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φ_0-STABILITY IN TERMS OF TWO MEASURES FOR DIFFERENTIAL EQUATIONS BY PERTURBING LYAPUNOV FUNCTIONS

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Abstract: The stability in terms of two measures for nonlinear differential equations is studied. Perturbing cone-valued Lyapunov functions have been applied. Comparison scalar ordinary differential equations have been employed.

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1. Introduction

The problems of stability of solutions of differential equations via Lyapunov method have beed successfully investigated in the past. One type of stability, very useful in real world problems, deals with two different measures. The stability in terms of two measures of differential equations have been studied by means of various types of Lyapunov functions (see [2], [6]).

In the present paper the stability in terms of two different measures of the solutions of ordinary differential equations is studied. Cone-valued Lyapunov functions are employed as well as comparison results. Our results also imply that in the case when Lyapunov function does not satisfy all the desired conditions it is fruitful to perturbed that function rather that discard it. New type of stability in terms of two measures is defined and sufficient conditions are obtained.

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2. Preliminary Notes and Definitions

Consider the system of nonlinear differential equations

$$x' = F(t, x) \quad \text{for} \quad t \geq t_0, \tag{1}$$

with initial condition

$$x(t_0) = x_0, \tag{2}$$

where $x \in \mathbb{R}^n$, $F : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$.

Let $x_0 \in \mathbb{R}^n$. We denote by $x(t; t_0, x_0)$ the solution of the initial value problem (1), (2).

Consider the following sets

$$K = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] \ : \ a(s) \text{ is strictly increasing and } a(0) = 0\};$$
$$CK = \{b \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+] \ : \ b(t, .) \in K \text{ for any fixed } t \geq 0\};$$
$$\mathcal{G} = \{H \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+] \ : \ \inf_{s \in \mathbb{R}} H(t, s) = 0 \text{ for each } t \geq 0\}.$$

Let $x, y \in \mathbb{R}^n$. Denote by $(x \bullet y)$ the dot product of both vectors x and y.

Let $\mathcal{K} \subset \mathbb{R}^n$ be a cone. Consider the set

$$\mathcal{K}^* = \{\varphi \in \mathbb{R}^n : \ (\varphi \bullet x) \geq 0 \text{ for any } \ x \in \mathcal{K}\}.$$

We assume that $\mathcal{K}^*$ is a cone.

Let ρ be positive constant, $\varphi_0 \in \mathcal{K}^*$, $H \in \mathcal{G}$. Define sets:

$$\tilde{S}(H, \rho, \varphi_0) = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : \quad H(t, (\varphi_0, x)) < \rho\};$$
$$\tilde{S}^C(H, \rho, \varphi_0) = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : \quad H(t, (\varphi_0, x)) \geq \rho\}.$$

We will study stability in terms of two measures of ordinary differential equations. In the case when cone-valued Lyapunov functions are applied, both measures are from the set $\mathcal{G}$. In this case we introduce the definition of a new type of stability that combines the ideas of stability in terms of two measures for ordinary differential equations (see [6], [7]) and φ_0-stability (see [1]).

Definition 1. Let $\varphi_0 \in \mathcal{K}^*$, $H, H_0 \in \mathcal{G}$. System (1) is said to be:

(S1) φ_0-*stable* in terms of measures H and H_0 if for every $\epsilon > 0$ and for any $t_0 \geq 0$ there exists $\delta = \delta(t_0, \epsilon) > 0$ such that inequality $H_0(t, (\varphi_0 \bullet x_0)) < \delta$ implies $H(t, (\varphi_0 \bullet x(t; t_0, x_0))) < \epsilon$ for $t \geq t_0$;

(S2) *uniformly* φ_0-*stable* in terms of measures H and H_0 if conditions (S1) are satisfied, where δ is independent on t_0;

In our further investigations we will use the following comparison scalar ordinary differential equations

$$u' = g_1(t, u), \quad t \geq t_0, \tag{3}$$

and

$$v' = g_2(t, v), \quad t \geq t_0, \tag{4}$$

where $u, v \in \mathbb{R},\ g_i(t, 0) \equiv 0,\ (i = 1, 2)$.

Definition 2. (see [6]) Let $H, H_0 \in \mathcal{G}$. Function H_0 is *uniformly finer* than H if there exists a constant $\delta > 0$ and a function $a \in K$ such that for any point $(t, s) \in \mathbb{R}_+ \times \mathbb{R}$ such that $H_0(t, s) < \delta$ the inequality $H(t, s) \leq a(H_0(t, s))$ holds.

We introduce the following class of functions:

Definition 3. We will say that the cone-valued function $V(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathcal{K}$ belongs to the class $\mathcal{L}$ if:

1. $V(t, x) \in C^1[\mathbb{R}_+ \times \mathbb{R}^n, \mathcal{K}]$;

2. There exist constants $M_i > 0,\ i = 1, 2, \ldots, n$, such that $|V_i(t, x) - V_i(t, y)| \leq M_i ||x - y||$ for any $t \in \mathbb{R}_+,\ x, y \in \mathbb{R}^n$, where $V = (V_1, V_2, \ldots, V_n)$.

We introduce some properties of the functions from class $\mathcal{L}$.

Definition 4. Let $\varphi_0 \in \mathcal{K}^*$, $H \in \mathcal{G}$. Function $V \in \mathcal{L}$ is said to be:

(S3) φ_0-*weakly* H-*decrescent*, if there exists a constant $\delta > 0$ and a function $a \in CK$ such that for any point $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n :\quad H(t, (\varphi_0 \bullet x)) < \delta$ the inequaity $(\varphi_0 \bullet V(t, x)) \leq a(t, H(t, (\varphi_0 \bullet x)))$ holds.

(S4) φ_0-*strongly* H-*decrescent*, if there exists a constant $\delta > 0$ and a function $a \in K$ such that for any point $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n :\quad H(t, (\varphi_0 \bullet x)) < \delta$ the inequaity $(\varphi_0 \bullet V(t, x)) \leq a(H(t, (\varphi_0 \bullet x)))$ holds.

Let function $V \in \mathcal{L},\ V = (V_1, V_2, \ldots, V_n)$. We will use the following derivative $D^+_{(1)} V(t, x) = (D^+_{(1)} V_1(t, x), D^+_{(1)} V_2(t, x), \ldots, D^+_{(1)} V_n(t, x))$ of the function V through the trajectory of the system (1)

$$D^+_{(1)} V_i(t, x) = \frac{\partial V_i(t, x)}{\partial t} + \sum_{j=1}^{n} \frac{\partial V_i(t, x)}{\partial x_j} F_j(t, x), \quad i = 1, 2, \ldots, n.$$

In the further investigations we will use the following comparison result:

Lemma 1. (see Theorem 1.4.1 in [4]) *Let $E \subset \mathbb{R} \times \mathbb{R}$ be an open set and:*

1. Function $g_1 \in C[E, \mathbb{R}]$.

2. Function $m \in C[[t_0, t_0 + a) \times \mathbb{R} \bigcap E, \mathbb{R}]$ *satisfies the inequalities*

$$m' \le g_1(t, m), \quad t \in [t_0, t_0 + a), \quad m(t_0) \le u_0.$$

3. Function $r^*(t) = r^*(t; t_0, u_0)$ *is the maximal solution of (3) with initial condition* $r^*(t_0) = u_0$*, defined for* $t \in [t_0, t_0 + a)$.

Then $m(t) \le r^*(t), \quad t \in [t_0, t_0 + a)$.

3. Main Results

We will obtain sufficient conditions for φ_0-stability in terms of two measures of systems of differential equations. We will employ two types of Lyapunov functions from class $\mathcal{L}$. The proof is based on the second method of Lyapunov with pertubing Lyapunov functions combined by comparison results for scalar ordinary differential equations.

Some of the discussed results in the paper improve Theorem 2, [5], Theorem 2.1, [3]. We note that φ_0-stability of differential equations is studied also in [1].

Theorem 1. *Let the following conditions be fulfiled:*

1. Function $F \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$.

2. Functions $H_0, H \in \mathcal{G}$, H_0 *is uniformly finer than* H.

3. Vector $\varphi_0 \in \mathcal{K}^*$.

4. There exists a function $V_1 \in \mathcal{L}$ *that is* φ_0*-weakly* H_0*-decrescent, and*

(i) $(\varphi_0 \bullet D^+_{(1)} V_1(t, x)) \le g_1(t, (\varphi_0 \bullet V_1(t, x)))$ *for* $(t, x) \in \tilde{\mathcal{S}}(H, \rho, \varphi_0)$,

where $g_1(t, u) \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$, $g_1(t, 0) \equiv 0$, *and* $\rho > 0$ *is a constant.*

5. For any number $\mu > 0$ *there exists a function* $V_2^{(\mu)} \in \mathcal{L}$ *that satisfies for* $(t, x) \in \tilde{\mathcal{S}}(H, \rho, \varphi_0) \bigcap \tilde{\mathcal{S}}^C(H_0, \mu, \varphi_0)$ *following conditions:*

(ii) $b(H(t, (\varphi_0 \bullet x))) \le (\varphi_0 \bullet V_2^{(\mu)}(t, x)) \le a(H_0(t, (\varphi_0 \bullet x)))$,

where $a, b \in K$;

(iii)
$$\Big(\varphi_0 \bullet \big\{D^+_{(1)} V_1(t, x) + D^+_{(1)} V_2^{(\mu)}(t, x)\big\}\Big) \le g_2\Big(t, \big(\varphi_0 \bullet \big\{V_1(t, x) + V_2^{(\eta)}(t, x)\big\}\big)\Big)$$

holds, where $g_2 \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$, $g_2(t, 0) \equiv 0$.

6. For any initial conditions the systems (1) and the scalar differential equations (3) and (4) have solutions, defined on $[t_0, \infty)$.

7. Zero solution of the scalar differential equation (3) is equi-stable.

8. Zero solution of scalar differential equation (4) is uniformly stable.

Then the system of differential equations (1) is φ_0-stable in terms of measures H and H_0.

Proof. Since function $V_1(t,x)$ is φ_0-weakly-H_0-decrescent, there exists a constant $\rho_1 \in (0,\rho)$ and a function $\psi_1 \in CK$ such that $H_0(t,(\varphi_0 \bullet x)) < \rho_1$ implies

$$(\varphi_0 \bullet V_1(t,x)) \leq \psi_1(t, H_0(t,(\varphi_0 \bullet x))). \tag{5}$$

Since function H_0 is uniformly finer than H, there exists a constant $\rho_0 \in (0,\rho_1)$ and a function $\psi_2 \in K$ such that $H_0(t,s) < \rho_0$ implies

$$H(t,s) \leq \psi_2(H_0(t,s))\,, \tag{6}$$

where $\psi_2(\rho_0) < \rho_1$.

Let $\epsilon > 0$ be a positive number, $\epsilon < \rho$, and $t_0 \geq 0$ be a fixed number.

Since zero solution of scalar differential equation (4) is uniformly stable there exists a function $\delta_1 = \delta_1(\epsilon) \in CK$ such that inequality $|v_0| < \delta_1$ implies

$$|v(t;t_0,v_0)| < b(\epsilon), \quad t \geq t_0, \tag{7}$$

where $v(t;t_0,v_0)$ is the maximal solution of equation (4) with initial condition $v(t_0) = v_0$.

Since the functions $a \in K$ and $\psi_2 \in K$ we can find $\delta_2 = \delta_2(\epsilon) > 0$, $\delta_2 < \rho_0$ such that the inequalities

$$a(\delta_2) < \frac{\delta_1}{2}, \quad \psi_2(\delta_2) < \epsilon \tag{8}$$

hold.

Since the zero solution of the scalar differential equation (3) is equi-stable there exists $\delta_3 = \delta_3(t_0,\epsilon) > 0$ such that inequality $|u_0| < \delta_3$ implies

$$|u(t;t_0,u_0)| < \frac{\delta_1}{2}, \qquad t \geq t_0, \tag{9}$$

where $u(t;t_0,u_0)$ is the maximal solution of (3) with initial condition $u(t_0) = u_0$.

Since the function $\psi_1 \in CK$ there exists $\delta_4 = \delta_4(\delta_3) > 0$ such that for $|u| < \delta_4$ the inequality

$$\psi_1(t_0,u) < \delta_3 \tag{10}$$

holds.

From inequalities (5) and (10) follows that there exists $\delta_5 = \delta_5(t_0,\epsilon) > 0$,

$\delta_5 < \min(\delta_4, \rho_1)$ such that $H_0(t_0, (\varphi_0 \bullet x_0)) < \delta_5$ implies

$$(\varphi_0 \bullet V_1(t_0, x_0)) \leq \psi_1(t_0, H_0(t_0, (\varphi_0 \bullet x_0))) < \delta_3. \tag{11}$$

Let $\delta_6 = \min\{\delta_2, \delta_5\}$, $\delta_6 = \delta_6(t_0, \epsilon) > 0$ and the point $x_0 \in \mathbb{R}^n$ be such that

$$H_0(t_0, (\varphi_0 \bullet x_0)) < \delta_6. \tag{12}$$

From condition 2, inequality (6), and the choice of δ_2 and δ_6 follows that $H(t_0, (\varphi_0 \bullet x_0)) < \epsilon$.

We will prove that if inequality (12) is satisfied then

$$H(t, (\varphi_0 \bullet x(t; t_0, x_0))) < \epsilon, \qquad t \geq t_0, \tag{13}$$

where $x(t; t_0, x_0)$ is a solution of initial value problem (1), (2).

Suppose inequality (13) is not true. Therefore, there exists a point $t^* > t_0$ such that

$$H(t^*, (\varphi_0 \bullet x(t^*; t_0, x_0))) = \epsilon, \qquad H(t, (\varphi_0 \bullet x(t; t_0, x_0))) < \epsilon, \quad t \in [t_0, t^*). \tag{14}$$

Define $x(s) = x(s; t_0, x_0)$, $s \in [t_0, t^*]$.

If we assume that $H_0(t^*, (\varphi_0 \bullet x(t^*))) \leq \delta_2 < \rho$ then from the choice of δ_2 follows $H(t^*, (\varphi_0 \bullet x(t^*))) \leq \psi_2(H_0(t^*, (\varphi_0 \bullet x(t^*)))) \leq \psi_2(\delta_2) < \epsilon$ that contradicts (14).

Therefore

$$H_0(t^*, (\varphi_0 \bullet x(t^*))) > \delta_2, \quad H_0(t_0, (\varphi_0 \bullet x_0)) < \delta_6 \leq \delta_2. \tag{15}$$

From inequality (15) follows that there exists a point $t_0^* \in (t_0, t^*)$ such that $\delta_2 = H_0(t_0^*, (\varphi_0 \bullet x(t_0^*)))$ and $(t, x(t)) \in \tilde{\mathcal{S}}(H, \epsilon, \varphi_0) \bigcap \tilde{\mathcal{S}}^c(H_0, \delta_2, \varphi_0)$, for $t \in [t_0^*, t^*)$. From the choice of ϵ follows that $\tilde{\mathcal{S}}(H, \epsilon, \varphi_0) \subset \tilde{\mathcal{S}}(H, \rho, \varphi_0)$ and therefore

$$(t, x(t)) \in \tilde{\mathcal{S}}(H, \rho, \varphi_0) \bigcap \tilde{\mathcal{S}}^c(H_0, \delta_2, \varphi_0), \quad t \in [t_0^*, t^*). \tag{16}$$

Let $r_1(t; t_0, u_0)$ be the maximal solution of differential equation (3) with initial condition $u(t_0) = u_0$, where $u_0 = (\varphi_0 \bullet V_1(t_0, x_0))$.

Define function $p(t) = (\varphi_0 \bullet V_1(t, x(t)))$ for $t \in [t_0^*, t^*]$. Then according to condition (i) we obtain

$$\begin{aligned} p'(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big\{ \Big(\varphi_0 \bullet V_1(t+\epsilon, x(t+\epsilon)) \Big) - \Big(\varphi_0 \bullet V_1(t, x(t)) \Big) \Big\} \\ = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big(\varphi_0 \bullet \big\{ V_1(t+\epsilon, x(t+\epsilon)) - V_1(t+\epsilon, x(t)) \big\} \Big) \\ + \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big(\varphi_0 \bullet \big\{ V_1(t+\epsilon, x(t)) - V_1(t, x(t)) \big\} \Big) \end{aligned}$$

$$= (\varphi_0 \bullet D^+_{(1)} V_1(t, x(t))) \le g_1(t, (\varphi_0 \bullet V_1(t, x(t)))) = g_1(t, p(t)). \quad (17)$$

According to Lemma 1 the following inequality is satisfied

$$p(s) \le r_1(s; t_0, u_0), \quad s \in [t_0, t^*]. \quad (18)$$

Condition (12) for the point x_0 and inequaities (9), (11), (18) imply that

$$(\varphi_0 \bullet V_1(t_0^*, x(t_0^*))) < \frac{\delta_1}{2}. \quad (19)$$

Consider the function $V_2^{(\delta_2)}(t, x)$ that is defined in condition 5 of Theorem 1 and define the function

$$m(t, x) = V_1(t, x) + V_2^{(\delta_2)}(t, x), \qquad t \ge t_0^*. \quad (20)$$

From inequaity (8) and condition (ii) of Theorem 1 follows that

$$(\varphi_0 \bullet V_2^{(\delta_2)}(t_0^*, x(t_0^*))) \le a(H_0(t_0^*, (\varphi_0 \bullet x(t_0^*)))) = a(\delta_2) < \frac{\delta_1}{2}. \quad (21)$$

From inequalities (19) and (21) we obtain

$$(\varphi_0 \bullet m(t_0^*, x(t_0^*))) < \delta_1. \quad (22)$$

Define function $q(t) = (\varphi_0, m(t, x(t)))$ for $t \ge t_0^*$.

Let $t \in [t_0^*, t^*]$. Then using inclusion (16) and condition (iii) of Theorem 1 we obtain

$$q'(t) = \Big(\varphi_0 \bullet \Big\{D^+_{(1)} V_1(t, x(t)) + D^+_{(1)} V_2^{(\delta_2)}(t, x(t))\Big\}\Big) \le g_2(t, q(t)). \quad (23)$$

According to Lemma 1 from inequality (23) follows the validity of the inequality

$$q(t) = (\varphi_0, m(t, x(t))) \le r^*(t; t_0^*, w_0^*), \quad t \in [t_0^*, t^*], \quad (24)$$

where $r^*(t; t_0^*, w_0^*)$ is the maximal solution of (4) through the point (t_0^*, w_0^*), $w_0^* = (\varphi_0 \bullet m(t_0^*, x(t_0^*))$.

From inequality (22) follows that $|w_0^*| < \delta_1$ and therefore according to inequality (7)

$$r^*(t; t_0^*, w_0^*) < \beta(\epsilon), \quad t \ge t_0^*. \quad (25)$$

From inequalities (25), (24), the choice of the point t^*, and condition (iii) of Theorem 1 we obtain

$$b(\epsilon) > r^*(t^*; t_0^*, w_0^*) \ge (\varphi_0 \bullet m(t^*, x(t^*)))$$
$$\ge (\varphi_0 \bullet V_2^{(\delta_2)}(t^*, x(t^*))) \ge b(H(t^*, (\varphi_0 \bullet x(t^*)))) = b(\epsilon).$$

The obtained contradiction proves the validity of inequality (13).

Inequality (13) proves φ_0-stability in terms of measures H and H_0 of the considered system of differential equations. □

Theorem 2. *Let the conditions 1, 2, 3, 4, 5, 6, 7 of Theorem 1 be satisfied, where the function $V_1 \in \mathcal{L}$ is φ_0-strongly H_0-decreascent.*

If zero solutions of both scalar differential equations (3) and (4) are uniformly stable, then the system of differential equations (1) is uniformly φ_0-stable in terms of measures H and H_0.

The proof of Theorem 2 is similar to the proof of Theorem 1.

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