International Journal of Pure and Applied Mathematics
Volume 54 No. 1 2009, 129-140

ON THE COMPARATIVE GROWTH OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS

Sanjib Kumar Datta[1 §], Somnath Mandal[2]

[1]Department of Mathematics
University of North Bengal
Raja Rammohunpur, District Darjeeling
PIN Code 734013, West Bengal, INDIA
e-mail: sanjib_kr_datta@yahoo.co.in

[2]Department of Mathematics
Siliguri Institute of Technology
Sukna, Darjeeling, PIN Code 734009, West Bengal, INDIA
e-mail: somnathm11@yahoo.co.in

Abstract: In the paper we study the comparative growth properties of composite entire and meromorphic functions using L-(p,q)-th order and L^*-(p,q)-th order improving some earlier results where $L = L(r)$ is a slowly changing function and p, q are positive integers with $p > q$.

AMS Subject Classification: 30D35, 30D30
Key Words: entire and meromorphic function, L-order, L-lower order, L-(p,q)-th order, L-(p,q)-th lower order, L^*-(p,q)-th order, L^*-(p,q)-th lower order, composition, growth

1. Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [6] and [2]. In the sequel we use the following two notations:

Received: June 16, 2009

§Correspondence address: 25, School Road, Kalianibash, Barrackpore, P.O.: Nonachandanpukur, Dist.: 24 Parganas (North), P.S.: Titagarh, PIN Code: 743102, West Bengal, INDIA

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots; \log^{[0]} x = x$$
$$\text{and } \exp^{[k]} x = \exp(\exp^{[k-1]} x) \text{ for } k = 1, 2, 3, \cdots; \exp^{[0]} x = x.$$

The following definition is well known.

Definition 1. The order ρ_f and lower order λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire then

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Somasundaram and Thamizharasi [5] introduced the notions of L-order and L-type for entire functions where $L = L(r)$ is a positive continuous function increasing slowly, i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Their definitions are as follows:

Definition 2. (see [5]) The L-order ρ_f^L and L-lower order λ_f^L of an entire function f are defined as follows:

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]} \text{ and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}.$$

When f is meromorphic, then

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]} \text{ and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]}.$$

Juneja, Kapoor and Bajpai [3] defined the (p,q)-th order and (p,q)-th lower order of an entire function f respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r},$$

where p, q are positive integers and $p > q$.

When f is meromorphic, one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r},$$

where p, q are positive integers and $p > q$.

So with the help of the above notion one can easily define the L-(p,q)-th order and L-(p,q)-th lower order of entire and meromorphic functions.

Definition 3. The L-(p,q)-th order $\rho_f^L(p,q)$ and L-(p,q)-th lower order

$\lambda_f^L(p,q)$ of an entire function f are defined as

$$\rho_f^L(p,q)=\limsup_{r\to\infty}\frac{\log^{[p]}M(r,f)}{\log^{[q]}[rL(r)]} \text{ and } \lambda_f^L(p,q)=\liminf_{r\to\infty}\frac{\log^{[p]}M(r,f)}{\log^{[q]}[rL(r)]},$$

where p,q are positive integers and $p>q$.

When f is meromorphic, one can easily see that

$$\rho_f^L(p,q)=\limsup_{r\to\infty}\frac{\log^{[p-1]}T(r,f)}{\log^{[q]}[rL(r)]} \text{ and } \lambda_f^L(p,q)=\liminf_{r\to\infty}\frac{\log^{[p-1]}T(r,f)}{\log^{[q]}[rL(r)]},$$

where p,q are positive integers and $p>q$.

In this paper we intend to establish some results relating to the growth properties of composite entire and meromorphic functions on the basis of L-(p,q)-th order improving some previous results where p,q are positive integers and $p>q$.

The more generalised concept of L-(p,q)-th order and L-(p,q) th lower order of entire and meromorphic functions are L^*-(p,q) th order and L^*-(p,q)-th lower order respectively. In order to prove our results we require the following definitions:

Definition 4. The L^*-order, L^*-lower order and L^*-type of a meromorphic function are defined by

$$\rho_f^{L^*}=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log\left[re^{L(r)}\right]},\quad \lambda_f^{L^*}=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}$$

and

$$\sigma_f^{L^*}=\limsup_{r\to\infty}\frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}},\quad 0<\rho_f^{L^*}<\infty.$$

When f is entire, one can easily verify that

$$\rho_f^{L^*}=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log\left[re^{L(r)}\right]},\quad \lambda_f^{L^*}=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log\left[re^{L(r)}\right]}$$

and

$$\sigma_f^{L^*}=\limsup_{r\to\infty}\frac{\log M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}},\quad 0<\rho_f^{L^*}<\infty.$$

Definition 5. The L^*-(p,q)-th order $\rho_f^{L^*}(p,q)$ and L^*-(p,q)-th lower order $\lambda_f^{L^*}(p,q)$ of an entire function f are defined as

$$\rho_f^{L^*}(p,q)=\limsup_{r\to\infty}\frac{\log^{[p]}M(r,f)}{\log^{[q]}\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*}(p,q)=\liminf_{r\to\infty}\frac{\log^{[p]}M(r,f)}{\log^{[q]}\left[re^{L(r)}\right]},$$

where p, q are positive integers and $p > q$.

When f is meromorphic, one can easily verify that

$$\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}\left[re^{L(r)}\right]},$$

where p, q are positive integers and $p > q$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. (see [1]) *If f and g are two entire functions, then for all sufficiently large values of r*

$$M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \leq M(r, f \circ g) \leq M(M(r, g), f).$$

Lemma 2. (see [4]) *Let f be entire and g be a transcendental entire function of finite lower order. Then for any $\delta > 0$,*

$$M\left(r^{1+\delta}, f \circ g\right) \geq M(M(r, g), f) \quad (r \geq r_0).$$

3. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f be entire and g be transcendental entire with $\lambda_g^L(m,q) < \infty$. Then*

$$\rho_f(p,m)\,\lambda_g^L(m,q) \leq \rho_{f\circ g}^L(p,q) \leq \rho_f(p,m)\,\rho_g^L(m,q),$$

where p, q, m are positive integers such that $p > m > q$.

Proof. By Lemma 2,

$$\begin{aligned}
\rho_{f\circ g}^L(p,q) &= \limsup_{r\to\infty} \frac{\log^{[p]} M\left(r^{1+\delta}, f\circ g\right)}{\log^{[q]}\left[rL(r)\right]^{1+\delta}} \\
&\geq \limsup_{r\to\infty} \frac{\log^{[p]} M(M(r,g), f)}{\log^{[m]} M(r,g)} \cdot \liminf_{r\to\infty} \frac{\log^{[m]} M(r,g)}{\log^{[q]}\left[rL(r)\right]} \\
&= \rho_f(p,m) \cdot \lambda_g^L(m,q).
\end{aligned}$$

Again by Lemma 1,

$$\begin{aligned}\rho_{f\circ g}^{L}(p,q) &= \limsup_{r\to\infty}\frac{\log^{[p]}M(r,f\circ g)}{\log^{[q]}[rL(r)]}\\ &\le \limsup_{r\to\infty}\frac{\log^{[p]}M(M(r,g),f)}{\log^{[m]}M(r,g)}\cdot\limsup_{r\to\infty}\frac{\log^{[m]}M(r,g)}{\log^{[q]}[rL(r)]}\\ &= \rho_f(p,m)\cdot\rho_g^{L}(m,q).\end{aligned}$$

From the above two inequalities we get that

$$\rho_f(p,m)\,\lambda_g^{L}(m,q)\le\rho_{f\circ g}^{L}(p,q)\le\rho_f(p,m)\,\rho_g^{L}(m,q).$$

This proves the theorem. □

Corollary 1. *Under the same conditions of Theorem 1,*

$$\rho_{f\circ g}^{L}(p,q)\ge\lambda_f(p,m)\,\rho_g^{L}(m,q).$$

Proof. By Lemma 2,

$$\begin{aligned}\rho_{f\circ g}^{L}(p,q) &= \limsup_{r\to\infty}\frac{\log^{[p]}M\left(r^{1+\delta},f\circ g\right)}{\log^{[q]}[rL(r)]^{1+\delta}}\\ &\ge \liminf_{r\to\infty}\frac{\log^{[p]}M(M(r,g),f)}{\log^{[m]}M(r,g)}\cdot\limsup_{r\to\infty}\frac{\log^{[m]}M(r,g)}{\log^{[q]}[rL(r)]}\\ &= \lambda_f(p,m)\cdot\rho_g^{L}(m,q).\end{aligned}$$

Thus the corollary is established. □

Remark 1. Considering $f=z$, $g=\exp^{[2]}z$, $p=3$, $m=2$, $q=1$ and $L(r)=\frac{1}{P}\exp\left(\frac{1}{r}\right)$ for any positive real number P one can easily verify that the condition $\lambda_g^{L}(m,q)<\infty$ in Theorem 1 and Corollary 1 is essential.

Remark 2. Taking $f=g=\exp z$, $p=3$, $m=2$, $q=1$ and $L(r)=\frac{1}{P}\exp\left(\frac{1}{r}\right)$ where P is any positive real number one can easily show that the sign '$\le$' cannot be replaced by '$<$' only in Theorem 1 and Corollary 1.

Remark 3. The second part of Theorem 1 is also valid under the same conditions for meromorphic f and entire g.

Theorem 2. *If f is an entire function and g be transcendental entire with $\lambda_g^{L}(m,q)<\infty$. Then*

$$\lambda_{f\circ g}^{L}(p,q)\ge\lambda_f(p,m)\,\lambda_g^{L}(m,q),$$

where p,q,m are positive integers such that $p>m>q$.

Proof. By Lemma 2,

$$\begin{aligned}\lambda_{f\circ g}^{L}(p,q) &= \liminf_{r\to\infty}\frac{\log^{[p]} M\left(r^{1+\delta}, f\circ g\right)}{\log^{[q]}\left[rL(r)\right]^{1+\delta}} \\ &\geq \liminf_{r\to\infty}\frac{\log^{[p]} M\left(M(r,g), f\right)}{\log^{[m]} M(r,g)}\cdot\liminf_{r\to\infty}\frac{\log^{[m]} M(r,g)}{\log^{[q]}\left[rL(r)\right]} \\ &= \lambda_f(p,m)\,\lambda_g^{L}(m,q).\end{aligned}$$

This proves the theorem. □

Remark 4. The condition $\lambda_g^L(m,q) < \infty$ in Theorem 2 is necessary which can be verified by taking $f = z$, $g = \exp^{[2]} z$, $p = 3$, $m = 2$, $q = 1$ and $L(r) = \frac{1}{P}\exp\left(\frac{1}{r}\right)$ for any positive real P.

Remark 5. On considering $f = g = \exp z$, $p = 3$, $m = 2$, $q = 1$ and $L(r) = \frac{1}{P}\exp\left(\frac{1}{r}\right)$ for any positive real number P one can easily verify that the sign '$\geq$' cannot be replaced by '$>$' only in Theorem 2.

Theorem 3. *Let f be meromorphic and g be entire such that $0 < \lambda_{f\circ g}^L(p,q) \leq \rho_{f\circ g}^L(p,q) < \infty$ and $0 < \lambda_g^L(m,q) \leq \rho_g^L(m,q) < \infty$. Then for any positive number A,*

$$\begin{aligned}\frac{\lambda_{f\circ g}^L(p,q)}{\rho_g^L(m,q)} &\leq \liminf_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{\log^{[m]} T\left(r^A, g^{(k)}\right)} \leq \frac{\lambda_{f\circ g}^L(p,q)}{\lambda_g^L(m,q)} \\ &\leq \limsup_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{\log^{[m]} T\left(r^A, g^{(k)}\right)} \leq \frac{\rho_{f\circ g}^L(p,q)}{\lambda_g^L(m,q)},\end{aligned}$$

where p, q, m are positive integers such that $p > m > q$ and $k = 0, 1, 2, \cdots$.

Proof. From the definition of L-(p,q)-th order and L-(p,q)-th lower order we have for arbitrary positive ε and for all large values of r,

$$\log^{[p]} T(r, f\circ g) \geq \left(\lambda_{f\circ g}^L(p,q) - \varepsilon\right)\log^{[q]}\left[rL(r)\right] \tag{1}$$

and

$$\log^{[m]} T\left(r^A, g^{(k)}\right) \leq \left(\rho_g^L(m,q) + \varepsilon\right)\log^{[q]}\left[rL(r)\right]. \tag{2}$$

Now from (1) and (2) it follows for all large values of r,

$$\frac{\log^{[p]} T(r, f\circ g)}{\log^{[m]} T\left(r^A, g^{(k)}\right)} \geq \frac{\left(\lambda_{f\circ g}^L(p,q) - \varepsilon\right)}{\left(\rho_g^L(m,q) + \varepsilon\right)}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{\log^{[m]} T\left(r^A, g^{(k)}\right)} \geq \frac{\lambda_{f\circ g}^L(p,q)}{\rho_g^L(m,q)}. \tag{3}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} T\left(r, f \circ g\right) \leq \left(\lambda_{f\circ g}^{L}\left(p,q\right) + \varepsilon\right) \log^{[q]} \left[rL\left(r\right)\right] \tag{4}$$

and for all large values of r,

$$\log^{[m]} T\left(r^{A}, g^{(k)}\right) \geq \left(\lambda_{g}^{L}\left(m,q\right) - \varepsilon\right) \log^{[q]} \left[rL\left(r\right)\right]. \tag{5}$$

So combining (4) and (5) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} T\left(r, f \circ g\right)}{\log^{[m]} T\left(r^{A}, g^{(k)}\right)} \leq \frac{\left(\lambda_{f\circ g}^{L}\left(p,q\right) + \varepsilon\right)}{\left(\lambda_{g}^{L}\left(m,q\right) - \varepsilon\right)}.$$

Since $\varepsilon\left(>0\right)$ is arbitrary it follows that

$$\liminf_{r\to\infty} \frac{\log^{[p]} T\left(r, f \circ g\right)}{\log^{[m]} T\left(r^{A}, g^{(k)}\right)} \leq \frac{\lambda_{f\circ g}^{L}\left(p,q\right)}{\lambda_{g}^{L}\left(m,q\right)}. \tag{6}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[m]} T\left(r^{A}, g^{(k)}\right) \leq \left(\lambda_{g}^{L}\left(m,q\right) + \varepsilon\right) \log^{[q]} \left[rL\left(r\right)\right]. \tag{7}$$

Now from (1) and (7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} T\left(r, f \circ g\right)}{\log^{[m]} T\left(r^{A}, g^{(k)}\right)} \geq \frac{\left(\lambda_{f\circ g}^{L}\left(p,q\right) - \varepsilon\right)}{\left(\lambda_{g}^{L}\left(m,q\right) + \varepsilon\right)}.$$

Choosing $\varepsilon \to 0$ we get

$$\limsup_{r\to\infty} \frac{\log^{[p]} T\left(r, f \circ g\right)}{\log^{[m]} T\left(r^{A}, g^{(k)}\right)} \geq \frac{\lambda_{f\circ g}^{L}\left(p,q\right)}{\lambda_{g}^{L}\left(m,q\right)}. \tag{8}$$

Also for all large values of r,

$$\log^{[p]} T\left(r, f \circ g\right) \leq \left(\rho_{f\circ g}^{L}\left(p,q\right) + \varepsilon\right) \log^{[q]} \left[rL\left(r\right)\right]. \tag{9}$$

So from (5) and (9) it follows for all large values of r,

$$\frac{\log^{[p]} T\left(r, f \circ g\right)}{\log^{[m]} T\left(r^{A}, g^{(k)}\right)} \leq \frac{\left(\rho_{f\circ g}^{L}\left(p,q\right) + \varepsilon\right)}{\left(\lambda_{g}^{L}\left(m,q\right) - \varepsilon\right)}.$$

As $\varepsilon\left(>0\right)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p]} T\left(r, f \circ g\right)}{\log^{[m]} T\left(r^{A}, g^{(k)}\right)} \leq \frac{\rho_{f\circ g}^{L}\left(p,q\right)}{\lambda_{g}^{L}\left(m,q\right)}. \tag{10}$$

Thus the theorem follows from (3), (6), (8) and (10). □

Theorem 4. *Let f be meromorphic and g be entire such that $0 < \lambda_{f\circ g}^{L}\left(p,q\right) \leq \rho_{f\circ g}^{L}\left(p,q\right) < \infty$ and $0 < \rho_{g}^{L}\left(m,q\right) < \infty$. Then for any positive* *number A,*

$$\liminf_{r\to\infty}\frac{\log^{[p]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\leq\frac{\rho^{L}_{f\circ g}\left(p,q\right)}{\rho^{L}_{g}\left(m,q\right)}\leq\limsup_{r\to\infty}\frac{\log^{[p]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)},$$

where p, q, m are positive integers such that $p > m > q$ and $k = 0, 1, 2, \cdots$.

Proof. From the definition of L-(p,q)-th order we get for a sequence of values of r tending to infinity,

$$\log^{[m]}T\left(r^{A},g^{(k)}\right)\geq\left(\rho^{L}_{g}\left(m,q\right)-\varepsilon\right)\log^{[q]}\left[rL\left(r\right)\right].\tag{11}$$

Now from (9) and (11) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\leq\frac{\left(\rho^{L}_{f\circ g}\left(p,q\right)+\varepsilon\right)}{\left(\rho^{L}_{g}\left(m,q\right)-\varepsilon\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain

$$\liminf_{r\to\infty}\frac{\log^{[p]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\leq\frac{\rho^{L}_{f\circ g}\left(p,q\right)}{\rho^{L}_{g}\left(m,q\right)}.\tag{12}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]}T\left(r,f\circ g\right)\geq\left(\rho^{L}_{f\circ g}\left(p,q\right)-\varepsilon\right)\log^{[q]}\left[rL\left(r\right)\right].\tag{13}$$

So combining (2) and (13) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\geq\frac{\left(\rho^{L}_{f\circ g}\left(p,q\right)-\varepsilon\right)}{\left(\rho^{L}_{g}\left(m,q\right)+\varepsilon\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\limsup_{r\to\infty}\frac{\log^{[p]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\geq\frac{\rho^{L}_{f\circ g}\left(p,q\right)}{\rho^{L}_{g}\left(m,q\right)}.\tag{14}$$

Thus the theorem follows from (12) and (14). □

In view of Theorem 1, Theorem 2 and Theorem 3 we may state the following theorem without proof.

Theorem 5. *Let f be meromorphic and g be entire such that $0 < \lambda^{L}_{f\circ g}(p,q) \leq \rho^{L}_{f\circ g}(p,q) < \infty$ and $0 < \lambda^{L}_{g}(m,q) \leq \rho^{L}_{g}(m,q) < \infty$. Then for any positive number A,*

$$\frac{\lambda_{f}\left(p,m\right)\lambda^{L}_{g}\left(m,q\right)}{\rho^{L}_{g}\left(m,q\right)}\leq\frac{\lambda^{L}_{f\circ g}\left(p,q\right)}{\rho^{L}_{g}\left(m,q\right)}\leq\liminf_{r\to\infty}\frac{\log^{[p]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\leq\frac{\lambda^{L}_{f\circ g}\left(p,q\right)}{\lambda^{L}_{g}\left(m,q\right)}$$

$$\leq\limsup_{r\to\infty}\frac{\log^{[p]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\leq\frac{\rho^{L}_{f\circ g}\left(p,q\right)}{\lambda^{L}_{g}\left(m,q\right)}\leq\frac{\rho_{f}\left(p,m\right)\rho^{L}_{g}\left(m,q\right)}{\lambda^{L}_{g}\left(m,q\right)},$$

where p, q, m are positive integers such that $p > m > q$ and $k = 0, 1, 2, \cdots$.

The following theorem is a natural consequence of Theorem 3 and Theorem 4.

Theorem 6. *Let f be meromorphic and g be entire such that $0 < \lambda_{f\circ g}^{L}(p,q) \leq \rho_{f\circ g}^{L}(p,q) < \infty$ and $0 < \lambda_{g}^{L}(m,q) \leq \rho_{g}^{L}(m,q) < \infty$. Then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log^{[p]} T(r, f\circ g)}{\log^{[m]} T\left(r^{A}, g^{(k)}\right)} \leq \min\left\{\frac{\lambda_{f\circ g}^{L}(p,q)}{\lambda_{g}^{L}(m,q)}, \frac{\rho_{f\circ g}^{L}(p,q)}{\rho_{g}^{L}(m,q)}\right\}$$
$$\leq \max\left\{\frac{\lambda_{f\circ g}^{L}(p,q)}{\lambda_{g}^{L}(m,q)}, \frac{\rho_{f\circ g}^{L}(p,q)}{\rho_{g}^{L}(m,q)}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[p]} T(r, f\circ g)}{\log^{[m]} T\left(r^{A}, g^{(k)}\right)},$$

where p, q, m are positive integers such that $p > m > q$ and $k = 0, 1, 2, \cdots$.

In the following theorems we see some comparative growth properties of composite entire and meromorphic functions on the basis of L^{*}-(p, q)-th order and L^{*}-(p, q)-th lower order improving some previous results where p, q are positive integers with $p > q$.

Theorem 7. *Let f be entire and g be transcendental entire with $\lambda_{g}^{L^{*}}(m,q) < \infty$. Then*

$$\rho_{f}(p,m)\,\lambda_{g}^{L^{*}}(m,q) \leq \rho_{f\circ g}^{L^{*}}(p,q) \leq \rho_{f}(p,m)\,\rho_{g}^{L^{*}}(m,q),$$

where p, q, m are positive integers such that $p > m > q$.

Corollary 2. *Under the same conditions of Theorem 7,*

$$\rho_{f\circ g}^{L^{*}}(p,q) \geq \lambda_{f}(p,m)\,\rho_{g}^{L^{*}}(m,q).$$

The proofs of Theorem 7 and Corollary 2 are omitted because they can be carried out in the line of Theorem 1 and Corollary 1 respectively.

Remark 6. Considering $f = z$, $g = \exp^{[2]} z$, $p = 3$, $m = 2$, $q = 1$ and $L(r) = \frac{1}{P}\exp\left(\frac{1}{r}\right)$ for any positive real number P one can easily verify that the condition $\lambda_{g}^{L^{*}}(m,q) < \infty$ in Theorem 7 and Corollary 2 is essential.

Remark 7. Taking $f = g = \exp z$, $p = 3$, $m = 2$, $q = 1$ and $L(r) = \frac{1}{P}\exp\left(\frac{1}{r}\right)$ where P is any positive real number one can easily show that the sign '$\leq$' cannot be replaced by '$<$' only in Theorem 7 and Corollary 2.

Remark 8. The second part of Theorem 7 is also valid under the same conditions for meromorphic f and entire g.

Theorem 8. *If f is an entire function and g be transcendental entire with*

$\lambda_g^{L^*}(m,q) < \infty$. *Then*

$$\lambda_{f\circ g}^{L^*}(p,q) \geq \lambda_f(p,m)\,\lambda_g^{L^*}(m,q),$$

where p,q,m *are positive integers such that* $p > m > q$.

The proof of Theorem 8 is omitted because it runs parallel to that of Theorem 2.

Remark 9. The condition $\lambda_g^{L^*}(m,q) < \infty$ is necessary in Theorem 8 which can be verified by taking $f = z$, $g = \exp^{[2]} z$, $p = 3$, $m = 2$, $q = 1$ and $L(r) = \frac{1}{P}\exp\left(\frac{1}{r}\right)$ for any positive real P.

Remark 10. On considering $f = g = \exp z$, $p = 3$, $m = 2$, $q = 1$ and $L(r) = \frac{1}{P}\exp\left(\frac{1}{r}\right)$ for any positive real number P one can easily verify that the sign '$\geq$' cannot be replaced by '$>$' only in Theorem 8.

Theorem 9. *Let* f *be meromorphic and* g *be entire such that* $0 < \lambda_{f\circ g}^{L^*}(p,q) \leq \rho_{f\circ g}^{L^*}(p,q) < \infty$ *and* $0 < \lambda_g^{L^*}(m,q) \leq \rho_g^{L^*}(m,q) < \infty$. *Then for any positive number* A,

$$\begin{aligned}\frac{\lambda_{f\circ g}^{L^*}(p,q)}{\rho_g^{L^*}(m,q)} &\leq \liminf_{r\to\infty}\frac{\log^{[p]} T(r,f\circ g)}{\log^{[m]} T\left(r^A, g^{(k)}\right)} \leq \frac{\lambda_{f\circ g}^{L^*}(p,q)}{\lambda_g^{L^*}(m,q)}\\ &\leq \limsup_{r\to\infty}\frac{\log^{[p]} T(r,f\circ g)}{\log^{[m]} T\left(r^A, g^{(k)}\right)} \leq \frac{\rho_{f\circ g}^{L^*}(p,q)}{\lambda_g^{L^*}(m,q)},\end{aligned}$$

where p,q,m *are positive integers such that* $p > m > q$ *and* $k = 0,1,2,\cdots$.

The proof of Theorem 9 is omitted because it can be carried out in the line of Theorem 3.

Theorem 10. *Let* f *be meromorphic and* g *be entire such that* $0 < \lambda_{f\circ g}^{L^*}(p,q) \leq \rho_{f\circ g}^{L^*}(p,q) < \infty$ *and* $0 < \rho_g^{L^*}(m,q) < \infty$. *Then for any positive number* A,

$$\liminf_{r\to\infty}\frac{\log^{[p]} T(r,f\circ g)}{\log^{[m]} T\left(r^A, g^{(k)}\right)} \leq \frac{\rho_{f\circ g}^{L^*}(p,q)}{\rho_g^{L^*}(m,q)} \leq \limsup_{r\to\infty}\frac{\log^{[p]} T(r,f\circ g)}{\log^{[m]} T\left(r^A, g^{(k)}\right)},$$

where p,q,m *are positive integers such that* $p > m > q$ *and* $k = 0,1,2,\cdots$.

We omit the proof of Theorem 10 as it runs parallel to that of Theorem 4.

In view of Theorem 7, Theorem 8 and Theorem 9 we may state the following theorem without proof.

Theorem 11. *Let* f *be meromorphic and* g *be entire such that* $0 < \lambda_{f\circ g}^{L^*}(p,q) \leq \rho_{f\circ g}^{L^*}(p,q) < \infty$ *and* $0 < \lambda_g^{L^*}(m,q) \leq \rho_g^{L^*}(m,q) < \infty$. *Then for* *any positive number A,*

$$\frac{\lambda_f(p,m)\,\lambda_g^{L^*}(m,q)}{\rho_g^{L^*}(m,q)} \le \frac{\lambda_{f\circ g}^{L^*}(p,q)}{\rho_g^{L^*}(m,q)} \le \liminf_{r\to\infty}\frac{\log^{[p]} T(r,f\circ g)}{\log^{[m]} T\left(r^A,g^{(k)}\right)} \le \frac{\lambda_{f\circ g}^{L^*}(p,q)}{\lambda_g^{L^*}(m,q)}$$

$$\le \limsup_{r\to\infty}\frac{\log^{[p]} T(r,f\circ g)}{\log^{[m]} T\left(r^A,g^{(k)}\right)} \le \frac{\rho_{f\circ g}^{L^*}(p,q)}{\lambda_g^{L^*}(m,q)} \le \frac{\rho_f(p,m)\,\rho_g^{L^*}(m,q)}{\lambda_g^{L^*}(m,q)},$$

where p,q,m are positive integers such that $p>m>q$ and $k=0,1,2,\cdots$.

The following theorem is a natural consequence of Theorem 9 and Theorem 10.

Theorem 12. *Let f be meromorphic and g be entire such that $0<\lambda_{f\circ g}^{L^*}(p,q)\le\rho_{f\circ g}^{L^*}(p,q)<\infty$ and $0<\lambda_g^{L^*}(m,q)\le\rho_g^{L^*}(m,q)<\infty$. Then for any positive number A,*

$$\liminf_{r\to\infty}\frac{\log^{[p]} T(r,f\circ g)}{\log^{[m]} T\left(r^A,g^{(k)}\right)} \le \min\left\{\frac{\lambda_{f\circ g}^{L^*}(p,q)}{\lambda_g^{L^*}(m,q)},\frac{\rho_{f\circ g}^{L^*}(p,q)}{\rho_g^{L^*}(m,q)}\right\}$$

$$\le \max\left\{\frac{\lambda_{f\circ g}^{L^*}(p,q)}{\lambda_g^{L^*}(m,q)},\frac{\rho_{f\circ g}^{L^*}(p,q)}{\rho_g^{L^*}(m,q)}\right\} \le \limsup_{r\to\infty}\frac{\log^{[p]} T(r,f\circ g)}{\log^{[m]} T\left(r^A,g^{(k)}\right)},$$

where p,q,m are positive integers such that $p>m>q$ and $k=0,1,2,\cdots$.

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