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GENERIC ONE-PARAMETER VERSAL UNFOLDINGS OF SYMMETRIC HAMILTONIAN SYSTEMS IN 1:1 RESONANCE

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Abstract: We consider Hamiltonian systems in 1:1 resonance in the presence of symmetry. We give some new proofs for known results concerning the classification of generic one-parameter deformations of equivariant linear systems and the passing and splitting of eigenvalues. We show that for nonlinear systems in two degrees of freedom the bifurcation of periodic solutions in the generic passing cases can be linearized. We conclude with several examples.

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1. Introduction

The generic behavior of the eigenvalues of one-parameter deformations of symmetric linear Hamiltonian systems is considered in a number of papers [4], [5], [6]. In [4] the authors study in particular the one to one resonance, i.e. linear Hamiltonian systems having multiple eigenvalues $\pm i$. Generic bifurcations in (nonlinear) Hamiltonian systems are also subject of [9] and [17]. Recently similar studies were done on reversible and reversible Hamiltonian systems [14], [12], [1]. When varying the parameter in a Hamiltonian system with multiple eigenvalues $\pm i$ two things can happen; either the eigenvalues split but remain on the imaginary axes, which is called 'passing', or the eigenvalues split into the complex plane having nonzero real part, which is called 'splitting'. In addition to these phenomena in [1] also crossing (crossing the imaginary axes from the

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complex plane) is observed in the reversible case. In [4] a classification is given of the generic cases in which passing and splitting occurs. It turns out that one has to distinguish two cases according to whether the quadratic Hamiltonian associated to the system is positive definite or indefinite. In the following we will give alternative proofs of these earlier results in a new setting. Instead of concentrating on the behaviour of the eigenvalues we will work from the point of view of generic co-dimension one bifurcations, i.e. study one parameter universal deformations in the presence of symmetry. Related articles are [19] and [18]. The results in this paper rely on group theoretic results in [20] and [10]. We will use Lie algebras rather then Lie groups, i.e. exploit the Hamiltonian structure. By using general facts on Lie algebras we can avoid detailed computations on eigenvalues. The approach allows one to obtain co-dimensions by counting the dimensions of the Lie algebras involved. After having obtained our results concerning linear systems we will consider nonlinear normal forms, in particular systems on $\mathbb{R}^4$ with passing eigenvalues and conclude with some examples concerning nonlinear systems.

We will conclude this section with a statement of the results proven in the following sections. If we consider a Hamiltonian system in 1:1 resonance having compact symmetry group Γ and having a one-parameter Γ-universal deformation, then the generalized eigenspace $E_{\pm i} = U_1 \oplus U_2$ with U_1 and U_2 symplectically irreducible (Theorem 1). The opposite only holds in special cases as indicated in the following table where also the behavior of the eigenvalues is given.

	definite (semisimple)	indefinite semisimple	indefinite nonsemisimple
U_1, U_2 nonisomorphic	passing	passing	not possible
U_1, U_2 complex dual	passing	not generic	splitting
U_1, U_2 isomorphic, not complex	not generic	not generic	splitting
U_1, U_2 isomorphic, complex	not generic	passing	not possible

Furthermore it is shown that for nonlinear equivariant systems with two degrees of freedom having generically passing eigenvalues the nonlinear bifurcation is equivalent to the linear one.

2. Preliminaries

Although in general one uses group theory in the study of bifurcations, in the theory of bifurcations of Hamiltonian systems it has advantages to use the theory of Lie algebras and work with the Hamiltonian functions rather than with the vector fields. Consider the symplectic space $(I\!R^{2n}, \omega)$, with ω the standard symplectic form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$, and $z = (x_1, \cdots, x_n, y_1, \cdots, y_n)$ canonical coordinates on $I\!R^{2n}$. The Lie algebra $sp(n)$ of infinitesimal symplectic matrices on $I\!R^{2n}$ is as a Lie algebra isomorphic to the algebra $\mathcal{Q}[n]$ of homogeneous quadratic polynomials on $I\!R^{2n}$, with the Poisson bracket $\{\ ,\ \}$ associated to the symplectic form ω, if one maps each homogeneous quadratic Hamiltonian function $H_2(z)$ to the corresponding infinitesimal symplectic matrix $DX_{H_2}(z)$, with $X_{H_2}(z) = \{H2(z), z\}$ the Hamiltonian vector field corresponding to H_2. Let Γ be a compact Lie group acting linearly and symplectically on $I\!R^{2n}$ and let $Sp_\Gamma(n)$ be the group of all symplectic matrices on $I\!R^{2n}$ commuting with the action of Γ. The Lie algebra $sp_\Gamma(n)$ of $Sp_\Gamma(n)$ is isomorphic to the subalgebra $\mathcal{Q}_\Gamma[n]$ of $\mathcal{Q}[n]$ consisting of all quadratic polynomials invariant under Γ, i.e. $\mathcal{Q}_\Gamma[n]$ consists of all $H_2 \in \mathcal{Q}[n]$ with $H_2(\gamma z) = H_2(z)$ for all $\gamma \in \Gamma$.

We will now recall some definitions concerning deformations. A p-parameter deformation of a Hamiltonian $H : I\!R^{2n} \to I\!R$ is a Hamiltonian $\hat{H} : I\!R^{2n} \times I\!R^p \to I\!R \times I\!R^p$ such that $\hat{H}(z, 0) = H(z)$ with $z \in I\!R^{2n}$. Two p-parameter deformations $\hat{H}$ and $\tilde{H}$ of H are equivalent if there exists a symplectic parameter dependent diffeomorphism $\varphi(z, \lambda)$ of $I\!R^{2n} \times I\!R^p$ such that $\varphi(z, \lambda) = (\psi(z, \lambda), \lambda)$ and $\hat{H} = \tilde{H} \circ \varphi$. A deformation $\hat{H}$ is called trivial if it is equivalent to H itself. A deformation is universal if any other deformation is equivalent to it. If H is invariant under the symplectic action of some group Γ then we may consider equivalence with respect to the group of Γ-equivariant symplectic diffeomorphisms. With respect to this group we will speak of Γ-universal deformations. The minimal number of parameters necessary to obtain a universal deformation is the co-dimension of H. Note that $\hat{H}(z, \lambda)$ is also called an unfolding of $H(z)$. A generic unfolding corresponds to a universal deformation.

Actually the codimension of H is the co-dimension of the tangent space at H to the orbit under the group of symplectic diffeomorphisms through H. Let $\{\ ,\ \}$ denote the Poisson bracket induced by the symplectic form and consider the map ad_H defined by $ad_H(F) = \{H, F\}$ where $H, F \in C^\infty(I\!R^{2n}, I\!R)$. Then the tangent space at H is given by $\mathrm{im}(ad_H)$. Determining a basis for a complement of $\mathrm{im}(ad_H)$ then provides us with the directions in which H should be deformed (or unfolded) in order to obtain a universal deformation. Now universal deformations should at least detune the resonance. Consequently we may restrict to the linear case. On the linear level this reduces to computations involving linear symplectic maps and homogeneous quadratic polynomials. In this case the complement of $\mathrm{im}(ad_{H_2})$ is determined as follows. When $H_2 \in \mathcal{Q}[n]$ is semisimple (i.e. ad_{H_2} is semisimple) then the complement is $\ker(ad_{H_2})$, i.e. the centralizer $C_{\mathcal{Q}[n]}(H_2)$ of H_2 in $\mathcal{Q}[n]$. When $H_2 \in \mathcal{Q}[n]$ is nilpotent then we can find a nilpotent $M \in \mathcal{Q}[n]$ and a semisimple $T \in \mathcal{Q}[n]$ such that H, M, T span a Lie algebra isomorphic to $sl(2, I\!R)$. The complement is now $C_{\mathcal{Q}[n]}(M)$. When H_2 is nonsemisimple, i.e. has a nontrivial Jordan-Chevalley decomposition into a nonzero semisimple part S and a nonzero nilpotent part N then , as before, we can embed N in a Lie algebra isomorphic to $sl(2, I\!R)$.The complement is now $C_{\mathcal{Q}[n]}(S) \cap C_{\mathcal{Q}[n]}(M)$ (for more details see [15, 16, 3]). In the presence of a symmetry group Γ we can choose S, N and M in $\mathcal{Q}_\Gamma[n]$ and we have to take centralizers in $\mathcal{Q}_\Gamma[n]$ instead of in $\mathcal{Q}[n]$. Consequently the co-dimension of the orbit of H_2 under the action of Γ-equivariant linear symplectic mappings is the dimension of $\mathcal{C} = C_{\mathcal{Q}_\Gamma[n]}(S) \cap C_{\mathcal{Q}_\Gamma[n]}(M)$. Let $Q_1, \cdots Q_k$ be a basis for the subalgebra $\mathcal{C}$ considered as a module. Then the universal unfolding of H_2 will be $H_2 + \lambda_1 Q_1 + \cdots + \lambda_k Q_k$. Note that S can be chosen as one of the Q_i. Detuning with S scales the eigenvalues but does not detune the resonance. We do not want to count this as an unfolding direction. Consequently we have to extend our equivalence group to include such scalings and thus diminish the co-dimension by one. The unfolding terms are now to be found in $\mathcal{C} - < S >$.

3. Decomposition of the Generalized Eigenspace

Let H_2 denote an element of $\mathcal{Q}[n]$ such that the corresponding matrix in $sp(n, I\!R)$ has eigenvalues $\pm i$, i.e. the corresponding Hamiltonian system is in $1 : \cdots : 1, -1 : \cdots : -1$ resonance, with $2p$ i's and $2q$ $-i$'s. From [8], [11], [13] it is clear that already in the two degrees of freedom 1:-1 resonance H_2 will have a three parameter universal deformation. Therefore generic one-parameter deformations will in general only be found when considering symmetric (or equivariant) systems. The question is now, for which compact groups Γ acting linearly and symplectically on $I\!R^{2n}$ do we have a one-parameter Γ-universal deformation of H_2, i.e. for which compact groups Γ does H_2 have co-dimension one with respect to the group of linear Γ-equivariant symplectic transformations?

If we have a symmetry group Γ we must have that H_2 is invariant under Γ. Thus Γ is isomorphic to a subgroup of $U(p, q)$. Furthermore a deformation of H_2 will be $H_2 + \lambda_1 Q_1 + \lambda_2 Q_2 + \ldots$, with $Q_i \in C_{\mathcal{Q}_\Gamma[n]}(H_2)$.

Suppose we have a finite dimensional symplectic representation V of Γ. Because Γ is compact this representation is semisimple and thus completely reducible, i.e. $V = V_1 \oplus ... \oplus V_l$ with each V_i symplectically irreducible. Each V_i is a symplectic subspace and because H_2 is invariant under Γ, each V_i is invariant under the flow corresponding to H_2. Considering the restriction of the flow to V_i we find on each V_i an Hamiltonian $H_{2,i}$. Moreover $H_2 = H_{2,1} + \ldots + H_{2,l}$, i.e. H_2 splits over the V_i. In addition the $H_{2,i}$ are independent, as they live on different subspaces, and in $C_{\mathcal{Q}_\Gamma[n]}(H_2)$. Consequently a representation as above will give rise to at least a $l-1$-parameter universal deformation. This gives the following theorem (see also [4], Theorem 3.3).

Theorem 1. *Consider a linear Hamiltonian system in 1:1 resonance with symmetry group* Γ. *Suppose that the system has a one parameter* Γ-*universal deformation, i.e. has codimension one with respect to the group of* Γ-*equivariant linear symplectic transformations. Then the generalized eigenspace* $E_{\pm i} = U_1 \oplus U_2$ *with* U_1 *and* U_2 *symplectically irreducible.*

Symplectically irreducible means that the U_i are nonabsolutely irreducible of complex or quaternionic type, or $U_i = V \oplus V$ with V absolutely irreducible of real type [20]. Consequently the following cases are obtained according to the representation of the symmetry group Γ:

1. *Representations of real type.*

1.1. $V \oplus V \oplus V \oplus V$, V absolutely irreducible of real type.

1.2. $V_1 \oplus V_1 \oplus V_2 \oplus V_2$, V_1 and V_2 absolutely irreducible of real type and nonisomorphic.

2. *Representations of complex type.*

2.1. $W \oplus W$, W nonabsolutely irreducible of complex type.

2.2. $W_1 \oplus W_2$, W_1 and W_2 nonabsolutely irreducible of complex type, W_1 and W_2 dual.

2.3. $W_1 \oplus W_2$, W_1 and W_2 nonabsolutely irreducible of complex type, nondual and nonisomorphic.

3. *Representations of quaternionic type.*

3.1. $W \oplus W$, W nonabsolutely irreducible of quaternionic type.

3.2. $W_1 \oplus W_2$, W_1 and W_2 nonabsolutely irreducible of quaternionic type and nonisomorphic.

4. *Representations of mixed type.*

4.1. $V \oplus V \oplus W$, V absolutely irreducible of real type and W nonabsolutely irreducible of complex type.

4.2. $V \oplus V \oplus W$, V absolutely irreducible of real type and W nonabsolutely irreducible of quaternionic type.

4.3. $W_1 \oplus W_2$, W_1 nonabsolutely irreducible of complex type and W_2 nonabsolutely irreducible of quaternionic type.

4. One Parameter Universal Deformations

In general the opposite of theorem 1 is not true. However, if the representations U_1 and U_2 are nonisomorphic then any element in $C_{\mathcal{Q}_\Gamma[n]}(H_2)$ must have the same blockdiagonal structure. Consequently such an element can not be independent from the earlier constructed $H_{2,i}$ and in addition must be semisimple. Thus we have

Theorem 2. *Consider a linear Hamiltonian system in 1:1 resonance with Hamiltonian H_2 and symmetry group Γ. Suppose that the representation of Γ on the generalized eigenspace $E_{\pm i}$ is equal to $U_1 \oplus U_2$ with U_1 and U_2 nonisomorphic irreducible representations. Then H_2 is semisimple and has a one-parameter Γ-universal deformation.*

Note that Theorems 1 and 2 hold for the definite as well as the indefinite case. Sofar we only considered the nonisomorphic case. The isomorphic case splits naturally into two subcases; isomorphic but symplectically nonisomorphic (the case where U_1 and U_2 are complex irreducible dual representations), and symplectically isomorphic. Note that in the isomorphic case Γ has a one block isotypic decomposition. We have the following lemma

Lemma 3. (see [20]) *Let Γ have a one block isotypic decomposition, then:*

(i) $\mathcal{Q}_\Gamma[n] \approx sp(n, \mathbb{R})$ if we have $2n$ irreducible real blocks,

(ii) $\mathcal{Q}_\Gamma[n] \approx u(p, q)$ if we have $p + q = n$ irreducible complex blocks, q duals,

(iii) $\mathcal{Q}_\Gamma[2n] \approx \alpha u(n, \mathbb{H})$ if we have n irreducible quaternionic blocks.

Let us first consider the case of U_1, U_2 complex duals. If H_2 is definite then $C_{\mathcal{Q}[n]}(H_2) \approx u(n)$ while $\mathcal{Q}_\Gamma[n] \approx u(1,1)$. Any deformation must be in $C_{\mathcal{Q}_\Gamma[n]}(H_2)$ which is the intersection of these two algebras and which is isomorphic to the two-torus. Because H_2 is also in $C_{\mathcal{Q}_\Gamma[n]}(H_2)$ we have a one-parameter universal deformation. If H_2 is indefinite and semisimple then $C_{\mathcal{Q}[n](H_2)} \approx u(p, q)$ and the intersection of the two algebras is isomorphic to $u(1,1)$ and H_2 will have Γ-codimension three. If H_2 is indefinite and nonsemisimple then the semisimple and nilpotent parts S and N of H_2 are both in $u(p, q) \cap u(1,1) \approx u(1,1)$. Embedding N into a subalgebra of $u(1,1)$ isomorphic to $sl(2, \mathbb{R})$ we obtain that the codimension of H_2 in $u(1,1)$ is one. We have

Theorem 4. *Consider a linear Hamiltonian system in 1:1 resonance with Hamiltonian H_2 and symmetry group Γ. Suppose that the representation of Γ on the generalized eigenspace $E_{\pm i}$ is $U_1 \oplus U_2$ with U_1 and U_2 dual complex irreducible representations. If H_2 is definite or indefinite and nonsemisimple then the Γ-codimension is one, if H_2 is indefinite and semisimple then the Γ-codimension is three.*

Next consider the case where U_1 and U_2 are symplectically isomorphic irreducible complex representations. Then $\mathcal{Q}_\Gamma[n] \approx u(2)$. In the same way as above we obtain that if H_2 is definite and semisimple we have $C_{\mathcal{Q}_\Gamma[n]}(H_2)$ has dimension four, that is, H_2 has codimension three. The definite nonsemisimple case can not occur because $u(2)$ does not contain nilpotent elements. Finally the indefinite semisimple case has codimension one.

Theorem 5. *Consider a linear Hamiltonian system in 1:1 resonance with Hamiltonian H_2 and symmetry group Γ. Suppose that the representation of Γ on the generalized eigenspace $E_{\pm i}$ is $U_1 \oplus U_2$ with U_1 and U_2 symplectically isomorphic complex irreducible representations. Then H_2 is semisimple and has Γ-codimension one if it is indefinite and Γ-codimension three if it is definite.*

Finally we have to consider the case where U_1 and U_2 are isomorphic noncomplex irreducible representations, that is, both real or both quaternionic. In the first case $\mathcal{Q}_\Gamma[n] \approx sp(2, \mathbb{R})$ (4 by 4 infinitesimal symplectic matrices), and in the second case $\mathcal{Q}_\Gamma[n] \approx \alpha u(2, \mathbb{H})$. In both cases $C_{\mathcal{Q}_\Gamma[n]}(H_2)$ is equal to $u(2)$ in the definite case and equal to $u(1,1)$ in the indefinite case and thus the arguments used above can be applied. Cosequently in the semisimple case definite or indefinite we have codimension larger than one, while in the indefinite nonsemisimple case we have codimension one.

Theorem 6. *Consider a linear Hamiltonian system in 1:1 resonance with Hamiltonian H_2 and symmetry group Γ. Suppose that the representation of Γ on the generalized eigenspace $E_{\pm i}$ is $U_1 \oplus U_2$ with U_1 and U_2 isomorphic noncomplex irreducible representations. Then H_2 has Γ-codimension one if it is indefinite and nonsemisimple. If H_2 is semisimple the Γ-codimension is larger than one.*

Theorems 2, 4 , 5, and 6 give us the following theorem.

Theorem 7. *Consider a linear Hamiltonian system in 1:1 resonance with Hamiltonian H_2 and symmetry group Γ. Suppose that the representation of Γ on the generalized eigenspace $E_{\pm i}$ is $U_1 \oplus U_2$ with U_1 and U_2 irreducible symplectic representations. Then H_2 has Γ-co-dimension one in the following* *cases:*

(i) U_1 *and* U_2 *nonisomorphic.*

(ii) U_1 *and* U_2 *complex duals and* H_2 *definite or nonsemisimple.*

(iii) U_1 *and* U_2 *isomorphic complex and* H_2 *indefinite and semisimple.*

(iv) U_1 *and* U_2 *isomorphic noncomplex and* H_2 *indefinite and nonsemisimple.*

5. The Eigenvalues

On $I\!R^{2n}$ with the standard symplectic form a system in 1:1 resonance has the normal form

$$H_2 = \sum_{j=1}^{n} (x_j^2 + y_j^2)\,. \tag{1}$$

The Lie algebra $sp(n)$ of infinitesimal symplectic matrices on $I\!R^{2n}$ is as a Lie algebra isomorphic to the algebra $\mathcal{Q}[n]$ of homogeneous quadratic polynomials on $I\!R^{2n}$ with the Poisson bracket associated to the symplectic form if one maps each matrix to the corresponding homogeneous quadratic Hamiltonian. The subalgebra $\mathcal{Q}_{H_2}[n]$ of homogeneous quadratic polynomials commuting wit H_2 is isomorphic to $u(n)$. Formulated in a different way we have that the group of linear symplectic transformations leaving H_2 fixed is $SO(2n) \cap Sp(n) \approx U(n)$. Clearly the eigenvalues of a linear system associated to a quadratic polynomial in the polynomial representation $\mathcal{Q}_{H_2}[n]$ of $u(n)$ realized above are purely imaginary (including zero). Furthermore any nontrivial one-parameter deformation of H_2 is of the form $H_2 + \lambda Q$ with Q in $\mathcal{Q}_{H_2}[n]$ and Q independent from H_2. Consequently we have

Theorem 8. *Let* $H_2(\lambda)$ *be a nontrivial one-parameter deformation of a linear Hamiltonian system in 1:1 resonance with Hamiltonian* $H_2(0)$ *positive definite. When* λ *passes through zero the eigenvalues pass.*

From the preceding section it is clear that when H_2 has Γ-co-dimension one then in the semisimple case $C_{\mathcal{Q}_\Gamma[n]}(H_2)$ is isomorphic to the two-torus. Consequently the eigenvalues of the deformation pass. The nonsemisimple case is the Hamiltonian Hopf bifurcation [16] and the eigenvalues split.

Theorem 9. *Let* $H_2(\lambda)$ *be a one-parameter universal deformation of a linear Hamiltonian system in 1:1 resonance with Hamiltonian* $H_2(0)$*. When* $H_2(0)$ *is semisimple the eigenvalues pass when* λ *passes through zero. When*

$H_2(0)$ is nonsemisimple the eigenvalues split when λ passes through zero.

6. Bifurcations in Nonlinear Systems with Generically Passing Eigenvalues

We will restrict ourselves to systems with two degrees of freedom, i.e. systems on $\mathbb{R}^4$, having a stationary point at the origin. We will focus on local bifurcations of periodic solutions. So let

$$H(x,y) = H_2(x,y) + F(x,y)\,,$$

where $F(x,y)$ consists of higher order terms. We assume H to be symmetric with respect to symmetry group Γ. We furthermore assume to have a universal deformation of H_2

$$H_2^\lambda = H_2 + \lambda I\,,$$

where H_2 and I are functionally independent and commute with respect to the standard Poisson structure induced on $\mathcal{Q}[4]$ by the symplectic form on $\mathbb{R}^4$. Now we are in the passing case thus the infinitesimal symplectic matrix corresponding to H_2^λ has purely imaginary eigenvalues. Because H_2 and I commute they share eigenspaces and therefore therefore both must correspond to infinitesimal symplectic matrices with purely imaginary eigenvalues. Thus both H_2 as well as I generate an S^1-action and these actions commute. So on the linear level we have $C_{\mathcal{Q}_\Gamma[4]}(H_2) = T_2$. Consequently the space of $\Gamma \times S^1$ invariant polynomials, S^1 being the S^1 generated by H_2, is generated by H_2 and I. Thus the normal form for H is

$$\bar{H} = H_2 + \bar{F}(H_2, I)\,,$$

with universal deformation

$$\bar{H}^\lambda = H_2 + \lambda I + \bar{F}(H_2, I)\,.$$

Consequently the energy-momentum map $\bar{H}^\lambda \times I$ is equivalent to $H_2^\lambda \times I$ (see also [7]). That is, the bifurcation of periodic orbits is actually determined by the linear universal deformation. We have two families of periodic solutions emanating from the origin and this case does qualitatively not differ from the nonresonant case.

Theorem 10. *Consider a (nonlinear) Hamiltonian system in 1:1 resonance with Hamiltonian H having symmetry group Γ. Suppose that in the Γ-universal unfolding of H_2 the eigenvalues generically pass. Then the normal form for the bifurcation of periodic solutions is given by the linear Γ-universal deformation* *of* H_2.

7. Examples of One Parameter Universal Deformations for Real and Complex Representations

In the following we will consider a number of examples of symmetric Hamiltonian systems in 1:1 resonance, i.e. systems for which the infinitesimal symplectic matrix A corresponding to the linearized system has multiple eigenvalues $\pm i$. We will restrict to real and complex representations of the symmetry group Γ. An example of a quaternionic representation can be found in [17].

Example 1. As an example of a representation of type $V \oplus V \oplus V \oplus V$ with V absolutely irreducible of real type we may consider the $O(2)$ symmetric Hamiltonian Hopf bifurcation as considered in [17]. This is an example of a representation of type $V \oplus V \oplus V \oplus V$ with V absolutely irreducible of real type. This case is generically nonsemisimple. The quadratic Hamiltonian is indefinite and has a one-parameter $O(2)$-universal deformation. The eigenvalues of the deformation split.

Example 2. As an example of a representation of type $V_1 \oplus V_1 \oplus V_2 \oplus V_2$ with V_1 and V_2 absolutely irreducible of real type and nonisomorphic we consider the $O(2)$ action on $\mathbb{C}^4$ given by

$$\theta \cdot (z_1, z_2, z_3, z_4) = (e^{i\theta} z_1, e^{-i\theta} z_2, e^{2i\theta} z_3, e^{-2i\theta} z_4)\,, \tag{2}$$

$$\kappa \cdot (z_1, z_2, z_3, z_4) = (z_2, z_1, z_4, z_3)\,. \tag{3}$$

The θ invariant polynomials are generated by

$$\begin{aligned} &z_1\bar{z}_1\ ,\ z_2\bar{z}_2\ ,\ z_3\bar{z}_3\ ,\ z_4\bar{z}_4\ ,\\ &z_1z_2\ ,\ \bar{z}_1\bar{z}_2\ ,\ z_3z_4\ ,\ \bar{z}_3\bar{z}_4\ ,\\ &z_1^2\bar{z}_3\ ,\ \bar{z}_1^2z_3\ ,\ z_1^2z_4\ ,\ \bar{z}_1^2\bar{z}_4\ ,\\ &z_2^2z_3\ ,\ \bar{z}_2^2\bar{z}_3\ ,\ z_2^2\bar{z}_4\ ,\ \bar{z}_2^2z_4\ . \end{aligned} \tag{4}$$

Consequently the $O(2)$ invariant polynomials are generated by

$$\begin{aligned} &z_1\bar{z}_1 + z_2\bar{z}_2\ ,\ z_3\bar{z}_3 + z_4\bar{z}_4\ ,\ z_1z_2\ ,\bar{z}_1\bar{z}_2\ ,\ z_3z_4\ ,\ \bar{z}_3\bar{z}_4\ ,\\ &z_1^2\bar{z}_3 + z_2^2\bar{z}_4,\ \bar{z}_1^2z_3 + \bar{z}_2^2z_4\ . \end{aligned} \tag{5}$$

Note that we have the relations

$$\begin{aligned} (z_1\bar{z}_1 - z_2\bar{z}_2)^2 &= (z_1\bar{z}_1 + z_2\bar{z}_2)^2 - 4(z_1z_2)(\bar{z}_1\bar{z}_2)\ ,\\ (z_3\bar{z}_3 - z_4\bar{z}_4)^2 &= (z_3\bar{z}_3 + z_4\bar{z}_4)^2 - 4(z_3z_4)(\bar{z}_3\bar{z}_4)\ ,\\ (z_1^2\bar{z}_3 - z_2^2\bar{z}_4)^2 &= (z_1^2\bar{z}_3 + z_2^2\bar{z}_4)^2 - 4(z_1z_2)^2(\bar{z}3\bar{z}_4)\ , \end{aligned}$$

$$(\bar{z}_1^2 z_3 - \bar{z}_2^2 z_4)^2 = (\bar{z}_1^2 z_3 + \bar{z}_2^2 z_4)^2 - 4(\bar{z}_1 \bar{z}_2)^2 (z_3 z_4) \ . \tag{6}$$

The only possible choices for $O(2)$ invariant quadratic Hamiltonians with eigenvalues $\pm i$ are

$$\begin{aligned} S_1(z) &= \frac{1}{2}(z_1\bar{z}_1 + z_2\bar{z}_2) + \frac{1}{2}(z_3\bar{z}_3 + z_4\bar{z}_4) \ , \\ S_2(z) &= \frac{1}{2}(z_1\bar{z}_1 + z_2\bar{z}_2) - \frac{1}{2}(z_3\bar{z}_3 + z_4\bar{z}_4) \ . \end{aligned} \tag{7}$$

Let us first consider the generators of the polynomials that commute with S_1 (under standard Poisson bracket). By abuse of language we will refer to this polynomials as S_1-invariant polynomials. They are generated by

$$\begin{gathered} z_1\bar{z}_1 \ , \ z_2\bar{z}_2 \ , \ z_3\bar{z}_3 \ , \ z_4\bar{z}_4 \ , \\ z_1\bar{z}_2 \ , \ z_1\bar{z}_3 \ , \ z_1\bar{z}_4 \ , \ z_2\bar{z}_3 \ , z_2\bar{z}_4 \ , \ z_3\bar{z}_4 \ , \\ \text{and their complex conjugates.} \end{gathered} \tag{8}$$

Consequently the $O(2)$ and S_1-invariant polynomials are generated by

$$\begin{gathered} S_1(z) \ , \ S_2(z) \ , \ Q_1(z) = z_1 z_2 \bar{z}_3 \bar{z}_4 \ , \ Q_3(z) = \bar{z}_1 \bar{z}_2 z_3 z_4 \ , \\ R(z) = (z_1^2 \bar{z}_3 + z_2^2 \bar{z}_4)(\bar{z}_1^2 z_3 + \bar{z}_2^2 z_4) \ . \end{gathered}$$

The S_2-invariant polynomials are generated by

$$\begin{gathered} z_1\bar{z}_1 \ , \ z_2\bar{z}_2 \ , \ z_3\bar{z}_3 \ , \ z_4\bar{z}_4 \ , \\ z_1\bar{z}_2 \ , \ z_1 z_3 \ , \ z_1 z_4 \ , \ z_2 z_3 \ , z_2 z_4 \ , \ z_3\bar{z}_4 \ , \\ \text{and their complex conjugates.} \end{gathered} \tag{9}$$

The $O(2)$ and S_2-invariant polynomials are generated by

$$S_1(z) \ , \ S_2(z) \ , \ R(z) \ , \ Q_2(z) = z_1 z_2 z_3 z_4 \ , \ Q_4(z) = \bar{z}_1 \bar{z}_2 \bar{z}_3 \bar{z}_4 \ .$$

Consequently an $O(2)$ invariant Hamiltonian system with

$$H(z) = S_i(z) + F(z) \quad (F(z) \text{standing for terms of degree} > 2) \, , \tag{10}$$

has an $O(2) \times S_i$-invariant normal form

$$\hat{H}(z) = S_i(z) + F(S_1, S_2, R, Q_i, Q_{i+2}) \ . \tag{11}$$

The quadratic part has a one-parameter universal deformation, which gives

$$\hat{H}(z;\lambda) = S_i(z) + \lambda S_j(z) + F(S_1, S_2, R, Q_i, Q_{i+2}) \ , \ i \neq j \ . \tag{12}$$

Note that from the above complex description one easily obtains a real system by setting $z = x + iy$. One then gets a system on $\mathbb{R}^8$ with the standard symplectic form. We get a definite or indefinite quadratic Hamiltonian and in any case passing of the purely imaginary eigenvalues for λ going through zero. About the bifurcation of the non-linear system nothing can be said at this stage.

We now turn to representations of complex type. Consider the $SO(2)$ action on $\mathbb{C}^2$ with symplectic form $Re(i\sum_{j=1}^{2} z_j\bar{w}_j)$ given by

$$\theta \cdot (z_1, z_2) = (e^{i\alpha\theta} z_1, e^{i\beta\theta} z_2) \ , \alpha \in \mathbb{N} \ , \ \beta \in \mathbb{Z} \ . \tag{13}$$

This action is a linear symplectic action which corresponds to the flow of the homogeneous quadratic Hamiltonian

$$I(z) = \frac{1}{2}\alpha z_1\bar{z}_1 + \frac{1}{2}\beta z_2\bar{z}_2 \ . \tag{14}$$

Note that actually any linear symplectic $SO(2)$ action is the flow of a linear Hamiltonian system with purely imaginary resonant eigenvalues, and consequently corresponds to a Hamiltonian of the above form when put into normal form.

The above action gives an $SO(2)$ action on each complex component. These actions are isomorphic if $\alpha = \beta$, and dual if $\alpha = -\beta$ (see [20]).

Next consider a linear Hamiltonian system with eigenvalues $\pm i$ which is $SO(2)$-equivariant, i.e. its Hamiltonian commutes with $I(z)$. Because H and I commute we may suppose both systems to be in normal form. Consequently the Hamiltonian of our system is one of the following two

$$\begin{aligned} S_1(z) &= \frac{1}{2}z_1\bar{z}_1 + \frac{1}{2}z_2\bar{z}_2 \ , \\ S_2(z) &= \frac{1}{2}z_1\bar{z}_1 - \frac{1}{2}z_2\bar{z}_2 \ . \end{aligned} \tag{15}$$

Example 3. As an example of a representation of type $W \oplus W$ with W nonabsolutely irreducible of complex type consider a Hamiltonian $H(z) = S_1(z) + F(z)$, F of degree larger than 2, and suppose $\alpha = \beta$. Then the flows of $I(z)$ and $S_1(z)$ coincide and H has normal form $\hat{H}(z) = S_1(z) + \hat{F}(z_1\bar{z}_1, z_2\bar{z}_2, z_3\bar{z}_3, z_4\bar{z}_4)$. The linear universal deformation of S_1 is a three parameter deformation (see [8], [11], [13]), consequently this is not generic in one parameter families. The nonlinear system was studied in [2] who found that is was co-dimension seven. A wellknown example of this case is the Hénon-Heiles Hamiltonian which depends on one parameter and is obviously not generic. Not even if one takes into account the additional D_3-symmetry (see [20]).

In this case the set of generators for the S_1-invariant polynomials is isomorphic to the Lie algebra $u(2)$ which contains no nilpotent elements. Therefore there is no nonsemisimple analogue of the Hamiltonian Hopf bifurcation for this case.

Example 4. As another example of a representation of type $W \oplus W$ with W nonabsolutely irreducible of complex type consider a Hamiltonian $H(z) = S_2(z) + F(z)$, where F as before contains the higher order terms, and let $I(z)$

be as in the previous case. In this case dS_2 and dI are independent (except for a set of measure zero). A normal form which commutes with $S_2(z)$ as well as with $I(z)$ is

$$\hat{H}(z) = S_2(z) + \hat{F}(S_1(z), S_2(z)) \ . \tag{16}$$

The momentum mapping $H \times I$ is equivalent to $S_2 \times I$ (see [7]) which has the one parameter universal deformation

$$(S_2 + \lambda I) \times I \ . \tag{17}$$

We have passing eigenvalues.

Example 5. As an example of a representation of type $W_1 \oplus W_2$ with W_1 and W_2 nonabsolutely irreducible of complex type and W_1 and W_2 dual consider a Hamiltonian $H(z)$ as in Example 3. but now in $I(z)$ let $\beta = -\alpha$. This case is analogous to Example 4. We get as one parameter universal deformation $S_1(z) + \lambda I(z)$. We have passing eigenvalues.

Example 6. As another example of a representation of type $W_1 \oplus W_2$ with W_1 and W_2 nonabsolutely irreducible of complex type and W_1 and W_2 dual consider a Hamiltonian $H(z)$ as in Example 4. As in Example 3. the semisimple case is not generic in one parameter families. In this case however we may consider the nonsemisimple quadratic Hamiltonian

$$H_2(z) = S_2(z) + N_2(z) \ , \ N_2(z) = S_1(z) - \frac{1}{2} i(z_1 z_2 - \bar{z}_1 \bar{z}_2) \ . \tag{18}$$

This is the $SO(2)$ symmetric Hamiltonian Hopf bifurcation considered in [17]. Note that this case is qualitatively equivalent to the case of zero eigenvalues which is clear from the standard form obtained in [16]. We have splitting eigenvalues (a Hamiltonian Hopf bifurcation).

Example 7. As an example of a representation of type $W_1 \oplus W_2$ with W_1 and W_2 nonabsolutely irreducible of complex type and W_1 and W_2 nondual and nonisomorphic consider a Hamiltonian $H(z) = S_i(z) + F(z)$ and let $\alpha \neq |\beta|$ in $I(z)$. This case is analogous to Example 4. One obtains the one parameter universal deformation $S_i(z) + \lambda I(z)$.

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