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SHAPE SENSITIVITY ANALYSIS FOR WATERSLIDES IN CAD FLUME SECTIONS

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Abstract: This paper presents a shape design sensitivity analysis (DSA) method for recreational waterslides with flume sections represented in computer-aided design (CAD) environment. Key dimensions of the flume sections and the overall waterslide configurations are identified as design variables. A set of differential equations based on Lagrange's equation of motion that describe the motion of the riding object are derived. These coupled ordinary differential equations are solved numerically using *Mathematica*. In addition, analytical shape DSA method has been developed by taking the derivatives of performance of the riding object with respect to CAD design variables. The DSA expressions are stated in coupled differential equations. These equations are also solved using *Mathematica*. A real-world waterslide configuration is presented to demonstrate the feasibility and accuracy of the proposed method.

AMS Subject Classification: 70-02
Key Words: CAD, differential equations, waterslides, shape DSA

1. Introduction

Applying advanced computer-aided technology and software tools for design of recreational waterslides in theme parks has gained attention of industry. This is mainly due to the increasing competitiveness in the recreational equipment industry and stricter safety requirements from the public and government. An integrated modeling, analysis, and design method for waterslide safety has been developed [2], [3], in which the geometric shape of the waterslide was represented

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in B-spline surfaces [3]. By moving the control points of the B-spline surfaces, the geometric shape of the waterslide is changed; therefore, altering the path, velocity and acceleration of the riding object. However, due to the nature of the B-spline representation, minor local changes in geometry resulting from the movement of the control points individualizes the geometry of flume sections; consequently, posing difficulties in manufacturing. In this paper flume sections are constructed in computer-aided design (CAD) tools to alleviate the manufacturing issue. In addition, friction forces are included into motion analysis. Shape DSA formulations are derived and implemented. With friction forces in place, the accelerations of the riding object and the overall riding times are more realistic. The realistic acceleration supports the design trade-off that would include rider's excitement level in addition to safety measure.

2. Mathematical Representations of Basic Flume Sections

Basic flume sections, such as the straight, elbow, and curved, are developed to serve as the building blocks for composing waterslide configurations. Geometry of all sections is expressed in parametric surface forms in terms of the parametric coordinates u and w, using CAD geometric dimensions. The overall waterslide configuration can be expressed mathematically as

$$\overline{\mathbf{X}}(u,w) = \sum_{i}^{N} \mathbf{X}^i(u^i, w^i)\,, \tag{1}$$

where $\mathbf{X}^i(u^i, w^i)$ is the parametric equation of the i-th flume section, and N is the total number of sections. For each flume section (with superscript i removed),

$$\mathbf{X}(u,w) = [X_1(u,w), X_2(u,w), X_3(u,w)]^T\,, \tag{2}$$

where $X_j(u,w)$ is the j-th coordinate of any given point on the surface with prescribed parameters *(u,w)*. Note that these sections will be translated and oriented for an overall waterslide configuration by the following equation:

$$\mathbf{X}(u,w) = \mathbf{X}_0 + \mathbf{T}(\theta)\mathbf{x}(u,w) = \begin{bmatrix} X_{0_1} \\ X_{0_2} \\ X_{0_3} \end{bmatrix} + \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x_1(u,w) \\ x_2(u,w) \\ x_3(u,w) \end{bmatrix}\,, \tag{3}$$

where $\mathbf{T}(\theta)$ is the rotational matrix that orients the section by rotating through angle θ about the X_2 axis, X_{0_i} is the location of the local coordinate system of the section in the waterslide configuration, $\boldsymbol{x}$*(u,w)* is the surface function of the section referring to its local coordinate system.

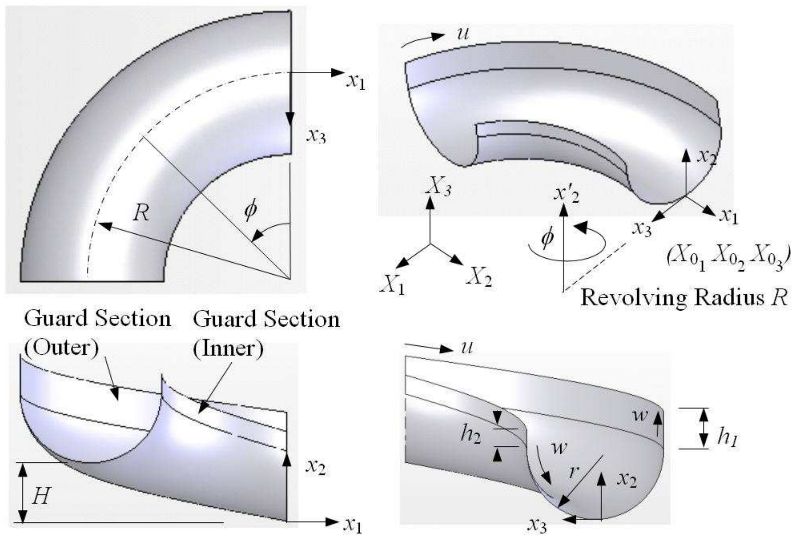

Figure 1: The elbow flume section

The mathematical representation of a flume section, for example, the elbow section shown in Figure 1, can be expressed using equation 3, where

$$\mathbf{x}(u,w) = [\cos u\phi(-R + r\cos w\pi), r(1-\sin w\pi) + H(1-u), \\ \sin u\phi(-R + r\cos w\pi + R]^T; \quad u, w \in [0,1]\,. \quad (4)$$

Note that in equation (4), R is the revolving radius, H is the height, and ϕ is the revolving angle measured along the x_2'-axis, which is offset along the x_2-axis by amount R. An example of the elbow section is shown in Figure 1. In order to ensure the safety of the riding object, guard sections are often added to both sides of the elbow section.

3. Equations of Motion

The well-known Lagrange's equation of motion based on Hamilton's principle for particle dynamics can be stated as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{Q}\,, \qquad (5)$$

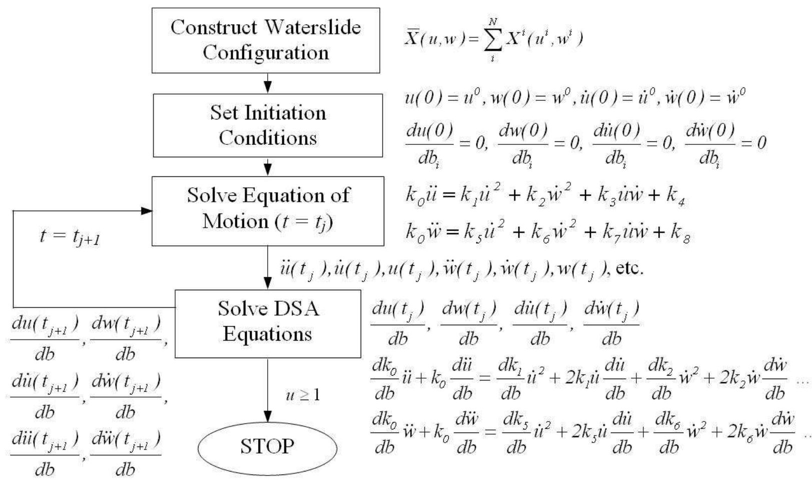

Figure 2: Computation flow for solving motion and DSA equations

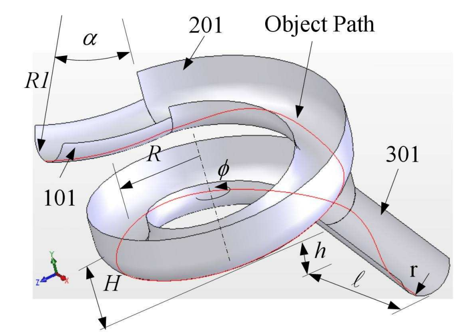

Figure 3: Object path and design variables

where the Lagrangian function L is defined as $L \equiv T - V, \dot{\mathbf{q}} = \partial \mathbf{q}/\partial t$, and the generalized coordinates $\boldsymbol{q}$ in this waterslide application are the parametric coordinates of the surface; i.e., $\mathbf{q} = [u, w]^T$. Also, $\mathbf{Q} = \mathbf{F}$, where $\mathbf{F}$ is the vector of generalized friction forces. For motion analysis, the kinetic energy T and potential energy U are, respectively,

$$T = \frac{m}{2}\|\dot{\mathbf{X}}(u, w)\|^2, \text{ and } U = mgX_2(u, w)\,, \tag{6}$$

where m is the particle mass and g is the gravitational acceleration. For the friction cases, the generalized friction forces $\mathbf{Q} = [f_u, f_w]^T$ can be derived as [4]

$$f_u = -\mu(\mathbf{g} + \mathbf{a}_n) \cdot \mathbf{n}(\mathbf{e}_t \cdot \mathbf{X}_{,u}), \text{ and } f_w = -\mu(\mathbf{g} + \mathbf{a}_n) \cdot \mathbf{n}(\mathbf{e}_t \cdot \mathbf{X}_{,w})\,, \tag{7}$$

where μ is the friction coefficient; $\boldsymbol{n}$ is the unit normal surface vector; $\mathbf{a}_n$ is the normal acceleration of the riding object; and $\mathbf{e}_t$ is the unit vector along the tangential direction of the object's path.

Following equation (5), two coupled second order ordinary differential equations that govern the particle motion can be obtained a

$$k_0\ddot{u} = k_1\dot{u}^2 + k_2\dot{w}^2 + k_3\dot{u}\dot{w} + k_4\,, \tag{8a}$$

$$k_0\ddot{w} = k_5\dot{u}^2 + k_6\dot{w}^2 + k_7\ddot{u}\dot{w} + k_8\,, \tag{8b}$$

where k_0 to k_8 consist of polynomials of u and w and their products. The initial conditions, including initial position and velocity of the riding object, must be provided in order to solve the equations of motion; i.e.,

$$u(0) = u^0, \quad w(0) = w^0, \quad \dot{u}(0) = \dot{u}^0, \text{ and } \dot{w} = \dot{w}^0\,. \tag{9}$$

The system of ordinary differential equations can be solved numerically for positions $u(t)$ and $w(t)$, velocities $\dot{u}(t)$ and $\dot{w}(t)$, and accelerations $\ddot{u}(t)$ and $\ddot{w}(t)$, of the riding object using, for example, *Mathematica*.

4. Shape Design Sensitivity Analysis

The sensitivity of the position of the riding object can be obtained by taking the total derivatives of equation (3) with respect to a design variable b_i; i.e.,

$$\frac{d\mathbf{X}(u,w)}{db_i} = \frac{\partial \mathbf{X}_0}{\partial b_i} + \frac{\partial \mathbf{T}(\theta)}{\partial b_i}\mathbf{x}(u,w) + \mathbf{T}(\theta)\frac{d\mathbf{x}(u,w)}{db_i}\,, \tag{10}$$

where

$$\frac{d\mathbf{x}(u,w)}{db_i} = \frac{\partial \mathbf{x}}{\partial u}\frac{du}{db_i} + \frac{\partial \mathbf{x}}{\partial w}\frac{dw}{db_i} + \frac{\partial \mathbf{x}}{\partial b_i}\,. \tag{11}$$

Note that the position of the riding object depends on the shape design variable in two ways. The first dependence is explicit, in which the geometric shape of the flume section varies due to design changes explicitly; i.e., $\partial\mathbf{x}/\partial b_i$ The second dependence is implicit, in which variations of the object path, velocity, and acceleration are varying due to variations of *u(t)* and *w(t)*; i.e., du/db_i and dw/db_i. The variations of *u(t)* and *w(t)* are determined by the equations of motion for the varied waterslide geometry.

By taking derivatives of equations (8a) and (8b) with respect to the design variable b_i, sensitivity equations of the motion problem can be obtained as follows

$$\frac{dk_0}{db_i}\ddot{u} + k_0\frac{d\ddot{u}}{db_i} = \frac{dk_1}{db_i}\dot{u}^2 + 2k_1\dot{u}\frac{d\dot{u}}{db_i} + \frac{dk_2}{db_i}\dot{w}^2 + 2k_2\dot{w}\frac{d\dot{w}}{db_i}$$

$$+\frac{dk_3}{db_i}\dot{u}\dot{w}+k_3\frac{d\dot{u}}{db_i}\dot{w}+k_3\dot{u}\frac{d\dot{w}}{db_i}+\frac{dk_4}{db_i}\,, \tag{12a}$$

$$\frac{dk_0}{db_i}\ddot{w}+k_0\frac{d\ddot{w}}{db_i}=\frac{dk_5}{db_i}\dot{u}^2+2k_5\dot{u}\frac{d\dot{u}}{db_i}+\frac{dk_6}{db_i}\dot{w}^2+2k_6\dot{w}\frac{d\dot{w}}{db_i}+\frac{dk_7}{db_i}\dot{u}\dot{w}+k_7\frac{d\dot{u}}{db_i}\dot{w}+k_7\dot{u}\frac{d\dot{w}}{db_i}+\frac{dk_8}{db_i}\,, \tag{12b}$$

where the expressions of individual terms can be found in [2].

Note that the design derivatives must be solved from equations (12a) and (12b) in order to calculate sensitivity coefficients of position, velocity and acceleration of the riding object. The initial conditions of the shape DSA equations for the waterslide configuration are defined as follows

$$\frac{du(0)}{db_i}=0,\quad \frac{dw(0)}{db_i}=0,\quad \frac{d\dot{u}(0)}{db_i}=0,\quad \text{and}\quad \frac{d\dot{w}(0)}{db_i}=0\,. \tag{13}$$

These conditions imply that the initial position and velocity of the riding object are assumed unchanged before and after design variations, which is physically meaningful. Note that *Mathematica* has been employed, specifically the *NDSolve* function, for solving the equations of motion (equation (8)) and DSA expressions (equation (12)). The computation flow is illustrated in Figure 2.

5. Numerical Example

The waterslide configuration shown in Figure 3 is employed to demonstrate the feasibility of the proposed DSA method. The friction coefficient is assumed μ=0.075. Eight design variables are identified. They are the radius of the cross section of all flume sections r; length ℓ and height h of the straight section (301); revolving radius R, angle ϕ and height H of the elbow section (201); and radius *R1* and sweep angle α of the curved section (101).

Shape DSA is carried out and verified, for example for design variable H. In Table 1, column A shows the flume section ids and time steps. Columns B and C list analysis results, i.e., acceleration in X_1-direction (Xtt) and the magnitude of the acceleration (Acc), respectively, at selected time steps. Columns D and E show the same information at the perturbed design, obtained from solving the equations of motion. In this example, the design perturbation is 0.1%. Therefore, for Table 1, the perturbation is $\delta H = 0.1\%, H = 0.75$ inch. Columns F and G show the differences of the analysis results due to design change, obtained using finite difference method; i.e., Column F = Column D – Column B, etc. Columns H and I list the sensitivity coefficients obtained from the shape

A	B	C	D	E	F	G	H	I	J	K
Time	Xtt(b)	Acc(b)	Xtt(b+db)	Acc(b+db)	del Xtt	del Acc	DSA Xtt	DSA Acc	%Xtt	%Acc
Section 101										
0.52	-2.53E-13	1.04E+02	-2.53E-13	1.04E+02	0.00E+00	1.42E-14	0.00E+00	0.00E+00	0.0	0.0
1.29	1.46E-12	5.57E+01	1.46E-12	5.57E+01	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.0	0.0
Section 201										
0.20	6.85E+01	8.00E+01	6.85E+01	8.00E+01	2.02E-02	2.59E-02	2.69E-01	3.45E-01	99.9	99.9
0.40	1.64E+02	1.81E+02	1.64E+02	1.81E+02	5.41E-02	4.47E-02	7.22E-01	5.96E-01	100.0	100.0
1.00	5.29E+01	1.19E+02	5.30E+01	1.19E+02	4.93E-02	1.09E-01	6.57E-01	1.45E+00	99.9	99.9
1.50	-6.20E+01	1.78E+02	-6.21E+01	1.78E+02	-1.36E-01	1.99E-01	-1.82E+00	2.66E+00	100.2	100.2
4.00	2.93E+02	3.01E+02	2.93E+02	3.02E+02	5.50E-01	4.15E-01	7.40E+00	5.59E+00	100.9	101.0
Section 301										
0.70	-6.41E+00	2.21E+02	-6.42E+00	2.21E+02	-9.64E-03	4.02E-01	-1.29E-01	5.37E+00	100.2	100.1
0.80	-6.05E+00	2.01E+02	-6.06E+00	2.02E+02	-1.12E-02	3.93E-01	-1.49E-01	5.28E+00	99.9	100.9

Table 1: Verification of acceleration DSA wrt height of elbow section H

DSA. The last two columns (J and K) show the accuracy of the proposed shape DSA method by comparing data in Columns H and I (multiplied with design perturbation) with those of Columns F and G, respectively. It is shown in Columns J and K that the sensitivity coefficients are accurate comparing with the finite difference results.

6. Conclusions and Future Research

In this paper, shape DSA for design of waterslides has been presented. Mathematical representations of commonly employed sections have been created in parametric surfaces, which are compatible to and can be directly implemented into CAD. The feasibility and accuracy of the proposed DSA method has been demonstrated using a real-world waterslide configuration. A gradient-based design optimization will naturally be the next step, especially for design problems with large amount of design variables.

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