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STABILITY OF CUBIC AND ADDITIVE FUNCTIONAL EQUATION IN QUASI-BANACH SPACES

K. Ravi[1 §], R. Kodandan[2], P. Narasimman[3]

[1]Department of Mathematics
Sacred Heart College
Tirupattur, 635 601, Tamil Nadu, INDIA
e-mail: shckravi@yahoo.co.in

[2]Department of Mathematics
Srinivasa Institute of Technology and Management Studies
Chittoor, 517 127, Andhra Pradesh, INDIA
e-mail: Rkodandan1979@rediff.co.in

[3]Department of Mathematics
Priyadarshini Engineering College
Vaniyambadi, 635 751, Tamil Nadu, INDIA
e-mail: simma_gtec@yahoo.co.in

Abstract: In this paper, we introduce and investigate the general solution of a new cubic and additive functional equation

$$3f(x+y+z)+f(-x+y+z)+f(x-y+z)+f(x+y-z)+4[f(x)+f(y)+f(z)] = 4[f(x+y)+f(x+z)+f(y+z)]$$

and discuss its Hyers-Ulam-Rassias and J.M. Rassias stability in quasi-Banach spaces.

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1. Introduction

In 1940, S.M. Ulam [27] proposed a number of important unsolved problems.

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§Correspondence author

Among those was the question concerning the stability of homomorphisms:

Let G_1 *be a group and let* G_2 *be a metric group with metric d(.,.). Given* $\epsilon > 0$ *does there exists a* $\delta > 0$ *such that if a function* $f : G_1 \to G_2$ *satisfies the inequality* $d(f(xy), f(x)f(y)) < \delta$ *for all* $x, y \in G_1$*, then there is a homomorphism* $H : G_1 \to G_2$ *with* $d(f(x), H(x)) < \epsilon$ *for all* $x \in G_1$*?*

In 1941, D.H. Hyers [8] answered Ulam's problem for the case of approximately additive functions under the assumption that G_1 and G_2 are Banach spaces. The result of Hyers was further generalized by Th.M. Rassias [20]. Th.M. Rassias permitted the Cauchy difference to become unbounded. He proved the following theorem by using a direct method.

Theorem 1.1. (see [20]) *If a function* $f : G_1 \to G_2$ *between Banach spaces satisfies inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta\,[||x||^p + ||y||^p]$$

for some $\theta \ge 0$, $0 \le p < 1$ *and for all* $x, y \in G_1$*, then there exists a unique additive function* $a : G_1 \to G_2$ *such that*

$$\|f(x) - a(x)\| \le \frac{2\theta}{2 - 2^p}||x||^p$$

for any $x \in G_1$*. Moreover, it* $f(tx)$ *is continuous in* t *for each fixed* $x \in G_1$ *then* a *is linear.*

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

Therefore the functional equation (1.1) is called the quadratic functional equation or the Euler-Lagrange functional equation. Note that, every solution of the quadratic functional equation (1.1) is called a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function $f(x) = B(x, x)$ for all x (see [1, 15]). The biadditive function B is given by

$$B(x, y) = \frac{1}{4}[f(x+y) - f(x-y)]. \tag{1.2}$$

Stability problem for the quadratic functional equation (1.1) was solved by many authors [2, 5, 6, 11]. Jun and Lee [12] proved the Hyers-Ulam-Rassias stability of the Pexiderized quadratic functional equation. The Hyers-Ulam-Rassias stability of a new quadratic functional equation

$$\begin{aligned} &f(x+y+z+w) + 2f(x) + 2f(y) + 2f(z) + 2f(w) \\ &\quad = f(x+y) + f(y+z) + f(z+x) + f(x+w) + f(y+w) + f(z+w) \end{aligned} \tag{1.3}$$

was investigated by I.S. Chang, E.M. Lee, H.M. Kim [4]. K.W. Jun and H.M.

Kim [13] investigated the generalized Hyers-Ulam stability problem for the quadratic and additive functional equation of the type

$$f(x+ay)+af(x-y)=f(x-ay)+af(x+y) \tag{1.4}$$

for any integer a with $a \neq -1,0,1$. The generalized quadratic and additive type functional equation of the form

$$f\left(\sum_{i=1}^{n} x_i\right)+(n-2)\sum_{i=1}^{n} f(x_i)=\sum_{i<j\leq n} f(x_i+y_j),(n>2) \tag{1.5}$$

was discussed by K.W. Jun and H.M. Kim [14] and studied its generalized Hyers-Ulam stability for the even or odd case in the n variabiles.

A. Najati [17] introduced a new cubic function

$$f(mx+y)+f(mx-y)=mf(x+y)+mf(x-y)+2(m^3-m)f(x)\,, \tag{1.6}$$

where m is a positive integer, $m \geq 2$, and studied its generalized Hyers-Ulam-Rassias stability.

Recently A. Najati, M.B. Moghimi [16] established the general solution and investigated the Hyers-Ulam-Rassias stability for a functional equation

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+2f(2x)-2f(x) \tag{1.7}$$

deriving from quadratic and additive functions in quasi-Banach spaces.

In this paper, the authors introduce a new cubic and additive functional equation

$$3f(x+y+z)+f(-x+y+z)+f(x-y+z)+f(x+y-z)+4[f(x)+f(y)+f(z)] = 4[f(x+y)+f(x+z)+f(y+z)] \tag{1.8}$$

and investigate its generalized Hyers-Ulam-Rassias, J.M. Rassias stability [24] in quasi-Banach spaces. It is easy to see that the function $f(x)=kx^3$ and $f(x)=ax$ are the solutions of the functional equation (1.8).

We now introduce some basic ideas in quasi-Banach spaces and some preliminary results that are useful for our further discussion.

Definition 1.2. (see [3, 25]) Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following:

(i) $\| x \| \geq 0$ for all $x \in X$ and $\| x \| = 0$ if and only if $x=0$.

(ii) $\| \lambda x \| = | \lambda | \| x \|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(iii) There is a constant $K \geq 1$ such that $\| x+y \| \leq K(\| x \| + \| y \|)$ for all $x,y \in X$.

It follows form (iii) that

$$\left\|\sum_{i=1}^{2n} x_i\right\| \leq K^n \sum_{i=1}^{2n} \|x_i\| \quad \text{and} \quad \left\|\sum_{i=1}^{2n+1} x_i\right\| \leq K^{n+1} \sum_{i=1}^{2n+1} \|x_i\|$$

for all integers $n \geq 1$ and all $x_1, x_2, \ldots, x_{2n+1} \in X$.

The pair $(X, \|\cdot\|)$ is called quasi-normed space if $\|\cdot\|$ is a quasi-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$. A quasi-Banach space is a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called $p-$norm $(0 < p \leq 1)$ if

$$\| x + y \|^p \leq \| x \|^p + \| y \|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called $p-$Banach space.

According to Aoki-Rolewicz Theorem [25] each quasi-norm is equivalent to some $p-$norm. Since it is much easier to work in $p-$norms than quasi-norms and hence we focus our attention on $p-$norms.

2. Solution of Equation (1.8)

In this section, let E_1 and E_2 denote real vectors spaces, we will prove the following two theorems.

Theorem 2.1. *If $f : E_1 \to E_2$ is a function satisfying (1.8) for all $x, y, z \in E_1$ then f is cubic.*

Proof. Substituting (x, y, z) by $(0, 0, 0)$ in (1.8), we obtain

$$f(0) = 0. \tag{2.1}$$

Replacing y, z by 0 in (1.8), we obtain

$$f(-x) = -f(x), \quad \forall \quad x \in E_1. \tag{2.2}$$

Again replacing (x, y, z) by (x, x, x) in (1.8), we arrive that

$$f(3x) = 4f(2x) - 5f(x), \quad \forall \quad x \in E_1. \tag{2.3}$$

Putting (x, y, z) by $(-2x, x, x)$ in (1.8), we arrive that

$$f(4x) = 10f(2x) - 16f(x), \quad \forall \quad x \in E_1. \tag{2.4}$$

Replacing z by $-x - y$ in (1.8), we obtain

$$f(2x+2y)+8f(x)+8f(y) = 8f(x+y)+f(2x)+f(2y), \quad \forall \quad x,\ y \in E_1. \tag{2.5}$$

Substituting $(x, y, x + y)$ by $(z - y, z - x, z)$ in (2.5), we obtain

$$f(2z) + 8f(z - y) + 8f(z - x) = 8f(z) + f(2z - 2y) + f(2z - 2x), \tag{2.6}$$

for all $x, y, z \in E_1$. Replacing $z-y$ by x, in (2.6), we obtain

$$f(2x) = 8f(x), \quad \forall \quad x \in E_1. \tag{2.7}$$

Again using (2.7) in (2.3), we obtain

$$f(3x) = 27f(x), \quad \forall \quad x \in E_1. \tag{2.8}$$

Using (2.8) in (2.4), we obtain

$$f(4x) = 64f(x), \quad \forall \quad x \in E_1. \tag{2.9}$$

Therefore, in general we have $f(nx) = n^3 f(x)$ for all $x \in E_1$. □

Theorem 2.2. *If $f : E_1 \to E_2$ be a function, satisfying (1.8) for all $x, y, z \in E_1$, then f is additive.*

Proof. Replace y by x in (1.8). We obtain

$$3f(2x+z) + 2f(z) + f(2x-z) + 8f(x) + 4f(z) = 4f(2x) + 8f(x+z), \tag{2.10}$$

for all $x, z \in E_1$. Replacing z by y in (2.10), we obtain

$$3f(2x+y) + 2f(y) + f(2x-y) + 8f(x) + 4f(y) = 4f(2x) + 8f(x+y), \tag{2.11}$$

for all $x, y \in E_1$. Again, replacing y by $-y$ in (2.11) and using oddness of f, we obtain

$$3f(2x-y) - 2f(y) + f(2x+y) + 8f(x) - 4f(y) = 4f(2x) + 8f(x-y), \tag{2.12}$$

for all $x, y \in E_1$. Adding (2.11) and (2.12), we arrive at

$$f(2x+y) + f(2x-y) + 4f(x) = 2f(2x) + 2f(x+y) + 2f(x-y), \quad \forall\, x, y \in E_1. \tag{2.13}$$

Replacing z by y in (1.8) and using (2.2), we obtain

$$3f(x+2y) - f(x-2y) + 6f(x) + 8f(y) = 8f(x+y) + 4f(2y), \quad \forall \quad x, y \in E_1. \tag{2.14}$$

Substituting (x, y) by (y, x) in (2.14), we obtain

$$f(2x-y) = 8f(x+y) + 4f(2x) - 3f(2x+y) - 6f(y) - 8f(x), \quad \forall \quad x, y \in E_1. \tag{2.15}$$

Using (2.15) and (2.13), we obtain

$$f(2x+y) = 3f(x+y) + f(2x) - 3f(y) - 2f(x) - f(x-y), \quad \forall \quad x, y \in E_1. \tag{2.16}$$

Interchanging (x, y) by (y, x) in (2.16) and using equation (2.2), we obtain

$$f(x+2y) = 3f(x+y) + f(2y) - 3f(x) - 2f(y) + f(x-y), \quad \forall \quad x, y \in E_1. \tag{2.17}$$

Adding (2.16) and (2.17), we obtain

$$f(2x+y) + f(x+2y) = 6f(x+y) + f(2x) + f(2y) - 5f(x) - 5f(y), \tag{2.18}$$

for all $x, y \in E_1$. Replacing y by $-y$ in (2.18) and using (2.2), we obtain

$$f(2x-y)+f(x-2y)=6f(x-y)+f(2x)-f(2y)-5f(x)+5f(y), \quad (2.19)$$

for all $x, y \in E_1$. Adding (2.18) and (2.19), we obtain

$$\begin{aligned} &f(2x+y)+f(x+2y)+f(2x-y)+f(x-2y) \\ &\qquad = 6[f(x+y)+f(x-y)]+2f(2x)-10f(x), \quad \forall \quad x, y \in E_1. \end{aligned} \quad (2.20)$$

Substituting $(x+y, x-y)$ by (u, v) in (2.20), we obtain

$$\begin{aligned} &f(x+u)+f(u+y)+f(x+v)+f(v-y) \\ &\qquad = 6[f(u)+f(v)]+2f(u+v)-10f\left(\frac{u+v}{2}\right), \end{aligned} \quad (2.21)$$

for all $x, y, u, v \in E_1$. Again replacing u, v, y by x in (2.21), we obtain

$$f(2x)=2f(x), \quad \forall \quad x \in E_1. \quad (2.22)$$

Replacing x by $\frac{x}{2}$ in (2.22), we obtain

$$f\left(\frac{x}{2}\right)=\frac{1}{2}f(x), \quad \forall \quad x \in E_1. \quad (2.23)$$

Using (2.22) in (2.3), we obtain

$$f(3x)=3f(x), \quad \forall \quad x \in E_1. \quad (2.24)$$

Using (2.22) in (2.4), we obtain

$$f(4x)=4f(x), \quad \forall \quad x \in E_1. \quad (2.25)$$

Using (2.22) in (2.13), we obtain

$$f(2x+y)+f(2x-y)=2[f(x+y)+f(x-y)], \quad \forall \quad x, y \in E_1. \quad (2.26)$$

Now replace x by $\frac{x}{2}$ in (2.26). We obtain

$$f(x+y)+f(x-y)=f(x+2y)+f(x-2y), \quad \forall\, x, y \in E_1. \quad (2.27)$$

Replacing $(x+y, x-y)$ by (u, v) in (2.27), we get

$$f(u)+f(v)=f(u+y)+f(v-y), \quad \forall \quad y, u, v \in E_1. \quad (2.28)$$

Again replacing (u, v) by (x, y) in (2.28) and using (2.1), we get

$$f(x+y)=f(x)+f(y), \quad \forall \quad x, y \in E_1. \quad (2.29)$$

Therefore the mapping $f : E_1 \to E_2$ is additive. □

3. Hyers-Ulam-Rassias Stability of Equation (1.8)

In this section, we assume that E_1 is a quasi-normed space with quasi-norm $||\cdot||_{E_1}$ and that E_2 is a $p-$Banach space with $p-$norm $||\cdot||_{E_2}$. Let k be the modulus of concavity of $||\cdot||_{E_2}$. We use the following notation:

$$D\ f(x,y,z) = 3f(x+y+z) + f(-x+y+z) + f(x-y+z) + f(x+y-z) + 4[f(x)+f(y)+f(z)] - 4[f(x+y)+f(x+z)+f(y+z)], \tag{3.1}$$

for all $x, y, z \in E_1$, and we state the following Lemma 3.1 [16] without proof, it will be useful in proving our theorems.

Lemma 3.1. (see [16]) *Let $0 < p \leq 1$ and let $x_1, x_2, \ldots, x_n$ be non negative real numbers then*

$$\left(\sum_{i=1}^{n} x_i\right)^p \leq \sum_{i=1}^{n} x_i^p. \tag{3.2}$$

Theorem 3.2. *Let $\phi : E_1 \times E_1 \times E_1 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 8^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{3.3}$$

and

$$\sum_{i=1}^{\infty} 8^{ip}\ \phi^p\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}\right) < \infty \tag{3.4}$$

for all $x, y, z \in E_1$ and for all $x, y \in \{0, -2x\}$ and $z \in \{-x, x\}$. Suppose that an cubic function $f : E_1 \to E_2$ satisfies the inequality

$$\|D\ f(x,y,z)\|_{E_2} \leq \phi(x,y,z), \quad \forall\ x, y, z \in E_1. \tag{3.5}$$

Then the limit

$$C(x) = \lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right) \tag{3.6}$$

exists for all $x \in E_1$ and $C : E_1 \to E_2$ is a unique cubic function satisfying

$$\|f(x) - C(x)\|_{E_2} \leq \frac{K}{8}\left[\tilde{\psi}_C(x)\right]^{\frac{1}{p}}, \quad \forall\ x \in E_1, \tag{3.7}$$

where

$$\widetilde{\psi}_C(x) = \sum_{i=1}^{\infty} 8^{ip}\left[\phi^p\left(0, 0, \frac{-x}{2^i}\right) + \phi^p\left(\frac{-x}{2^{i-1}}, \frac{-x}{2^{i-1}}, \frac{x}{2^i}\right)\right]. \tag{3.8}$$

Proof. Replacing z by $-x-y$ in (3.5), we obtain

$$\left\|f(2x+2y) + 8f(x) + 8f(y) - 8f(x+y) - f(2x) - f(2y)\right\|_{E_2} \leq \phi(x, y, -x-y), \tag{3.9}$$

for all $x, y \in E_1$. Replacing $(x, y, x+y)$ by $(z-y, z-x, z)$ in (3.9), we obtain

$$\left\|f(2z) + 8f(z-y) + 8f(z-x) - 8f(z) - f(2z-2y) - f(2z-2x)\right\|_{E_2}$$

$$\leq \phi(z-y, z-x, -z), \tag{3.10}$$

for all $x, y, z \in E_1$. Replacing z by $-z$ in (3.10) and using oddness, we obtain

$$\Big\| f(2z) + 8f(z+y) + 8f(z+x) - 8f(z) - f(2z+2y) - f(2z+2x) \Big\|_{E_2} \leq \phi(-z-y, -z-x, z), \tag{3.11}$$

for all $x, y, z \in E_1$. Adding equations (3.10) and (3.11), we obtain

$$\Big\| 2f(2z) + 8f(z-y) + 8f(z+y) + 8f(z-x) + 8f(z+x) - 16f(z) - f(2z-2y) - f(2z-2x) - f(2z+2y) - f(2z+2x) \Big\|_{E_2} \leq K \; [\phi(z-y, z-x, -z) + \phi(-z-y, -z-x, z)], \tag{3.12}$$

for all $x, y, z \in E_1$. Replacing z, y by x in (3.12) and using (2.4), we obtain

$$\|f(2x) - 8f(x)\|_{E_2} \leq \frac{K}{2} [\phi(0, 0, -x) + \phi(-2x, -2x, x)] \tag{3.13}$$

which can be written as

$$\|f(2x) - 8f(x)\|_{E_2} \leq K \; \psi(x), \quad \forall \; x \; \in E_1, \tag{3.14}$$

where

$$\psi(x) = \frac{1}{2} [\phi(0, 0, -x) + \phi(-2x, -2x, x)] \; , \quad \forall \; x \; \in E_1. \tag{3.15}$$

In equation (3.14), replacing x by $\frac{x}{2^{n+1}}$ and multiplying both sides by 8^n, we have

$$\left\| 8^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 8^n f\left(\frac{x}{2^n}\right) \right\|_{E_2} \leq K \; 8^n \; \psi\left(\frac{x}{2^{n+1}}\right), \quad \forall \; x \; \in E_1, \tag{3.16}$$

for all non-negative integers n. Since E_2 is a $p-$Banach space and using (3.16), we obtain

$$\left\| 8^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 8^m f\left(\frac{x}{2^m}\right) \right\|_{E_2}^p \leq \sum_{i=m}^{n} \left\| 8^{i+1} f\left(\frac{x}{2^{i+1}}\right) - 8^i f\left(\frac{x}{2^i}\right) \right\|_{E_2}^p \leq K^p \; \sum_{i=m}^{n} 8^{ip} \; \psi^p\left(\frac{x}{2^{i+1}}\right) \tag{3.17}$$

for all non-negative integers m and n with $n \geq m$ and all $x \in E_1$. Now $0 < p \leq 1$ and with the help of Lemma 3.1, the equation (3.15) can be written as

$$\psi^p(x) \leq \frac{1}{2^p} [\phi^p(0, 0, -x) + \phi^p(-2x, -2x, x)] \, , \quad \forall \; x \; \in E_1. \tag{3.18}$$

Therefore it follows from (3.4) and (3.18) that

$$\sum_{i=1}^{\infty} 8^{ip} \; \psi^p\left(\frac{x}{2^i}\right) < \infty, \quad \forall \; x \; \in E_1. \tag{3.19}$$

Hence, we conclude from (3.17) and (3.19) that the sequence $\{8^n\ f\left(\frac{x}{2^n}\right)\}$ is a Cauchy sequence for all $x \in E_1$, since E_2 is complete, the sequence $\{8^n\ f\left(\frac{x}{2^n}\right)\}$ converges for all $x \in E_1$. Now we define the mapping $C : E_1 \to E_2$ by (3.6) for all $x \in E_1$. Letting $m = 0$ and allowing $n \to \infty$ in (3.17), we get

$$\|f(x) - C(x)\|_{E_2}^p \leq K^p\ \sum_{i=0}^{\infty} 8^{ip}\ \psi^p\left(\frac{x}{2^{i+1}}\right)$$
$$= \frac{K^p}{8^p}\ \sum_{i=1}^{\infty} 8^{ip}\ \psi^p\left(\frac{x}{2^i}\right), \quad \forall\ x \in E_1. \quad (3.20)$$

Using (3.15) in the equation (3.20), we arrive at the result (3.7). Now, we show that C is a cubic it follows from (3.3), (3.5) and (3.6),

$$\|D\ C(x,y,z)\|_{E_2} = \lim_{n\to\infty} 8^n \left\|D\ f\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)\right\|_{E_2} \leq \lim_{n\to\infty} 8^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right),$$
$$\forall\ x, y, z\ \in E_1.$$

Therefore the mapping $C : E_1 \to E_2$ satisfies (1.8). Hence by Theorem 2.1, we obtain that the mapping $C : E_1 \to E_2$ is cubic. To prove the uniqueness of C. Let $T : E_1 \to E_2$ be another cubic mapping satisfying (3.7). Since

$$\lim_{n\to\infty} 8^{np} \sum_{i=1}^{\infty} 8^{ip}\phi^p\left(\frac{x}{2^{n+i}}, \frac{y}{2^{n+i}}, \frac{z}{2^{n+i}}\right) = \lim_{n\to\infty} \sum_{i=n+1}^{\infty} 8^{ip}\phi^p\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}\right) = 0,$$

for all $x, y, z \in E_1$ and for all $x, y \in \{0, -2x\}$ and $z \in \{-x, x\}$ then

$$\lim_{n\to\infty} 8^{np}\ \tilde{\psi}_C\ \left(\frac{x}{2^n}\right) = 0, \quad \forall\ x\ \in E_1. \quad (3.21)$$

It follows from (3.7) and (3.21),

$$\|C(x) - T(x)\|_{E_2}^p = \lim_{n\to\infty} 8^n \left\|f\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|_{E_2}^p$$
$$\leq \frac{k^p}{8^p} \lim_{n\to\infty} 8^{np}\ \tilde{\psi}_C\ \left(\frac{x}{2^n}\right) = 0$$

for all $x\ \in E_1$. So $C = T$. Hence the theorem is proved. □

Theorem 3.3. *Let $\phi : E_1 \times E_1 \times E_1 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{8^n}\phi(2^n x, 2^n y, 2^n z) = 0, \quad (3.22)$$

and

$$\sum_{i=0}^{\infty} \frac{1}{8^{ip}}\phi^p(2^i x, 2^i y, 2^i z) < \infty, \quad (3.23)$$

for all $x, y, z \in E_1$ and for all $x, y \in \{0, -2x\}$ and $z \in \{-x, x\}$. Suppose that an cubic function $C : E_1 \to E_2$ satisfies the inequality (3.5) for all $x, y, z \in E_1$, *the limit*

$$C(x)=\lim_{n\to\infty}\frac{1}{8^n}f(2^nx) \tag{3.24}$$

exists for all $x\in E_1$ *and* $C:E_1\to E_2$ *is a unique cubic function satisfying*

$$\|f(x)-C(x)\|_{E_2}\le\frac{K}{8}\left[\tilde{\psi}_C(x)\right]^{\frac{1}{p}},\qquad\forall\ x\ \in E_1, \tag{3.25}$$

where

$$\widetilde{\psi}_C(x)=\sum_{i=0}^{\infty}\frac{1}{8^{ip}}\left\{\ \phi^p(0,0,-2^ix)+\ \phi^p(-2^{i+1}x,-2^{i+1}x,-2^ix)\right\}. \tag{3.26}$$

Proof. If we replacing x by 2^nx in (3.14) and dividing by 8^{n+1} on both sides of (3.14), we obtain

$$\left\|\frac{1}{8^{n+1}}f(2^{n+1}x)-\frac{1}{8^n}f(2^nx)\right\|_{E_2}\le\frac{K}{8^{n+1}}\ \psi(2^nx) \tag{3.27}$$

for all $x\ \in E_1$ and for all non-negative integers n. Since E_2 is a $p-$Banach space, using (3.27), we obtain

$$\begin{aligned}\left\|\frac{1}{8^{n+1}}f(2^{n+1}x)-\frac{1}{8^m}f(2^mx)\right\|_{E_2}^p&\le\sum_{i=m}^{n}\left\|\frac{1}{8^{i+1}}f(2^{i+1}x)-\frac{1}{8^i}f(2^ix)\right\|_{E_2}^p\\&\le\frac{K^p}{8^p}\sum_{i=m}^{n}\frac{1}{8^{ip}}\ \psi^p(2^ix)\end{aligned} \tag{3.28}$$

for all non-negative integers m and n with $n\ge m$ and all $x\in E_1$. Since

$$\sum_{i=0}^{\infty}\frac{1}{8^{ip}}\ \psi^p(2^ix)<\infty$$

for all $x\in E_1$ then (3.28) implies that the sequence $\{\frac{1}{8^n}f(2^nx)\}$ is a Cauchy sequence for all $x\in E_1$. Since E_2 is complete, the sequence $\{\frac{1}{8^n}f(2^nx)\}$ converges for all $x\in E_1$. Now we define the mapping $C:E_1\to E_2$ by (3.24) for all $x\in E_1$. Letting $m=0$ and allowing $n\to\infty$ in (3.28), we get

$$\|f(x)-C(x)\|_{E_2}^p\le\frac{K^p}{8^p}\ \sum_{i=0}^{\infty}\frac{1}{8^{ip}}\psi^p(2^ix),\qquad\forall\ x\ \in E_1\,. \tag{3.29}$$

Using (3.15) in the equation (3.29), we arrive at the result (3.25).

Now using (3.29), (3.22) in the equation (3.4), we obtain

$$\|D\ C(x,y,z)\|_{E_2}\le\lim_{n\to\infty}\ \frac{1}{8^n}\phi(2^nx,2^ny,2^nz)=0,\qquad\forall\ x,y,z\ \in E_1.$$

Therefore the mapping $C:E_1\to E_2$ satisfies (1.8). Hence by Theorem 2.1, we obtain that the mapping c is cubic. Uniqueness is proved in similar manner, as in the proof of Theorem 3.2. □

Corollary 3.4. *Let θ, r, s, t be non negative real numbers, suppose that an cubic function $f : E_1 \to E_2$ satisfies the inequality*

$$\|Df(x,y,z)\|_{E_2} \leq \theta \left[\|x\|_{E_1}^{3r} + \|y\|_{E_1}^{3s} + \|z\|_{E_1}^{3t} + \|x\|_{E_1}^{r} \quad \|y\|_{E_1}^{s} \quad \|x\|_{E_1}^{t}\right], \quad \forall\, x, y, z \in E_1. \tag{3.30}$$

Then there exists a unique cubic function $C : E_1 \to E_2$ satisfies

$$\|f(x) - C(x)\|_{E_2} \leq \frac{K\ \theta}{8} \left[\left\{(2 + 2^{3\lambda p+1}) + 2^{2\lambda p}\right\} \frac{1}{|2^{3(\lambda-1)p} - 1|}\right]^{\frac{1}{p}} \quad \|x\|_{E_1}^{3\lambda},$$

where $\lambda = r = s = t \neq 1$.

Proof. In Theorem 3.2, take

$$\phi(x,y,z) = \theta \left[\|x\|_{E_1}^{3r} + \|y\|_{E_1}^{3s} + \|z\|_{E_1}^{3t} + \|x\|_{E_1}^{r} \quad \|y\|_{E_1}^{s} \quad \|x\|_{E_1}^{t}\right]$$

for all $x, y \in E_1$. Then using the equations (3.8) in (3.7) and substituting $\lambda = r = s = t \neq 1$, it becomes

$$\|f(x) - C(x)\|_{E_2} \leq \frac{K\ \theta}{8} \left[\left\{(2 + 2^{3\lambda\ p+1}) + 2^{2\lambda p}\right\} \frac{1}{|2^{3(\lambda-1)p} - 1|}\right]^{\frac{1}{p}} \quad \|x\|_{E_1}^{3\lambda}. \ \square$$

Theorem 3.5. *Let $\phi : E_1 \times E_1 \times E_1 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 3^n \phi\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) = 0, \tag{3.31}$$

and

$$\sum_{i=1}^{\infty} 3^{ip}\ \phi^p\left(\frac{x}{3^i}, \frac{y}{3^i}, \frac{z}{3^i}\right) < \infty, \tag{3.32}$$

for all $x, y, z \in E_1$ and for all $y \in \{x, x\}$, $z \in \{x, -x\}$. Suppose that an odd function $f : E_1 \to E_2$ satisfies the inequality (3.5) for all $x, y, z \in E_1$ then the limit

$$A(x) = \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right) \tag{3.33}$$

exists for all $x \in E_1$ and $A : E_1 \to E_2$ is a unique additive function satisfying

$$\|f(x) - A(x)\|_{E_2} \leq \frac{K}{3} \left[\tilde{\varphi}_A(x)\right]^{\frac{1}{p}}, \quad \forall\ x\ \in E_1, \tag{3.34}$$

where

$$\widetilde{\varphi}_A(x) = \sum_{i=1}^{\infty} 3^{ip} \left\{\ \phi^p\left(\frac{x}{3^i}, \frac{x}{3^i}, \frac{x}{3^i}\right) +\ \phi^p\left(\frac{x}{3^i}, \frac{x}{3^i}, \frac{-x}{3^i}\right)\right\}. \tag{3.35}$$

Proof. Substituting y by x in (3.5), we obtain

$$\|3f(2x+z)+2f(z)+f(2x-z)+8f(x)+4f(z)-4f(2x)-8f(x+z)\|_{E_2} \leq \phi(x,x,z), \quad (3.36)$$

for all $x, z \in E_1$. Replacing z by y in (3.36), we obtain

$$\|3f(2x+y)+2f(y)+f(2x-y)+8f(x)+4f(y)-4f(2x)-8f(x+y)\|_{E_2} \leq \phi(x,x,y), \quad (3.37)$$

for all $x, y \in E_1$. Again replacing y by $-y$ in (3.37) and using oddness, we obtain

$$\|3f(2x-y)-2f(y)+f(2x+y)+8f(x)-4f(y)-4f(2x)-8f(x-y)\|_{E_2} \leq \phi(x,x,-y), \quad (3.38)$$

for all $x, y \in E_1$. Adding (3.37) and (3.38) and the definition of quasi-norm, we obtain

$$\|4f(2x+y)+4f(2x-y)+16f(x)-8[f(2x)+f(x+y)+f(x-y)]\|_{E_2} \leq K\left[\phi(x,x,y)+\phi(x,x,-y)\right], \quad \forall\, x, y \in E_1. \quad (3.39)$$

Again replacing y by x in (3.39) and using (2.22), we obtain

$$\|f(3x)-3f(x)\|_{E_2} \leq \frac{K}{4}\left[\phi(x,x,x)+\phi(x,x,-x)\right], \quad \forall\, x \in E_1\,. \quad (3.40)$$

(3.40) can also be written as,

$$\|f(3x)-3f(x)\|_{E_2} \leq K\ \varphi(x), \quad \forall\, x \in E_1.$$

where

$$\varphi(x) = \frac{1}{4}\left[\phi(x,x,x)+\phi(x,x,-x)\right], \quad \forall\, x \in E_1.$$

If we replace x by $\frac{x}{3^{n+1}}$ in (3.40) and multiply both sides of (3.40) by 3^n, we get

$$\left\|3^{n+1}f\left(\frac{x}{3^{n+1}}\right)-3^n f\left(\frac{x}{3^n}\right)\right\|_{E_2} \leq K\ 3^n\ \varphi\left(\frac{x}{3^{n+1}}\right), \quad \forall\, x \in E_1\,, \quad (3.41)$$

and for all non-negative integers n. Since E_2 is a $p-$Banach space and using (3.41), we obtain

$$\left\|3^{n+1}f\left(\frac{x}{3^{n+1}}\right)-3^m f\left(\frac{x}{3^m}\right)\right\|_{E_2}^p \leq \sum_{i=m}^{n}\left\|3^{i+1}f\left(\frac{x}{3^{i+1}}\right)-3^i f\left(\frac{x}{3^i}\right)\right\|_{E_2}^p \leq K^p\ \sum_{i=m}^{n} 3^{ip}\ \varphi^p\left(\frac{x}{3^{i+1}}\right) \quad (3.42)$$

for all non-negative integers m and n with $n \geq m$ and all $x \in E_1$. But

$$\varphi^p(x) = \frac{1}{4^p}\left[\phi^p(x,x,x)+\phi^p(x,x,-x)\right], \quad \forall\, x \in E_1. \quad (3.43)$$

Therefore it follows from (3.42) and (3.43) that

$$\sum_{i=1}^{\infty} 3^{ip} \varphi^p\left(\frac{x}{3^i}\right) < \infty, \quad \forall\, x \in E_1, \tag{3.44}$$

Hence we conclude from (3.42) and (3.43) that the sequence $\{3^n f\left(\frac{x}{3^n}\right)\}$ is a Cauchy sequence for all $x \in E_1$, since E_2 is complete, the sequence $\{3^n f\left(\frac{x}{3^n}\right)\}$ converges for all $x \in E_1$. Now we define the mapping $A : E_1 \to E_2$ by (3.33) for all $x \in E_1$ and letting $m = 0$ and allowing $n \to \infty$ in (3.42), we get

$$\begin{aligned} \|f(x) - A(x)\|_{E_2}^p &\leq k^p \sum_{i=0}^{\infty} 3^{ip} \varphi^p\left(\frac{x}{3^{i+1}}\right) \\ &= \frac{K^p}{3^p} \sum_{i=1}^{\infty} 3^{ip} \varphi^p\left(\frac{x}{3^i}\right), \quad \forall\, x \in E_1. \end{aligned} \tag{3.45}$$

Therefore (3.34) follows from (3.43) and (3.45). We will now show that A is additive.

Using the equations (3.5), (3.31) and (3.33), we obtain

$$\begin{aligned} \|D\, A(x,y,z)\|_{E_2} &= \lim_{n\to\infty} 3^n \left\|D\, f\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right)\right\|_{E_2} \\ &\leq \lim_{n\to\infty} 3^n \phi\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) = 0 \end{aligned}$$

for all $x, y, z \in E_1$, therefore the mapping $A : E_1 \to E_2$ satisfies (1.8). Hence by Theorem 2.2, we get that the mapping $A : E_1 \to E_2$ is additive.

To prove uniqueness of A, Consider $T : E_1 \to E_2$ be another additive mapping satisfying (3.34). But

$$\begin{aligned} &\lim_{n\to\infty} 3^{np} \sum_{i=1}^{\infty} 3^{ip} \varphi^p\left(\frac{x}{3^{n+i}}, \frac{y}{3^{n+i}}, \frac{z}{3^{n+i}}\right) \\ &\qquad = \lim_{n\to\infty} \sum_{i=n+1}^{\infty} 3^{ip} \phi^p\left(\frac{x}{3^i}, \frac{y}{3^i}, \frac{z}{3^i}\right) = 0, \end{aligned} \tag{3.46}$$

for all $x, y, z \in E_1$, and for all $y \in \{x, x\}$,$z \in \{x, -x\}$ and then apply (3.46) in equation (3.35), we get

$$\lim_{n\to\infty} 3^{np}\, \tilde{\varphi}_A \left(\frac{x}{3^n}\right) = 0\,, \quad \forall\, x \in E_1. \tag{3.47}$$

Hence it follows from (3.35) and (3.47),

$$\|A(x) - T(x)\|_{E_2}^p = \lim_{n\to\infty} 3^{np} \left\|f\left(\frac{x}{3^n}\right) - T\left(\frac{x}{3^n}\right)\right\|_{E_2}^p$$

$$\leq \frac{k^p}{4^p} \lim_{n\to\infty} 3^{np}\ \tilde{\varphi}_A\ \left(\frac{x}{3^n}\right) = 0$$

for all $x\ \in E_1$, so $A = T$. Thus the proof of the theorem is complete. □

Theorem 3.6. *Let $\phi : E_1 \times E_1 \times E_1 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{3^n}\phi(3^n x, 3^n y, 3^n z) = 0, \tag{3.48}$$

and

$$\sum_{i=0}^{\infty} \frac{1}{3^{ip}}\phi^p(3^i x, 3^i y, 3^i z) < \infty, \tag{3.49}$$

for all $x, y, z \in E_1$ and for all $y \in \{x, x\}$ and $z \in \{x, -x\}$. Suppose that an odd function $f : E_1 \to E_2$ satisfies the inequality (3.5) for all $x, y, z \in E_1$, the limit

$$A(x) = \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) \tag{3.50}$$

exists for all $x \in E_1$ and $A : E_1 \to E_2$ is a unique additive function

$$\|f(x) - A(x)\|_{E_2} \leq \frac{K}{3}\, [\tilde{\varphi}_A(x)]^{\frac{1}{p}}\,, \quad \forall\ x\ \in E_1\,, \tag{3.51}$$

where

$$\tilde{\varphi}_A(x) = \sum_{i=1}^{\infty} \frac{1}{3^{ip}} \left\{\ \phi^p\left(3^i x, 3^i x, 3^i x\right) +\ \phi^p\left(3^i x, 3^i x, -3^i x\right)\right\}. \tag{3.52}$$

Proof. If we replacing x by $3^n x$ in (3.40) and dividing by 3^{n+1} on both sides, we obtain

$$\left\|\frac{1}{3^{n+1}} f(3^{n+1}x) - \frac{1}{3^n} f(3^n x)\right\|_{E_2} \leq \frac{K}{3^{n+1}}\ \varphi(3^n x), \quad \forall\ x\ \in E_1\,, \tag{3.53}$$

and for all non-negative integers n. Since E_2 is a $p-$Banach space, using (3.53), we obtain

$$\begin{aligned}\left\|\frac{1}{3^{n+1}} f(3^{n+1}x) - \frac{1}{3^m} f(3^m x)\right\|_{E_2}^p &\leq \sum_{i=m}^{n} \left\|\frac{1}{3^{i+1}} f(3^{i+1}x) - \frac{1}{3^i} f(3^i x)\right\|_{E_2}^p \\ &\leq \frac{K^p}{3^p} \sum_{i=m}^{n} \frac{1}{3^{ip}}\ \varphi^p(3^i x)\end{aligned} \tag{3.54}$$

for all non-negative integers m and n with $n \geq m$ and all $x \in E_1$. Since

$$\sum_{i=0}^{\infty} \frac{1}{3^{ip}}\ \varphi^p(3^i x) < \infty, \quad \forall\ \ x \in E_1\,,$$

then (3.54) implies that the sequence $\{\frac{1}{3^n} f(3^n x)\}$ is a Cauchy sequence for all $x \in E_1$. Since E_2 is complete, the sequence $\{\frac{1}{3^n} f(3^n x)\}$ converges for all $x \in E_1$. Now we define the mapping $A : E_1 \to E_2$ by (3.50) for all $x \in E_1$. The rest of the proof is similar to the proof of Theorem 3.5. □

Corollary 3.7. *Let θ, r, s, t be non negative real numbers, suppose that an additive function $f : E_1 \to E_2$ satisfies the inequality (3.30) for all $x, y, z \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$ satisfies*

$$\|f(x) - A(x)\|_{E_2} \leq \frac{K\ \theta}{3} \left[\frac{8}{|3^{(3\lambda-1)p} - 1|}\right]^{\frac{1}{p}} \ \|x\|_{E_1}^{3\lambda} \ ,$$

where $\lambda = r = s = t \neq \frac{1}{3}$.

Proof. In Theorem 3.5, using the equation (3.35) in (3.34) and substituting $\lambda = r = s = t$, it becomes

$$\|f(x) - A(x)\|_{E_2} \leq \frac{K\ \theta}{3} \left[\frac{8}{|3^{(3\lambda-1)p} - 1|}\right]^{\frac{1}{p}} \ \|x\|_{E_1}^{3\lambda} .$$ □

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