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GENERALIZED QUASI HAMILTONIAN SEMIGROUPS

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Abstract: Super Hamiltonian semigroups were considered by K.P. Shum and X.M. Ren in 2004. In their previous paper, it was proved that a semigroup is a super Hamiltonian if and only if it is expressible as a strong semilattice of quasi-groups. However, the above result does not hold in general. In this paper, we consider the quasi periodic groups and accordingly, we give a characterization theorem for the generalized quasi Hamiltonian semigroups. Some subclasses of generalized quasi Hamiltonian semigroups are discussed and some results obtained by Cherubini and Varisco on quasi Hamiltonian semigroups are extended.

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Key Words: generalized Hamiltonian semigroups, super Hamiltonian semigroups, strong semilattice of semigroups

1. Introduction

The following semigroups have been investigated by a number of authors in the literature, for example, see [1]-[3], [6] and [13]:

(i) quasi commutative if $(\forall a, b \in S)$ $(\exists r \in \mathbf{N}, r > 1)$ such that $ab = b^r a$.

(ii) σ-reflexive if $(\forall a, b \in S)$ $(\exists n \in \mathbf{N}, n > 1)$ such that $(ab)^n = ba$.

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(iii) Hamiltonian if $(\forall a, b \in S)$ $(\exists r, s \in \mathbf{N}, r+s > 2)$ such that $ab = b^r a^s$.

(iv) generalized quasi Hamiltonian if $(\forall a, b \in S)$ $(\exists r, s, n \in \mathbf{N}, r+s \neq 2n)$ such that $(ab)^n = b^r a^s$.

We point out here that in the above definitions, the integers n, r, s are related to the elements a, b in the semigroup S. Moreover, in the definition of generalized quasi Hamiltonian semigroups, the condition $r+s \neq 2n$ can be replaced by the condition $r+s < 2n$ as stated in [13]. It is clear that the class of generalized quasi Hamiltonian semigroups is a general class which contains the classes of quasi communicative semigroups, the σ-reflexive semigroups and the quasi Hamiltonian semigroups, respectively as its special subclasses. The generalized quasi Hamiltonian semigroups were first investigated by Shum and Ren [13], in particular, they proved that a semigroups S is a supper Hamiltonian semigroups if and only if S can be expressed as a strong semilattice of quasi-groups. However, their result is not true in general. In this paper, we study the generalized quasi Hamiltonian semigroups and introduce the concept of quasi periodic groups. By using the above concepts, we can accordingly amend and modify a previous result of Shum and Ren on a super Hamiltonian semigroup. In addition, some subclasses of generalized quasi Hamiltonian semigroups are considered and some related results given by Cherubini and Varisco in 1983 on quasi Hamiltonian semigroups are extended.

For notations and definitions not given in this paper, the reader is referred to [3] and [13].

2. Quasi Periodic Groups

A semigroup S generated by one element a, i.e., $S = < a >$, is called *monogenic*. If the semigroup S is finite, then there exist some positive integers $m, r \in \mathbf{N}$ such that $a^m = a^{m+r}$. Such integers m and r are called the index of a and periodicity of a, respectively.

We call a semigroup S a quasi group if S itself is a quasi regular semigroup and has one and only one idempotent, or equivalently, S contains a subgroup G and for each element $a \in S$, there exists a positive integer n such that $a^n \in G$.

Proposition 2.1. *Let S be a quasi group. Then there exists a maximum subgroup G of S which consists of all regular elements of S and is denoted by $G = H_e$, where e is the unique idempotent of the quasi group S.*

Proof. If $x \in S$ is a regular element, then there exists $x' \in S$ such that

xx' and $x'x$ are idempotents. Because S is a quasi group, S contains only one idempotent e. Hence, $xx' = x'x = e$. This shows that $x\,\mathcal{H}\,e$ and $x \in H_e$. In other words, all the regular elements of S are in the subgroup H_e of S. Clearly, all elements in H_e is regular. Thereby, the group H_e consists of all regular elements of S. If $a \in S$ is not regular, then a cannot belong to any subgroup of S. In fact, if a belongs to a subgroup of S, then a must be a regular element. This is clearly a contradiction. Thus, H_e is the maximum subgroup of S. □

We call a group G *periodic* if $< g >$ is finite for every element $g \in G$.

We now formulate the following definition.

Definition 2.2. A semigroup S is called a quasi periodic group if S is a quasi group and the maximum subgroup of S is periodic.

The following theorem is a characterization theorem for quasi periodic groups.

Theorem 2.3. *Let S be a quasi group. Then S is a generalized quasi Hamiltonian semigroup if and only if S is a quasi periodic group.*

Proof. "⇒" Let S be a generalized quasi Hamiltonian semigroup. Then for any $a \in S$, there exist $n, r, s \in \mathbf{N}$ with $n \neq r+s$ such that $(a^2)^n = a^{r+s}$. Since $2n \neq r+s$, $< a >$ is a monogenic semigroup. Because S itself is a quasi group, the maximum subgroup G of S is clearly periodic and hence, by definition, S is a quasi periodic group.

"⇐" Let S be a quasi periodic group and G the maximum subgroup of S. Then G is a periodic subgroup of S. Hence, for any $a_i \in S$ with $i = 1, 2$, there exists $n_i \in \mathbf{N}$ such that $a_i^{n_i} \in G$. Since G is a group, there exists $m_i \in \mathbf{N}$ such that $a_i^{n_i+m_i} = 1_G$ (the identity of G). Similarly, for a_1a_2, there exists $n, m \in \mathbf{N}$ such that $(a_1a_2)^n \in G$ and $(a_1a_2)^{n+m} = 1_G$. Thus we can easily deduce that
$(a_1a_2)^{(n+m)(n_1+m_1)(n_2+m_2)} = 1_G = a_2^{2(n+m)(n_1+m_1)(n_2+m_2)} a_1^{(n+m)(n_1+m_1)(n_2+m_2)}$.
Clearly, $2(n+m)(n_1+m_1)(n_2+m_2) \neq 3(n+m)(n_1+m_1)(n_2+m_2)$. Hence S is a generalized quasi Hamiltonian semigroup. The proof is completed. □

3. Generalized Quasi Hamiltonian Semigroups

In this section, we investigate the structure of generalized quasi Hamiltonian semigroups. Although a structure theorem of the super Hamiltonian semigroup were given by Shum and Ren in [13], there exists a gap in their proof. In this section, we shall amend their gap and give a general theorem for the generalized quasi Hamiltonian semigroups.

We first make the following observation:

Proposition 3.1. *The set of all idempotents $E(S)$ of a generalized quasi Hamiltonian semigroup S forms a semilattice.*

Proof. Since S is a generalized quasi Hamiltonian semigroup, there exist $n_1, r_1, s_1 \in \mathbf{N}$ such that $(ef)^{n_1} = f^{r_1}e^{s_1} = fe$, for any $e, f \in E(S)$. Similarly, there exist $n_2, r_2, s_2 \in \mathbf{N}$ such that $(fe)^{n_2} = e^{r_2}f^{s_2} = ef$. Thus, we deduce that

$$(ef)^{n_1n_2}e = ((ef)^{n_1})^{n_2}e = (f^{r_1}e^{s_1})^{n_2}e = (fe)^{n_2} = e^{r_2}f^{s_2} = ef\,,$$

$$(ef)^{n_1n_2}e = e((fe)^{n_2})^{n_1} = e(e^{r_2}f^{s_2})^{n_1} = (ef)^{n_1} = f^{r_1}e^{s_1} = fe\,.$$

Consequently, $ef = fe$ and $(ef)^2 = ef^2e = efe = e^2f = ef$. This shows that $E(S)$ is a commutative semigroup and thereby, $E(S)$ is a semilattice. □

Proposition 3.2. *Let S be a generalized quasi Hamiltonian semigroup. Then for any $a \in S$, $< a >$ is a finite monogenic semigroup.*

Proof. Since S is a generalized quasi Hamiltonian semigroup, there exist $n, r, s \in \mathbf{N}$ such that $a^{2n} = (a^2)^n = a^ra^s = a^{r+s}$. By the definition of generalized quasi Hamiltonian semigroups, we have $2n \neq r+s$. Hence $< a >$ is a finite monogenic semigroup. □

In a generalized quasi Hamiltonian semigroup S, for any $a \in S$, we denote the maximum subgroup of $< a >$ by G_a. And we denote the set of all regular elements of S by $\mathrm{Reg}(S)$.

Proposition 3.3. *Let S be a generalized quasi Hamiltonian semigroup. Then, for $e \in E(S)$, the subset*

$$S_e = \{a \in S | \exists m \in \mathbf{N} \text{ such that } a^m = e\}$$

is a subsemigroup of S. Moreover, S_e is a quasi periodic group.

Proof. We first show that S_e is a subsemigroup of S. Let $a, b \in S_e$. Then by the definition of S_e, there exist $m_1, m_2 \in \mathbf{N}$ such that $a^{m_1} = b^{m_2} = e$. Since S is a generalized quasi Hamiltonian semigroup, there exist $n_i, r_i, s_i \in \mathbf{N}$,

$i = 1, 2, \cdots$ such that

$$\begin{array}{ccc} (ab)^{n_1} & = & b^{r_1}a^{s_1}, \\ (b^{r_1}a^{s_1})^{n_2} & = & a^{s_1s_2}b^{r_1r_2}, \\ \cdots & \cdots & \cdots \\ (a^{s_1\cdots s_{2k}}b^{r_1\cdots r_{2k}})^{n_{2k+1}} & = & b^{r_1\cdots r_{2k+1}}a^{s_1\cdots s_{2k+1}}, \\ (b^{r_1\cdots r_{2k+1}}a^{s_1\cdots s_{2k+1}})^{n_{2k+2}} & = & a^{s_1\cdots s_{2k+2}}b^{r_1\cdots r_{2k+2}}, \\ \cdots & \cdots & \cdots . \end{array} \tag{1}$$

By the above equalities, we observe that there exist some sufficiently large integers n, r, s satisfying $(ab)^n = b^r a^s$ and $b^r \in G_b$, $a^s \in G_a$. Since $< ab >$ is a finite monogenic semigroup, there exists $m \in \mathbf{N}$ such that $(ab)^m = 1_{G_{ab}}$. Thus we deduce that

$$1_{G_{ab}} = (ab)^m = (ab)^{nm} = (b^r a^s)^m = (b^r a^s)^m e = (ab)^{nm} e = 1_{G_{ab}} e = e1_{G_{ab}}.$$

Observe that e can be written as

$$e = \overbrace{a^{m_1-s}b^{m_2-r} \cdots a^{m_1-s}b^{m_2-r}}^{m} \ (b^r a^s)^m,$$

we have

$$\begin{aligned} e1_{G_{ab}} &= \overbrace{a^{m_1-s}b^{m_2-r} \cdots a^{m_1-s}b^{m_2-r}}^{m} (b^r a^s)^m (b^r a^s)^m \\ &= \overbrace{a^{m_1-s}b^{m_2-r} \cdots a^{m_1-s}b^{m_2-r}}^{m} (b^r a^s)^m = e. \end{aligned}$$

Hence $(ab)^m = 1_{G_{ab}} = e$. This shows that $ab \in S_e$ and thereby S_e is indeed a subsemigroup of S. Since a^{m_1} is an idempotent, it is regular and this implies that S_e is a quasi regular semigroup. We now show that S_e has only one idempotent e. If $f \in S_e$ is another idempotent, then there exists $m \in \mathbf{N}$ such that $f^m = e$. This leads to $f = e$ and this shows that e is the unique idempotent of S_e. Thus S_e must be a quasi group. Since for every element a of S, $< a >$ is a finite monogenic semigroup. Therefore, S_e is a quasi periodic group. $\square$

In the following, we denote the maximum subgroup of S_e by G_e.

Proposition 3.4. *Let S be a generalized quasi Hamiltonian semigroup. Then for any $a \in G_e$, the class L_a (R_a) contains an unique idempotent e in S.*

Proof. Since $a \in G_e$, $G_e \subseteq H_a \subseteq$. If $f \in L_a$ is an idempotent, then $e = xf$ and $f = ye$ for some $x, y \in S$. Thus, we have

$$ef = xff = xf = e, \ \ fe = yee = ye = f.$$

Since $E(S)$ is a semilattice (see Proposition 3.1), $ef = fe$ and consequently, $e = f$. This shows that the class L_a contains an unique idempotent e. Similarly, the class R_a also contains an unique idempotent e. □

Proposition 3.5. *Let S be a generalized quasi Hamiltonian semigroup. Then $a \in S_e$ is a regular element of S_e if and only if a is a regular element of S.*

Proof. Clearly, if a is a regular element of S_e, then a must be a regular element of S. Conversely, if we let a be a regular element of S and $a^m = e$, then there exists $x \in S$ such that $axa = a$ and $xax = x$. Since S is a generalized quasi Hamiltonian semigroup, there exist some sufficiently large integers n, r, s satisfying $(xa)^n = x^r a^s$ and $a^s \in G_a$, where G_a is the maximum subgroup of $< a >$ (see the equalities (1) in the proof of Proposition 3.3). Obviously, xa is an idempotent and $G_a \subseteq G_e$. Thus we immediately deduce that

$$a = axa = a(xa)^n xa = aa^r x^s xa = aea^r x^s xa$$

$$= aexaxa = aexa = eaxa = ea = a^{m+1}.$$

Hence $a \in G_a \subseteq G_e$ and thereby a is a regular element of S_e. □

Proposition 3.6. *Let S be a generalized quasi Hamiltonian semigroup. Then $G = \cup_{e \in E(S)} G_e$ consists of all regular elements of S. Moreover, G is a subsemigroup of S.*

Proof. It is clear that an element of S contained in a subgroup of S must be a regular element of S. Hence $G \subseteq \mathrm{Reg}(S)$. Conversely, if $a \in S_e$ is a regular element of S, then by Proposition 3.5, a is also a regular element of S_e. Since S_e is a quasi periodic group, by Proposition 2.1, G_e consists of all regular elements of S_e. Hence $a \in G_e \subseteq G$, i.e., $\mathrm{Reg}(S) \subseteq G$. Thus G is a subsemigroup. Now, we let $a \in G_e$ and $b \in G_f$. Then there exist some inverses a^{-1} and b^{-1} of a and b in G_e and G_f, respectively. This leads to

$$ab(b^{-1}a^{-1})ab = a(bb^{-1})(a^{-1}a)b = afeb = aefb = ab.$$

Hence, we can see immediately that the element ab is regular. Since G consists of all regular elements of S, $ab \in G$ and so G must be a subsemigroup of S. □

Proposition 3.7. *Let S be a generalized quasi Hamiltonian semigroup. Then for $e, f \in E(S)$ with $e \neq f$, we have*

$$G_e \cap G_f = \emptyset$$

and

$$G_e G_f \subseteq G_{ef}.$$

Proof. If there exists $a \in G_e \cap G_f$, then there exists $m \in \mathbf{N}$ such that a^m is idempotent. Since any group contains one and only one idempotent which is the identity of the group, we have $e = a^m = f$. Hence, $G_e \cap G_f = \emptyset$ whenever $e \neq f$.

In order to show that $G_eG_f \subseteq G_{ef}$, we let $a \in G_e$, $b \in G_f$ with a^{-1} the inverse of a in G_e and b^{-1} the inverse of b in G_f. Moreover, we denote the idempotent of $< ab >$ and $< ba >$ by $(ab)^m$ and $(ba)^{m'}$, respectively. Thus $(ab)^m \in H_{ab} = G_{(ab)^m}$ and $(ba)^{m'} \in H_{ba}$. It is now easy to see that $baa^{-1}b^{-1}$ is an idempotent and $ba \, \mathcal{R} \, baa^{-1}b^{-1}$. This leads to that $(ba)^{m'} \, \mathcal{R} \, baa^{-1}b^{-1}$. Since R_{ba} contain only one idempotent (see Proposition 3.4), we have $(ba)^{m'} = baa^{-1}b^{-1}$. By the above observation, we can deduce that

$$\begin{aligned} ef = fef = b^{-1}baa^{-1}b^{-1}b = b^{-1}(baa^{-1}b^{-1})b = b^{-1}(ba)^{mm'}b = f(ab)^{mm'} \\ = (ab)^{mm'}f = (ab)^{mm'} = (ab)^m. \end{aligned}$$

Hence $ab \in G_{(ab)^m} = G_{ef}$ and thereby, $G_eG_f \subseteq G_{ef}$. □

Proposition 3.8. *Let S be a generalized quasi Hamiltonian semigroup. Then $E(S)$ lies in the center of $G = \cup_{e \in E(S)} G_e$.*

Proof. Let $a \in G_e$ and $f \in E(S)$. Then by Proposition 3.7, $af, fa \in G_{ef}$. Consequently, there exists $m_1, m_2 \in \mathbf{N}$ such that $(af)^{m_1} = (fa)^{m_2} = ef$. Let $m = m_1m_2$. Then we have $(af)^m = (fa)^m = ef$. Consequently, we have

$$\begin{aligned} af = (ef)(af)(ef) = (fe)(af)(fa)^m = f(eaf)(fa)^m = f(af)(fa)^{m-1}(fa) \\ = (fa)(fa)^{m-1}(fa) = (fa)^m(fa) = (ef)(fa) = fa. \end{aligned}$$

This shows that $E(S)$ lies in the center of G. □

Corollary 3.9. *Let S be a generalized quasi Hamiltonian semigroup. Then* Reg(S) *is a Clifford semigroup.*

Proof. By Proposition 3.6, $\mathrm{Reg}(S) = G$, and whence $\mathrm{Reg}(S)$ is a union of groups. By Proposition 3.8, $E(S)$ lies in the center of G. Thereby, $\mathrm{Reg}(S)$ is a Clifford semigroup. □

Since G is a Clifford semigroup, G is a strong semilattice of groups. Now, if we let $E(S)$ be the structure semilattice then the groups $\{G_e | e \in E(S)\}$ can be regarded as the group components of the strong semilattice expression of G. We have the following proposition.

Proposition 3.10. *Let S be a generalized quasi Hamiltonian semigroup. Then for $e, f \in E(S)$ with $e \neq f$, we have*

$$S_e \cap S_f = \emptyset$$

and

$$S_eS_f \subseteq S_{ef}.$$

Proof. Let $e, f \in E(S)$ with $e \neq f$ and suppose that $S_e \cap S_f \neq \emptyset$, say $a \in S_e \cap S_f$. Then there exists $m \in \mathbf{N}$ such that $a^m \in G_e \cap G_f$. This contradicts $G_e \cap G_f = \emptyset$ (see Proposition 3.7). Hence $S_e \cap S_f = \emptyset$ holds.

Now we prove that $S_eS_f \subseteq S_{ef}$. Let $a \in S_e$ and $b \in S_f$. Then by equalities (1) in the proof of Proposition 3.3, there exist some sufficiently large integers n, r, s satisfying $(ab)^n = b^r a^s$ and hence by Proposition 3.7, there exists $k \in \mathbf{N}$ such that $((ab)^n)^k = ef$. This leads to $ab \in S_{ef}$ and hence, $S_eS_f \subseteq S_{ef}$. □

In the above proposition, we have shown that a generalized quasi Hamiltonian semigroup S can be expressed a semilattice of a quasi periodic groups, i.e., $S = \cup_{e \in E(S)} S_e$, which is the semilattice $E(S)$ of quasi periodic groups $\{S_e | e \in E(S)\}$. We note that the converse part of the above proposition is also true. We now state the following theorem.

Theorem 3.11. *A semigroup S is a generalized quasi Hamiltonian semigroup if and only if $E(S)$ of S forms a semilattice and S can be expressed by $S = \cup_{e \in E(S)} S_e$, which is a semilattice $E(S)$ of some quasi periodic groups $\{S_e | e \in E(S)\}$.*

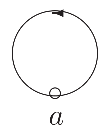
Proof. The necessity part has already been proved. We only need to prove the sufficiency part. Let $S = \cup_{e \in E(S)} S_e$ be the semilattice $E(S)$ of quasi periodic groups S_e, and let $a \in S_e$, $b \in S_f$. Since S_e and S_f are quasi periodic groups, there exist $m_1, m_2 \in \mathbf{N}$ such that $a^{m_1} = e$ and $b^{m_2} = f$. By the semilattice structure of S, we see that $ab \in S_{ef}$. Hence, there exists some $m \in \mathbf{N}$ such that $(ab)^m = ef$. This leads to

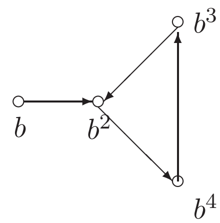
$$(ab)^m = ef = fe = b^{m_2} a^{m_1}.$$

Now, we can select an integer m such that $2m \neq m_1 + m_2$ because $(ab)^m = (ab)^{2m} = (ab)^{3m} = \cdots$. Hence S is indeed a generalized quasi Hamiltonian semigroup. □

Remark 3.12. A generalized quasi Hamiltonian semigroup, in general, need not be a strong semilattice of quasi periodic groups. We provide below an example which is a generalized quasi Hamiltonian semigroup but it is not a strong semilattice of quasi periodic groups.

Example 3.13. Consider $\beta < \alpha$. Let $S_\alpha = < a >$ and $S_\beta = < b >$ be the monogenic semigroups mounted on the semilattice α and β expressed by the following diagram.





S_α S_β

Define the multiplication on $S = S_\alpha \cup S_\beta$ as the multiplications of S. Clearly, the defined multiplication on S is well-defined and is associative. Thus, S forms a semigroup. It is clear that S is commutative and obviously $E(S)$ lies in the center of S. Moreover, we have $(ab^i)^4 = b^{4i} = b^i = b^i a$ and hence S is a generalized quasi Hamiltonian semigroup which is commutative (this semigroup was called *super Hamiltonian* in [13]). However, there does not exist any homomorphism $\phi : S_\alpha \to S_\beta$ such that

$$b^i \phi(a) = b^i a = b^i. \tag{2}$$

In fact, in order to make that equality (2) holds for $i = 1$, we must define $\phi(a) = 1$, however, in this case, ϕ is not a homomorphism which maps from S_α into S_β. Thus S is not a strong semilattice of semigroups $S'_\alpha s$. This example shows that Theorem 2.8 in [13] is not true. We note that the authors in [13] forgot to verify that the multiplication of the strong semilattice of semigroups and the multiplication in the semigroup itself should be the same! Moreover, we note here that a strong semilattice of quasi-groups is not necessarily a super Hamiltonian semigroup. For instance, a non-periodic Abelian group is not necessarily a super Hamiltonian semigroup.

4. Some Special Generalized quasi Hamiltonian Semigroups

In this Section, we discuss the properties of some subclasses of generalized quasi Hamiltonian semigroups. In particular, we consider the quasi Hamiltonian semigroups, quasi commutative semigroups and σ-reflexive semigroups.

We first formulate the following definition.

Definition 4.1. Let S be a semigroup. Then S is called:

(i) a uniform quasi commutative semigroup if

$$(\exists r \in \mathbf{N}, r > 1)(\forall a, b \in S)\ ab = b^r a;$$

(ii) a uniform σ-reflexive semigroup if

$$(\exists n \in \mathbf{N}, n > 1)(\forall a, b \in S)\ (ab)^n = ba;$$

(iii) a uniform quasi Hamiltonian semigroup if

$$(\exists r, s \in \mathbf{N}, r + s > 2)(\forall a, b \in S)\ ab = b^r a^s.$$

Proposition 4.2. *Let S be a uniform σ-reflexive semigroup. Then S is a commutative semigroup.*

Proof. Since S is a uniform σ-reflexive semigroup, there exists $n \in \mathbf{N}$ such that for all $a, b \in S$, $(ab)^n = ba$. Thus we deduce that

$$baa = (ab)^n a = a(ba)^n = aab,$$

and

$$aab = (b(aa))^n = ((ba)a)^n = aba.$$

Consequently, we have

$$\begin{array}{rcl} ba & = & (ab)^n \\ & = & abab \cdots ababab \\ & = & (aba)b \cdots ababab \\ & = & (baa)b \cdots ababab \\ & = & (ba)ab \cdots ababab \\ & \cdots & \\ & = & (baba \cdots ba)abab \\ & = & (baba \cdots)b(aab)ab \\ & = & (baba \cdots)b(aba)ab \\ & = & (baba \cdots ba)b(aab) \\ & = & (baba \cdots ba)b(aba) \\ & = & (ba)^n \\ & = & ab. \end{array}$$

Hence S is a commutative semigroup. □

By Proposition 4.2, we see that the concept of uniform σ-reflexive semigroups is not weaker than the concept of commutative semigroups, on the contrary, it is a stronger concept. In the following, we consider an important subclass of quasi periodic groups.

We give the following definition.

Definition 4.3. Let S be a quasi periodic group (periodic group).

(i) If the periodicity of each $a \in S$ is a factor of n and the index of a is less than or equal to m, then S is called a *quasi periodic group* (periodic group) with periodicity n and index m (if S is a group, then the index is always 1 and is not necessary to mention).

(ii) If $S^2 = G$ (G is the maximum subgroup of S), then S is called a *near periodic group.*

Theorem 4.4. *Let S be a semigroup with $S^2 = S$. Then S is a uniform σ-reflexive semigroup with $(ab)^n = ba$ if and only if $S = (Y; G_\alpha; \phi_{\alpha,\beta})$ is a strong semilattice of commutative periodic groups with period $n-1$.*

Proof. "$\Rightarrow$" Clearly, S is a generalized quasi Hamiltonian semigroup. By Theorem 3.11, we can write $S = \cup_{e \in E(S)} S_e$ which is the semilattice of the quasi periodic groups $S_e | e \in E(S)$. Now, by Proposition 4.2, S is commutative semigroup. Since for every $e \in E(S)$, G_e is a commutative periodic group with period $n-1$. Also, because $(E(S)$ lies in the center of S, we can easily see that $S = \cup_{e \in E(S)} G_e$ is a semilattice of commutative periodic groups with periodicity n-1 and thereby, S is a Clifford semigroup. This shows that $S = (Y; G_\alpha; \phi_{\alpha,\beta})$ is a strong semilattice of commutative periodic groups with period $n-1$.

"$\Leftarrow$" Let $S = (Y; G_\alpha; \phi_{\alpha,\beta})$ be a strong semilattice of commutative periodic groups with periodicity $n-1$. Then for $a \in G_\alpha$ and $b \in G_\beta$, we have $a = a^n, b = b^n$. This leads to

$$\begin{aligned} a \circ b = a^n \circ b^n &= (a^n \phi_{\alpha,\alpha\beta})(b^n \phi_{\beta,\alpha\beta}) \\ &= (a\phi_{\alpha,\alpha\beta})^n (b\phi_{\beta,\alpha\beta})^n = ((a\phi_{\alpha,\alpha})(b\phi_{\beta,\alpha\beta}))^n = (a \circ b)^n = (b \circ a)^n. \end{aligned}$$

Hence S is a uniform σ-reflexive semigroup. ☐

Theorem 4.5. *Let S be a semigroup. Then S is a uniform σ-reflexive semigroup with $(ab)^n = ba$ if and only if $S = (Y; S_\alpha; \phi_{\alpha,\beta})$ is a strong semilattice of commutative near periodic groups with periodicity $(n-1)$, where $\phi_{\alpha,\beta}$, $\alpha, \beta \in Y$ with $\alpha > \beta$, are homomorphisms from S_α into G_β (the maximum subgroup of S_β).*

Proof. "$\Rightarrow$" Clearly, S^2 is a subsemigroup of S and S^2 is also a uniform σ-reflexive semigroup. Moreover, $E(S^2) = E(S)$ and $(S^2)_e = G_e$. Now, by Theorem 4.4, $S^2 = (E(S); G_e; \phi_{e,f})$ is a strong semilattice of commutative periodic groups with periodicity $n-1$. Now, we define a homomorphisms $\psi_{e,f} : S_e \to S_f$, where $e \geq f$. If $e = f$, then we let $\psi_{e,e}$ be the identical automorphism. If $e > f$, then we can extend the domain of the homomorphism $\phi_{e,f}$ from G_e to S_e in the following way: for any $a \in S_e$, define $a\psi_{e,f} = a^n \phi_{e,f}$. Since the index of a

is 1 or 2 and $n > 1$, we have $a^n \in G_e$ and

$$(ab)\psi_{e,f} = (ab)^n \phi_{e,f} = (a^n \phi_{e,f})(b^n \phi_{e,f}) = (a\psi_{e,f})(b\psi_{e,f})\,.$$

Thus, $\psi_{e,f}$ is indeed a homomorphism. Let $e, f, g \in E(S)$ with $e \geq f \geq g$. Then we can deduce that $\psi_{e,f}\psi_{f,g} = \psi_{e,g}$. In fact, for any $a \in S_e$, we have

$$\begin{aligned} a\psi_{e,f}\psi_{f,g} &= a^n \phi_{e,f}\psi_{f,g} = (a\phi_{e,f})^n \psi_{f,g} \\ &= (a\phi_{e,f})^n \phi_{f,g} = a^n \phi_{e,f}\phi_{f,g} = a^n \phi_{e,g} = a\psi_{e,g}. \end{aligned}$$

Hence, $\psi_{e,f}\psi_{f,g} = \psi_{e,g}$ and thereby, $\{\psi_{e,f} | e, f \in E(S)$ with $e \geq f\}$ are the structure homomorphisms of the strong semilattice. It is clear that the range of $\psi_{e,f}$ $(e > f)$ is contained in G_f. The final and the most vital step is to verify that the multiplication in the semigroup S and the multiplication in the strong semilattice of semigroups are the same. For this purpose, we let $a \in S_e$ and $b \in S_f$. If $e = f$, then it is clear that $(a\psi_{e,ef})(b\psi_{f,ef}) = ab$. Now we suppose that $e \neq f$. Since $ab = (ba)^n \in G_{ef}$, $ab = abef = (ae)(bf) = a^n b^n$. Thus, we deduce that

$$(a\psi_{e,ef})(b\psi_{f,ef}) = (a^n \phi_{e,ef})(b^n \phi_{f,ef}) = a^n b^n = ab.$$

Hence the multiplication in the semigroup S and the multiplication in the strong semilattice are indeed the same. Since for $a, b \in S_e$, $ab = (ba)^n = a^n b^n \in G_e$, we have $(S_e)^2 = G_e$. By summing up the above discussion, we can easily prove that $S = (E(S); S_e; \psi_{e,f})$ is a strong semilattice of commutative near periodic groups whose periodicity are $n - 1$, where $\psi_{e,f}$'s $(e > f)$ are the structural homomorphisms which map from S_e into G_f.

"$\Leftarrow$" Let $S = (Y; S_\alpha; \phi_{\alpha,\beta})$ be a strong semilattice of commutative near periodic groups with periodicity $n - 1$, and $\phi_{\alpha,\beta}$'s $(\alpha > \beta)$ are homomorphisms which map from S_α into G_β. For $a, b \in S_\alpha$, since S_α is a commutative near periodic group with periodicity $n - 1$, we have $a \circ b = ab \in G_\alpha$. Moreover, for any $x \in S_\alpha$, $(x^{n-1})^2 = x^{2n-2} = x^{n-1}$. Hence $1_{G_\alpha} = x^{n-1}$. Thus we deduce that

$$a \circ b = ab = eabe == a^n b^n,$$

and

$$a \circ b = (a\phi_{\alpha,\alpha})(b\phi_{\alpha,\alpha}) = ab = eabe = a^n b^n = (ba)^n = (b \circ a)^n.$$

Also, for $a \in S_\alpha$, $b \in S_\beta$, $\alpha \neq \beta$, we have

$$\begin{aligned} a \circ b &= (a\psi_{\alpha,\alpha\beta})(b\psi_{\beta,\alpha\beta}) \\ &= (a\psi_{\alpha,\alpha\beta})^n (b\psi_{\beta,\alpha\beta})^n = ((a\psi_{\alpha,\alpha\beta})(b\psi_{\beta,\alpha\beta}))^n = (b \circ a)^n. \end{aligned}$$

Hence S is a uniform σ-reflexive semigroup. □

In the following, we are going to show that the concepts of uniform σ-reflexive semigroups, uniform quasi commutative semigroups and uniform quasi

Hamiltonian semigroups are equivalent.

Theorem 4.6. *Let S be a semigroup. Then the following statements are equivalent:*

(i) S is a uniform quasi commutative semigroup with $ab = b^n a$ $(n > 1)$.

(ii) S is a uniform σ-reflexive semigroup with $ab = (ba)^n$ $(n > 1)$.

(iii) S is a uniform quasi Hamiltonian semigroup with $ab = b^n a^s (n > 1)$ or $ab = b^s a^n$ $(n > 1)$.

Proof. "(ii)⇒(i)" Let S be a uniform σ-reflexive semigroup with $ab = (ba)^n$. Then by Theorem 4.5, $S = (E(S); S_e; \phi_{e,f})$ is a strong semilattice of commutative near periodic groups. By Proposition 4.2, S is commutative. Since for $a \in S_e$, $a^2 = a^{2n}$, $a^3 = a^{3n}$, the index of S_e is 2, the periodicity of S_e is a common factor of $2n - 2$ and $3n - 3$ and hence it is $n - 1$. Furthermore, $e = a^{2(n-1)} = a^{3(n-1)}$. Thus for $a \in S_e$ and $b \in S_f$, we have

$$ab = abef = (ba)^n ef = b^n a^n ef = (b^n a)a^{3(n-1)} f = b^n aef = b^n a.$$

Hence S is a uniform quasi commutative semigroup with $ab = b^n a$ $(n > 1)$.

"(iii)⇒(ii)" Let S be a uniform quasi Hamiltonian semigroup with $ab = b^n a^s$ $(n > 1)$ for all $a, b \in S$. Then by Theorem 3.11, $S = \cup_{e \in E(S)} S_e$ is the semilattice $E(S)$ of quasi periodic groups. Clearly, for $a \in S_e$, $a^2 = a^{n+s}$, $a^3 = a^{2n+s} = a^{n+2s}$. This means that the index of $< a >$ is 2, the periodicity of $< a >$ is a common factor of $n+s-1$, $2n+s-3$ and $n+2s-3$ and thereby it is a common factor of $(2n+s-3)-(n+s-2) = n-1$ and $(n+2s-3)-(n+s-2) = s-1$. we denote it by p. Clearly, we have $e = a^{2p}$. Now let $a \in S_e$ and $b \in S_f$. Then from $ab = b^n a^s = a^{ns} b^{ns}$ and $ba = a^n b^s = b^{ns} a^{ns}$, we know that $a^{ns} \in G_e$ and $b^{ns} \in G_f$ because $ns \geq 2$. Moreover, by Proposition 3.7, we see that $ab, ba \in G_{ef}$ and by Proposition 3.8, $E(S)$ lies in the center of G. Thus we deduce that

$$ab = efab = efb^n a^s = fb^n a^s e = b^{2p+n-1}(ba)a^{2p+s-1} = ba.$$

Thereby, S is commutative semigroup with $ab = b^n a^n = (ba)^n$. This shows that S is a uniform σ-reflexive semigroup with $ab = (ba)^n$ $(n > 1)$. For $n = 1$, $s > 1$, the proof is similar and we hence omit the proof.

Since the class of uniform quasi commutative semigroups with $ab = b^n a$ $(n > 1)$ forms a subclass of uniform quasi Hamiltonian semigroups with $ab = b^n a^s$ $(n > 1)$, we see that from (iii)⇒ (ii), we have (i) ⇒ (ii) and from (ii)⇒(i), we have (ii)⇒ (iii). Thus, the proof is completed. □

The following are some special classes of uniform quasi commutative semigroups (uniform σ-reflexive semigroups, uniform quasi Hamiltonian semigroups).

They all have simpler structures.

Corollary 4.7. *A semigroup S satisfies one of the following conditions:*

(i) $ab = b^2a$ for all $a, b \in S$,

(ii) $ab = (ba)^2$ for all $a, b \in S$,

(iii) $ab = b^2a^s$ for all $a, b \in S$, where s is an arbitrary fixed positive integer if and only if $S = \cup_{\alpha\in Y} S_\alpha$ is a semilattice of null semigroups, and for $a \in S_\alpha$, $b \in S_\beta$, $ab = 0_{\alpha\beta}$ (the zero element of $S_{\alpha\beta}$).

Proof. By Theorem 4.6, the condition (i), (ii) and (iii) are equivalent. Hence we only need to prove that the conclusion holds if S satisfies condition (i).

"$\Rightarrow$" Clearly, S is a uniform quasi commutative semigroup. By Theorem 4.6, S is also a uniform σ-reflexive semigroup. Now, by Theorem 4.5, we see that $S = (Y; S_\alpha; \phi_{\alpha,\beta})$ is a strong semilattice of commutative near periodic groups. Since $a^2 = a^3$, for all $a \in S$, the periodicity of the semigroup S_α is 1 and G_α has only one element. Notice that since S_α is a near periodic group, S_α is a null semigroup. Denote the idempotent of S_α by 0_α, for $a \in S_\alpha$ and $b \in S_\beta$. Then, we have

$$ab = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}) = 0_{\alpha\beta}.$$

Hence the necessity part is proved.

"$\Leftarrow$" Since $S = (Y; S_\alpha; \phi_{\alpha,\beta})$ is a semilattice of null semigroups, and for $a \in S_\alpha$, $b \in S_\beta$, $ab = 0_{\alpha\beta}$, we have $ab = 0_{\alpha\beta} = b^2a$ and hence the sufficiency part is proved. □

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