: The graph between U and μ.

G1: $\lambda = 5, \gamma = 3, q = 0.5$ G2: $\lambda = 5, \gamma = 3, q = 0.4$

G3: $\lambda = 6, \gamma = 3, q = 0.4$ G4: $\lambda = 5, \gamma = 2, q = 0.4$

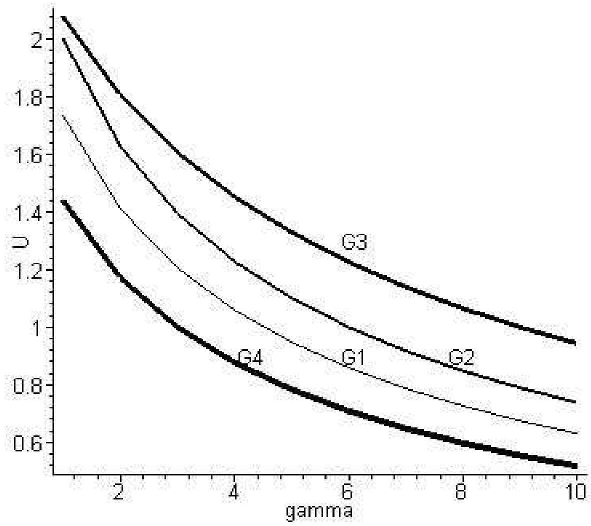

Figure 6: The graph between U and γ.
G1: $\lambda = 4, \mu = 5, q = 0.5$ G2: $\lambda = 5, \mu = 5, q = 0.6$
G3: $\lambda = 7, \mu = 5, q = 0.5$ G4: $\lambda = 4, \mu = 4, q = 0.5$

which simplifies to

$$\Pi_0 = 1 - \left[\frac{(\lambda + \mu q + \gamma) - \sqrt{\lambda^2 + \mu^2 q^2 + \gamma^2 + 2\lambda\gamma + 2\mu q\gamma - 2\lambda\mu q}}{2\mu q}\right].$$

Now for $n \geq 1$, multiplying (2.17) by z on both sides and taking the limit as $z \to 0$, we obtain,

$$\lim_{z \to 0} z P_n^*(z) = \left(\frac{\omega - \sqrt{\omega^2 - 4\lambda\mu q}}{2\mu q}\right)^n \lim_{z \to 0} z P_0^*(z)\,,$$

$$\begin{aligned}\Pi_n &= \lim_{z \to 0} z P_n^*(z) \\ &= \Pi_0 \left[\frac{(\lambda + \mu q + \gamma) - \sqrt{\lambda^2 + \mu^2 q^2 + \gamma^2 + 2\lambda\gamma + 2\mu q\gamma - 2\lambda\mu q}}{2\mu q}\right]^n.\end{aligned}$$

Thus equations (3.1)-(3.3) provide the steady-state distribution for the system size. Obviously, the steady state distribution exists if and only if $\rho < 1$. □

Remark. When $\gamma = 0$, $\rho = \frac{\lambda}{\mu q} < 1$, which is the well known steady state condition for an M/M/1 feedback queue (see [33]).

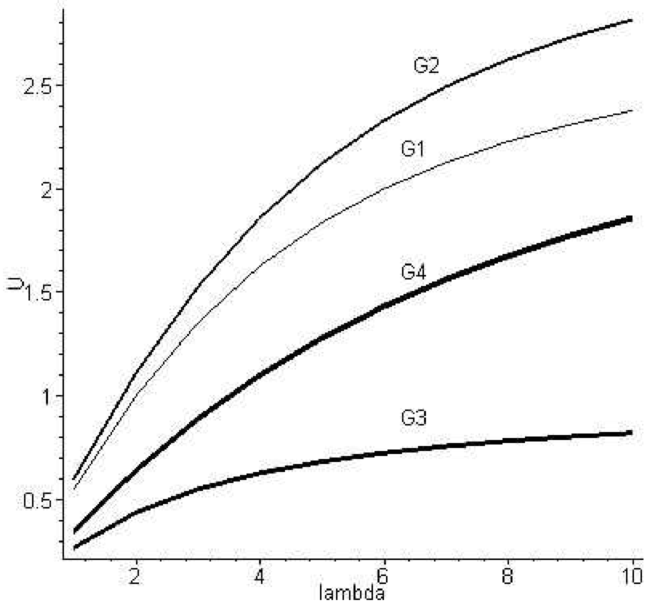

Figure 7: The graph between U and λ.
G1: $\mu = 6, \gamma = 2, q = 0.5$ G2: $\mu = 6, \gamma = 2, q = 0.6$
G3: $\mu = 2, \gamma = 2, q = 0.5$ G4: $\mu = 6, \gamma = 5, q = 0.5$

4. Moments under Steady State

In this section, we obtain the moments connected with the steady state obtained earlier in this paper. The steady state moments are often good approximations to the transient counterpart expressions even when time t is of moderate size.

Theorem 5. *For $\gamma > 0$, the steady state probability generating function $\Pi(s)$ is given by*

$$\Pi(z) = \frac{1-\rho}{1-\rho z}. \tag{4.1}$$

The mean, the variance of the system size namely $E(Q), Var(Q)$ and the mean of the queue size $E(Q_q)$ are obtained as

$$E(Q) = \frac{\rho}{1-\rho}, \tag{4.2}$$

$$Var(Q) = \frac{\rho}{(1-\rho)^2}, \tag{4.3}$$

$$\text{and } E(Q_q) = \frac{\rho^2}{(1-\rho)}, \tag{4.4}$$

where ρ is given by equation (3.3).

Proof. Multiplying equations (3.1) and (3.2) by s^n and summing over n, after simple simplification, we derive (4.1). The results (4.2) and (4.3) follow directly from (4.1) on differentiation with respect to s and setting $s = 1$.

Similarly, the higher factorial moments can be obtained from (4.1) by successive differentiation with respect to s and evaluating at $s = 1$.

Now $E(Q_q) = \sum_{n=1}^{\infty}(n-1)\Pi_n = \frac{\rho^2}{1-\rho}$. □

Remark. The mean number of customers in the system $E(X)$ includes the down time when there are no customers in the system.

$$\text{P(Server is busy)} = \sum_{n=1}^{\infty}\Pi_n = \rho\,, \tag{4.5}$$

$$and\text{P(Server is idle)} = \Pi_0 = (1-\rho)\,, \tag{4.6}$$

where ρ is as given in (3.3).

Theorem 6. *For $\gamma > 0$, under steady state, the system throughput, U is given as*

$$U = \mu q\ \rho\,, \tag{4.7}$$

where ρ is given by equation (3.3).

Proof. The system throughput, U, is the rate at which customers exit the queue whenever there are one or more customers in the system. With the exit rate μq, we obtain

$$U = [1-\Pi_0]\,\mu q = \mu q\ \rho\,, \tag{4.8}$$

where we have used equation (3.1). □

Remark. The number of customers that were ejected per unit time is

$$\gamma\sum_{n=1}^{\infty}n\Pi_n = \gamma E(X) = \frac{\gamma\rho}{1-\rho}. \tag{4.9}$$

Therefore, the fraction of customers who entered the queue and were ejected when the server failed due to catastrophe is

$$\frac{\gamma E(X)}{\lambda} = \frac{\gamma\rho}{\lambda(1-\rho)}. \tag{4.10}$$

The probability that an arriving customer will complete service, conditioned on the arrival during the working state of the server is given by

$$\frac{U}{\lambda} = \frac{\mu q}{\lambda}\ \rho. \tag{4.11}$$

5. Busy Period Analysis

A busy period is defined as the interval of time commencing at the instant 0 when a customer arrives at an empty counter and terminating at the instant when the server becomes free for the first time. Let the length of the interval which is a random variable be T and $\{N^*(t)\}$ be the stochastic process denoting the number of customers present at the instant t during the busy period.

We have $\{N^*(0) = 1\}$ and the duration of the busy period is the first t for which $\{N^*(t) = 0\}$. Let $q_n(t) = P\{N^*(t) = n|N^*(0) = 1\}, n = 0, 1, \ldots$ be the zero-avoiding state probabilities. Then $q_1(0) = 1, q_n(0) = 0,\ n = 1, 2, \ldots$.

Now $q_n(t)$ will satisfy the following difference differential equations.

$$q_1^{'}(t) = -(\lambda + \mu q + \gamma)q_1(t) + \mu q\ q_2(t), \tag{5.1}$$

$$q_n^{'}(t) = \lambda q_{n-1}(t) - (\lambda + \mu q + \gamma)q_n(t) + \mu q\ q_{n+1}(t), \quad n = 2, 3, \ldots. \tag{5.2}$$

We solve the above equations using continued fractions methodology. By similar arguments as in Section 2, the solution for the above equations is given by

$$q_n(t) = \left(\frac{\lambda}{\mu q}\right)^{n/2} \frac{n}{\lambda t} e^{-(\lambda + \mu q + \gamma)t} I_n(2t\sqrt{\lambda\mu q}),\ n = 1, 2, \ldots.$$

Conditioning on the number of customers present at instant t, all of which complete their services in $(t, t + dt)$, we have

$$\begin{aligned} b(t)dt &\simeq P\{t \le T < t + dt\} \\ &= \sum_{j=1}^{\infty} P\{t \le T < t + dt|N^*(t) = j\}P\{N^*(t) = j\} \\ &= P\{t \le T < t + dt|N^*(t) = 1\}P\{N^*(t) = 1\} \\ &\quad + \sum_{j=2}^{\infty} P\{t \le T < t + dt|N^*(t) = j\}P\{N^*(t) = j\}. \end{aligned}$$

The first term implies that there is only one customer (at the instant t) whose service is completed between $(t, t+dt)$, the probability of this event being $\mu q dt + o(dt)$. The second term implies service completion of two or more customers in $(t, t + dt)$ and the probability of this event is $o(dt)$. Thus taking the limit as $dt \to 0$,

$$b(t) = [\mu q q_1(t)] = \mu q \left(\frac{\lambda}{\mu q}\right)^{1/2} \frac{1}{\lambda t} e^{-(\lambda+\mu q+\gamma)t} I_1(2t\sqrt{\lambda\mu q}). \tag{5.3}$$

The Laplace transform of T is given by

$$\begin{aligned} b^*(z) &= L\{b(t)\} = \mu q L\{q_1(t)\} = \mu q q_1^*(z) \\ &= \left[\frac{(z+\lambda+\mu q+\gamma) - \sqrt{(z+\lambda+\mu q+\gamma)^2 - 4\lambda\mu q}}{2\lambda}\right]. \end{aligned}$$

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$$\begin{aligned} E(T) &= -\frac{d}{dz} b^*(z)|_{z} = 0 \\ &= \frac{1}{2\lambda}\left[\frac{\lambda+\mu q+\gamma - \sqrt{(\lambda+\mu q+\gamma)^2 - 4\lambda\mu q}}{\sqrt{(\lambda+\mu q+\gamma)^2 - 4\lambda\mu q}}\right], \end{aligned} \tag{5.4}$$

$$E(T^2) = -\frac{d}{dz} b^*(z)|_{z} = 0 = \frac{2\mu q}{[(\lambda+\mu q+\gamma)^2 - 4\lambda\mu q]^{\frac{3}{2}}}, \tag{5.5}$$

$$V(T) = \frac{8\mu q\lambda^2 - \omega_1\left[(\lambda+\mu q+\gamma)^2 - \omega_1\right]^2}{4\lambda^2\left[(\lambda+\mu q+\gamma)^2 - 4\lambda\mu q\right]^{\frac{3}{2}}}, \tag{5.6}$$

where $\omega_1 = \sqrt{(\lambda+\mu q+\gamma)^2 - 4\lambda\mu q}$.

6. Numerical Illustrations

In the Sections 3 and 4, we have obtained explicit expressions for $\Pi_n,\ n = 0, 1, \cdots,$ the steady state probabilities of the system size, $E(X)$, the mean number of customers in the system, U, the system throughput. In this section, we present some numerical illustrations to show the effect of the catastrophe parameter γ and the arrival rate λ on $\Pi_0\, E(X)$ and U.

In Figures 1 and 2, we plot the behavior of the probability Π_0 of no customers in the system as a function of λ and γ respectively. Obviously, Π_0 decreases as the arrival rate λ increases and increases with the catastrophe rate γ.

Figure 3 highlights the effect of the catastrophe rate γ on the mean number of customers $E(X)$ in the system. It is found that $E(X)$ is a decreasing function of γ. Figure 4 illustrates the behavior of $E(X)$ as a function of the arrival rate λ. It is found that $E(X)$ is an increasing function of λ.

Figures 5, 6 and 7 show the nature of the throughput U. It is found that U is an increasing function of the service rate μ as shown in Figure 5. In Figure 6, it is seen that U is an decreasing function of catastrophe parameter γ. From Figure 7, one observes that U increases as arrival rate λ increases.

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