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FIBONACCI-LIKE POLYNOMIALS: COMPUTATIONAL EXPERIMENTS, PROOFS, AND CONJECTURES

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Abstract: This article introduces a new class of polynomials arising in a course of exploring a qualitative behavior of orbits of a two-parametric difference equation. The use of symbolic computations and computational experiments in the context of *Maple* made it possible to prove polynomial generalizations of Cassini's identity for Fibonacci numbers and formulate conjectures about polynomial forms of Catalan's identity.

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1. Introduction of Fibonacci-Like Polynomials

Several authors [7], [4], [9], [3] refer to polynomials $F_n(x)$ defined recursively

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$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x); \quad F_0(x) = 1, \quad F_1(x) = x\,, \tag{1}$$

as Fibonacci polynomials. The following are a few polynomials defined by formula (1)

$$F_0(x) = 1,\ F_1(x) = x,\ F_2(x) = x^2 + 1,\ F_3(x) = x^3 + 2x,$$
$$F_4(x) = x^4 + 3x^2 + 1;\ F_5(x) = x^5 + 4x^3 + 3x.$$

One can note that polynomials $F_n(x)$ assume values of Fibonacci numbers F_n at $x = 1$, alternate the properties of even and odd functions, polynomials of even degree do not have real roots, and zero is the only real root of odd degree polynomials.

In this paper, a new class of polynomials will be introduced. To this end, consider two polynomials of degree n in variable x

$$P_{2n-1}(x) = \sum_{i=0}^{n} C_{2n-i}^{i} x^{n-i} \tag{2}$$

and

$$P_{2n}(x) = \sum_{i=0}^{n} C_{2n-i+1}^{i} x^{n-i}. \tag{3}$$

The following two propositions establish recursive relationships that polynomials (2) and (3) satisfy.

Proposition 1. *For all* $n = 1, 2, 3, 4, \ldots$

$$P_{2n}(x) = P_{2n-1}(x) + P_{2n-2}(x)\,. \tag{4}$$

Proof. To prove identity (4), it can be first re-written in the form

$$\sum_{i=0}^{n} C_{2n-i+1}^{i} x^{n-i} = \sum_{i=0}^{n} C_{2n-i}^{i} x^{n-i} + \sum_{i=0}^{n} C_{2n-i-1}^{i} x^{n-i}.$$

Expanding the sums and applying the identity $C_m^k = C_{m-1}^k + C_{m-1}^{k-1}$ yields

$$C_{2n+1}^0 x^n + (C_{2n-1}^1 + C_{2n-1}^0)x^{n-1} + (C_{2n-2}^2 + C_{2n-2}^1)x^{n-2} + \ldots + C_n^n + C_n^{n-1}$$
$$= C_{2n}^0 x^n + (C_{2n-1}^1 + C_{2n-1}^0)x^{n-1} + (C_{2n-2}^2 + C_{2n-2}^1)x^{n-2} + \ldots + C_n^n + C_n^{n-1}\,.$$

Noting that $C_{2n+1}^0 = C_{2n}^0$, competes the proof of identity (4). □

Proposition 2. *For all* $n = 1, 2, 3, \ldots$

$$P_{2n+1}(x) = xP_{2n}(x) + P_{2n-1}(x). \tag{5}$$

Proof. In order to prove identity (5), one has to show that

$$xP_{2n}(x)+P_{2n-1}(x)=x\sum_{i=0}^{n}C_{2n-i+1}^{i}x^{n-i}+\sum_{i=0}^{n}C_{2n-1}^{i}x^{n-i}$$
$$=\sum_{i=0}^{n}(C_{2n-i+1}^{i}x^{n-i+1}+C_{2n-i}^{i}x^{n-i})=\sum_{i=0}^{n+1}C_{2n-i+1}^{i}x^{n-i+1}=P_{2n+1}(x).$$

Indeed,

$$C_{2n+1}^{0}x^{n+1}+C_{2n}^{1}x^{n}+C_{2n-1}^{2}x^{n-1}+\ldots+C_{n}^{n}x$$
$$+C_{2n}^{0}x^{n}+C_{2n-1}^{1}x^{n-!}+C_{2n-2}^{2}x^{n-2}+\ldots+C_{n}^{n}$$
$$=C_{2n+1}^{0}x^{n+1}+(C_{2n}^{1}+C_{2n}^{0})x^{n}+(C_{2n-1}^{2}+C_{2n-1}^{1})x^{n-1}+\ldots+(C_{n+1}^{n}+C_{n+1}^{n-1})xC_{n}^{n}$$
$$=C_{2n+1}^{0}x^{n+1}+C_{2n+1}^{1}x^{n}+C_{2n}^{2}x^{n-1}+\ldots+C_{n+2}^{n}x+C_{n}^{n}$$
$$=C_{2n+2}^{0}x^{n+1}+C_{2n+2-1}^{1}x^{n+1-1}+C_{2n+2-2}^{2}x^{n+1-2}+\ldots+C_{2n+2-n}^{n}x^{n+1-n}$$
$$+C_{2n+2-(n-1)}^{n+1}=\sum_{i=0}^{n+1}C_{2n-i+2}x^{n-i+1}.$$

This completes the proof of identity (5). □

Remark 1. Identity (4) does not hold and identity (5) holds for Fibonacci polynomials $F_n(x)$ defined by formula (1).

Now, noting that $P_0(x)=1$ and defining $P_{-1}(x)=1$, one unite relations (2) and (3) using the following single recursive formula

$$P_n(x)=x^{\bmod (n,2)}P_{n-1}(x)+P_{n-2}(x);\quad P_0(x)=1,\quad P_1(x)=x+1\,. \tag{6}$$

Here, the $\bmod (n,2)$ notation represents a function that returns a remainder of n divided by 2; i.e., in the case of integer n, the function returns either 0 or 1. The following are a few polynomials defined by formula (6)

$$P_0(x)=1,\quad P_1(x)=x+1,\quad P_2(x)=x+2,\quad P_3(x)=x^2+3x+1,$$
$$P_4(x)=x^2+4x+3;\quad P_5(x)=x^3+5x^2+6x+1.$$

One can see that just like polynomials $F_n(x)$, polynomials $P_n(x)$ assume values of Fibonacci numbers at $x=1$. However, whereas polynomials $F_n(x)$ can be put into one-to-one correspondence with their degree n, this is not the case for polynomials $P_n(x)$. Polynomials $P_n(x)$ will be referred to below as Fibonacci-like polynomials.

As was shown elsewhere [1], this new class of polynomials arises in a course of exploring a qualitative behavior of orbits of a two-parametric non-linear difference equation $f_{k+1}=af_k+bf_{k-1}, f_0=f_1=1$, where a and b are real parameters [6], [5]. Fibonacci-like polynomials possess many interesting properties that can be discovered through computational experiments. In what follows, using Fibonacci-like polynomials, the celebrated Cassini's and Catalan's identities for Fibonacci numbers [8] are generalized to polynomial forms. The use of computer algebra system *Maple* [2] made it possible to provide mathematical induction proof of the polynomial forms of Cassini's identity and formulate conjectures about polynomial forms of Catalan's identity.

2. Polynomial Generalizations of Cassini's Identity

Let $R(n,x)$ =const be a statement to be proved by mathematical induction for all $n = 1, 2, 3, \ldots$. If the statement $R(1,x)$ =const is true and as a way of testing the transition from n to $n+1$ the simplification of the difference $R(n,x)-R(n+1,x)$ yields zero, one can conclude that the statement $R(n,x)$ =const holds true for all $n = 1, 2, 3, \ldots$. This approach was used in proving with the help of *Maple* the following three propositions.

Proposition 3. *For all* $n = 0, 1, 2, 3, \ldots$

$$P_{2n+1}(x)P_{2n-1}(x) - x(P_{2n}(x))^2 = 1. \tag{7}$$

Proof. To begin note that when $n = 0$ relation (7) is true. Indeed,

$$P_1(x)P_{-1}(x) - x(P_0(x))^2 = (x+1) - x = 1.$$

To test the transition from n to $n+1$ the following *Maple* code can be used.

$P1 := (x,n) \to sum(binomial(2 \cdot n - i + 1, i) \cdot x^{n-i}, i = 0, \ldots, n)$

$P2 := (x,n) \to sum(binomial(2 \cdot n - i, i) \cdot x^{n-i}, i = 0, \ldots, n)$

$P3 := (x,n) \to sum(binomial(2 \cdot n + 2 - i, i) \cdot x^{n+1-i}, i = 0, \ldots, n+1)$

$stepcur := P2(x,n) \cdot P3(x,n) - x \cdot (P1(x,n))^2 :$

$stepnext := P2(x,n+1) \cdot P3(x,n+1) - x \cdot (P1(x,n+1))^2 :$

$dd := (stepcur - stepnext);$

$simplify(dd)$

The last command yields zero. This completes the proof. □

Remark 2. Identity (7) does not hold for Fibonacci polynomials $F_n(x)$ defined by formula (1). Instead, $F_{2n+1}(x)F_{2n-1}(x) - (F_{2n}(x))^2 = -1$.

Corollary 1. *For all* $n = 0, 1, 2, 3, \ldots$ *Fibonacci numbers* F_n *satisfy the equality*

$$F_{2n+2}F_{2n} - (F_{2n+1})^2 = 1.$$

Proof. The equality follows from Proposition 3 and the fact that $P_n(1) = F_{n+1}$ for all $n = -1, 0, 1, 2, \ldots$. □

Proposition 4. *For all* $n = 1, 2, 3, \ldots$

$$xP_{2n}(x)P_{2n-2}(x) - (P_{2n-1}(x))^2 = -1. \tag{8}$$

Proof. To begin note that when $n = 0$ relation (8) is true. Indeed,

$$xP_2(x)P_0(x) - (P_1(x))^2 = x(x+2) - (x+1)^2 = -1.$$

To test the transition from n to $n+1$ the following *Maple* code can be used.

$P1 := (x, n) \to sum(binomial(2 \cdot n - i + 1, i) \cdot x^{n-i}, i = 0, \ldots, n)$

$P2 := (x, n) \to sum(binomial(2 \cdot n - i, i) \cdot x^{n-i}, i = 0, \ldots, n)$

$P4 := (x, n) \to sum(binomial(2 \cdot n - 1 - i, i) \cdot x^{n-1-i}, i = 0, \ldots, n-1)$

$stepcur := x \cdot P2(x, n) \cdot P4(x, n) - (P1(x, n))^2 :$

$stepnext := x \cdot P2(x, n+1) \cdot P4(x, n+1) - (P1(x, n+1))^2 :$

$dd := (stepcur - stepnext);$

$simplify(dd)$

The last command yields zero. This completes the proof. □

Remark 3. Identity (8) does not hold for Fibonacci polynomials $F_n(x)$ defined by formula (1). Instead, $F_{2n}(x)F_{2n-2}(x) - (F_{2n-1}(x))^2 = 1$.

Corollary 2. *For all* $n = 0, 1, 2, 3, \ldots$ *Fibonacci numbers* F_n *satisfy the identity* $F_{2n+1}F_{2n-1} - (F_{2n})^2 = -1$.

Proof. The equality follows from Proposition 4 and the fact that $P_n(1) = F_{n+1}$ for all $n = -1, 0, 1, 2, \ldots$. □

Corollary 3. *For all* $k = 1, 2, 3, \ldots$

$$x^{\bmod (k+1,2)} P_k(x)P_{k-2}(x) - x^{\bmod (k,2)} (P_{k-1}(x))^2 = (-1)^{k+1}.$$

Corollary 4. (Cassini's Identity) *For all* $n = 1, 2, 3, \ldots$ *Fibonacci numbers* F_n *satisfy the identity*

$$F_{n+1}F_{n-1} - (F_n)^2 = (-1)^{n+1}.$$

Proof. The identity follows from Corollaries 1 and 2. □

Proposition 5. *For all* $n = 1, 2, 3, \ldots$

$$(P_2(x))^2 - P_{2n-2}(x)P_{2n+2}(x) = 1. \tag{9}$$

Proof. When $n = 1$ relation (9) is true. Indeed,

$$(P_2(x))^2 - P_0(x)P_4(x) = (x+2)^2 - (x^2 + 4x + 3) = 1.$$

To test the transition from n to $n+1$ the following *Maple* code can be used.

$P1 := (x,n) \to sum(binomial(2\cdot n - i + 1, i)\cdot x^{n-i}, i = 0, \ldots, n)$

$P4 := (x,n) \to sum(binomial(2\cdot n - 1 - i, i)\cdot x^{n-1-i}, i = 0, \ldots, n-1)$

$P5 := (x,n) \to sum(binomial(2\cdot n - i + 3, i)\cdot x^{n+1-i}, i = 0, \ldots, n+1)$

$stepcur := (P1(x,n))^2 - P4(x,n)\cdot P5(x,n):$

$stepnext := (P1(x,n+1))^2 - P4(x,n+1)\cdot P5(x,n+1):$

$dd := (stepcur - stepnext);$

$simplify(dd)$

The last command yields zero. This completes the proof. □

Corollary 5. *For all $n = 1, 2, 3, \ldots$ Fibonacci numbers F_n satisfy the identity $(F_{2n+1})^2 - F_{2n-1}F_{2n+3} = 1$.*

Proof. Substituting $x = 1$ in (9) and using the relationship $P_n(1) = F_{n+1}$ completes the proof. □

3. Conjecturing Polynomial Forms of Catalan's Identity

Propositions formulated in this section were motivated by the use of *Maple*. They represent the polynomial generalizations of Catalan's identity (see Corollary 9 below). Yet, they appear to be beyond the *Maple*'s computational capability to provide a symbolic proof in the case of double mathematical induction required in the case of such generalizations. The use of the software, however, made it possible to verify the results for all $x \in R$ and for different (sufficiently large) values of two integer variables involved. In that way, the polynomials generalizations of Catalan's identity offered below remain technology-motivated conjectures.

Proposition 6. *For all $n = 1, 2, 3, \ldots$; $m = 1, 2, 3, \ldots$; $n \geq m$ $(P_{2n}(x))^2 - P_{2(n-m)}(x)P_{2(n+m)}(x) = (P_{2(m-1)}(x))^2$.*

Corollary 6. *For all $n = 1, 2, 3, \ldots$, $m = 1, 2, 3, \ldots$, $n \geq m$, the following identity among Fibonacci numbers*

$$(F_{2n+1})^2 - F_{2n+1-2m}F_{2n+1+2m} = (F_{2m-1})^2 \tag{10}$$

holds true. Here $F_0 = F_1 = 1$.

Proposition 7. *For all $n = 1, 2, 3, \ldots$; $m = 1, 2, 3, \ldots$; $n \geq m$, $(P_{2n+1}(x))^2 - P_{2n+1-2m}(x)P_{2n+1+2m}(x) = -x(P_{2m-2}(x))^2$.*

Corollary 7. *For all* $n = 1, 2, 3, \ldots$, $m = 1, 2, 3, \ldots$, $n \geq m$, *the following identity among Fibonacci numbers*

$$(F_{2n})^2 - F_{2n-2m}F_{2n+2m} = -(F_{2m-1})^2 \tag{11}$$

holds true. Here $F_0 = F_1 = 1$.

Proposition 8. *For all* $n = 1, 2, 3, \ldots$, $m = 1, 2, 3, \ldots$, $n \geq m$,

$$x(P_{2n}(x))^2 - P_{2n-(2m+1)}(x)P_{2n+(2m+1)}(x) = -(P_{2m-1}(x))^2. \tag{12}$$

Proposition 9. *For all* $n = 1, 2, 3, \ldots$; $m = 1, 2, 3, \ldots$; $n \geq m$,

$$(P_{2n+1}(x))^2 - xP_{2n+1-(2m+1)}(x)P_{2n+1+2m+1}(x) = (P_{2m-1}(x))^2. \tag{13}$$

Corollary 8. *For all* $n = 1, 2, 3, \ldots$, $m = 1, 2, 3, \ldots$, $n \geq m$, *the following identities among Fibonacci numbers*

$$(F_{2n+1})^2 - F_{2n+1-(2m+1)}F_{2n+1+2m+1} = -(F_{2m})^2 \tag{14}$$

$$(F_{2n})^2 - F_{2n-(2m+1)}F_{2n+2m+1} = (F_{2m})^2 \tag{15}$$

hold true.

Proof. Substituting $x = 1$ in formulas (12) and (13) and using Proposition 2 yield identities (14) and (15). □

Corollary 9. *Identities (10), (11), (14), and (15) are special cases of Catalan's identity*

$$(F_k)^2 - F_{k+r}F_{k-r} = (-1)^{k-r+1}(F_{r-1})^2$$

for all values of k *and even values of* r. *Here* F_k *is a Fibonacci number and* $F_0 = F_1 = 1$.

Proof. The case $k = 2n + 1$, $r = 2m$ yields identity (10); the case $k = 2n$, $r = 2m$ yields identity (11); the case $k = 2n + 1$, $r = 2m + 1$ yields identity (14); the case $k = 2n$, $r = 2m + 1$ yields identity (15). □

4. Concluding Remarks

In conclusion, the authors would like to formulate a different kind of conjecture about Fibonacci-like polynomials. It deals with the number of roots of the polynomials. Motivated by *Maple*-enabled computational experiments, the following conjecture can be formulated: *Every Fibonacci-like polynomial of degree n has exactly n different roots, all located in the interval (-4, 0).* This conjecture was verified for all $n \leq 100$.

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