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THE OPERATOR $\otimes$ AND ITS SPECTRUM RELATED TO HEAT EQUATION

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Abstract: In this paper, we study the equation

$$\frac{\partial}{\partial t}u(x,t) + c^2 \otimes u(x,t) = 0$$

with the initial condition

$$u(x,0) = f(x)$$

for $x \in \mathbb{R}^n$ — the n-dimensional Euclidean space. The operator is

$$\begin{aligned}\otimes &= \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3 \\ &= \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 + \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)\right. \\ &\qquad \left.\cdot\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right) + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right] = \frac{3}{4}\Diamond\triangle + \frac{1}{4}\Box^3,\end{aligned}$$

where

$$\begin{aligned}\triangle &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}, \\ \Box &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2},\end{aligned}$$

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$$\diamondsuit = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2,$$

$p+q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is the given generalized function and c is a positive constant.

On the suitable conditions for f and u, we obtain the uniqueness solution of such equation. Moreover, if we put $q = 0$ we obtain the solution of heat equation

$$\frac{\partial}{\partial t} u(x,t) + c^2 \triangle^3 u(x,t) = 0.$$

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1. Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \triangle u(x,t) \tag{1}$$

with the initial condition

$$u(x,0) = f(x)\,,$$

where $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, we obtain

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4c^2 t}\right) f(y)dy \tag{2}$$

as the solution of (1).

Now, (2) can be written $u(x,t) = E(x,t) * f(x)$ where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4c^2 t}\right). \tag{3}$$

$E(x,t)$ is called *the heat kernel*, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$, see [2], pp. 208-209.

In 1996, A. Kananthai [3] has introduced the Diamond operator $\diamondsuit$ defined by

$$\diamondsuit = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 , \ p+q=n\,,$$

or $\diamondsuit$ can be written as the product of the operators in the form $\diamondsuit = \triangle\Box = \Box\triangle$, where $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and $\Box = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}$ is the ultra-hyperbolic. The Fourior transform of the Diamond operator also has been studied and the elementary solution of such operator, see [4].

Next, K. Nonlaopon and A. Kananthai (see [5]) studied the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box u(x,t)\,.$$

Now, the purpose of this work is to study the equation

$$\frac{\partial}{\partial t} u(x,t) + c^2 \otimes u(x,t) = 0 \tag{4}$$

with the initial condition

$$u(x,0) = f(x)$$

for $x \in \mathbb{R}^n$ — the n-dimensional Euclidean space. The operator is

$$\begin{aligned}\otimes &= \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^3 \\ &= \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)\left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 + \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)\right. \\ &\quad \left.\cdot \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right) + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right] = \frac{3}{4}\diamondsuit\triangle + \frac{1}{4}\Box^3\,,\end{aligned}$$

$p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is the given generalized function and c is a positive constant. We obtain $u(x,t) = E(x,t) * f(x)$ as a solution of (4), where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2\left[\left(\sum_{i=1}^{p} \xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^3\right] t + i(\xi,x)\right] d\xi\,, \tag{5}$$

and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t>0$. The function $E(x,t)$ is the elementary solution of (4).

All properties of $E(x,t)$ will be studied in details.

Now, if we put $q = 0$ in (4), then (4) reduces to the equation

$$\frac{\partial}{\partial t} u(x,t) + c^2 \triangle^3 u(x,t) = 0$$

which is related to the heat equation.

2. Preliminaries

Definition 2.1. Let $f(x) \in L_1(\mathbb{R}^n)$ — the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)\, dx\,, \tag{6}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ and $dx = dx_1\, dx_2 \ldots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(\xi)\, d\xi. \tag{7}$$

If f is a distribution with compact supports by [6], Theorem 7.4-3, p. 187, equation (2.1) can be written as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \left\langle f(x), e^{-i(\xi,x)} \right\rangle. \tag{8}$$

Definition 2.2. The spectrum of the kernel $E(x,t)$ of (5) is the bounded support of the Fourier transform $\widehat{E(\xi,t)}$ for any fixed $t > 0$.

Definition 2.3. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a point in $\mathbb{R}^n$ and denote by $\Gamma_+ = \{\xi \in \mathbb{R}^n : \xi_1^2 + \xi_2^2 + \ldots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \ldots - \xi_{p+q}^2 > 0 \text{ and } \xi_1 > 0\}$ the set of an interior of the forward cone, and $\overline{\Gamma}_+$ denotes the closure of Γ_+.

Let Ω be spectrum of $E(x,t)$ defined by Definition 2.2 for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $\widehat{E(\xi,t)}$ be the Fourier transform of $E(x,t)$ and define

$$\widehat{E(\xi,t)} = \begin{cases} \dfrac{1}{(2\pi)^{n/2}} \exp\left[c^2 \left(\left(\displaystyle\sum_{i=1}^{p} \xi_i^2 \right)^3 - \left(\displaystyle\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right) t \right] & \text{for } \xi \in \Gamma_+, \\ 0 & \text{for } \xi \notin \Gamma_+. \end{cases} \tag{9}$$

Lemma 1. (The Fourier Transform of $\otimes\delta$)

$$\mathcal{F}\otimes\delta = \frac{(-1)^3}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3-\left(\xi_{p+1}^2+\xi_{p+2}^2-\ldots+\xi_{p+q}^2\right)^3\right]$$

where $\mathcal{F}$ is the Fourier transform defined by equation (6) and if the norm of ξ is given by $\|\xi\| = \left(\xi_1^2+\xi_2^2+\ldots+\xi_n^2\right)^{1/2}$, then

$$|\mathcal{F}\otimes\delta| \le \frac{3}{(2\pi)^{n/2}}\|\xi\|^6\,,$$

that is $\mathcal{F}\otimes$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution. Moreover, by equation (7)

$$\otimes\delta = \mathcal{F}^{-1}\frac{1}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3-\left(\xi_{p+1}^2+\xi_{p+2}^2-\ldots+\xi_{p+q}^2\right)^3\right].$$

Proof. By equation (2.3)

$$\begin{aligned}
\mathcal{F}\otimes\delta &= \frac{1}{(2\pi)^{n/2}}\left\langle\otimes\delta, e^{-i(\xi,x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, \otimes e^{-i(\xi,x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, \left(\frac{3}{4}\diamondsuit\triangle+\frac{1}{4}\Box^3\right)e^{-i(\xi,x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, \frac{3}{4}\diamondsuit\triangle e^{-i(\xi,x)}\right\rangle+\frac{1}{(2\pi)^{n/2}}\left\langle\delta, \frac{1}{4}\Box^3 e^{-i(\xi,x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\Bigg\langle\delta, \frac{3}{4}(-1)^2\left[\left(\sum_{i=1}^p\xi_i^2\right)^2-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right] \\
&\quad\cdot(-1)\left(\sum_{i=1}^n\xi_i^2\right)e^{-i(\xi,x)}\Bigg\rangle \\
&\quad+\frac{1}{(2\pi)^{n/2}}\left\langle\delta, \frac{1}{4}(-1)^3\left[\left(\sum_{i=1}^p\xi_i^2\right)-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)\right]^3 e^{-i(\xi,x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left[\frac{3}{4}(-1)^3\left[\left(\sum_{i=1}^p\xi_i^2\right)^2-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right]\left(\sum_{i=1}^n\xi_i^2\right)\right] \\
&\quad+\frac{1}{(2\pi)^{n/2}}\left(\frac{1}{4}(-1)^3\left[\left(\sum_{i=1}^p\xi_i^2\right)-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)\right]^3\right)
\end{aligned}$$

$$= \frac{(-1)^3}{(2\pi)^{n/2}}\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right]$$

$$= \frac{(-1)^3}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3 - \left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right].$$

Now,

$$\begin{aligned}
|\mathcal{F}\otimes\delta| &= \frac{1}{(2\pi)^{n/2}}\left|\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3 - \left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right| \\
&\le \frac{1}{(2\pi)^{n/2}}\left|\xi_1^2+\ldots+\xi_n^2\right| \\
&\quad \cdot\left|\left(\xi_1^2+\ldots+\xi_n^2\right)^2+\left(\xi_1^2+\ldots+\xi_n^2\right)^2+\left(\xi_1^2+\ldots+\xi_n^2\right)^2\right| \\
&\le \frac{3}{(2\pi)^{n/2}}\|\xi\|^6,
\end{aligned}$$

where $\|\xi\| = \left(\xi_1^2+\xi_2^2+\ldots+\xi_n^2\right)^{1/2}$, $\xi_i(i=1,2,\ldots,n)\in\mathbb{R}$. Hence we obtain $\mathcal{F}\otimes\delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution.

Since $\mathcal{F}$ is $1-1$ transformation from the space $\mathcal{S}'$ of the tempered distribution to the real space $\mathbb{R}$, then by equation (7)

$$\otimes\delta = \mathcal{F}^{-1}\frac{1}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3 - \left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right].$$

That completes the proof. □

Lemma 2. *Given the operator*

$$\mathrm{L} = \frac{\partial}{\partial t} + c^2\left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right], \tag{10}$$

where

$$\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3 = \frac{3}{4}\Diamond\triangle + \frac{1}{4}\Box^3,$$

$p+q=n$ is the dimension of $\mathbb{R}^n$, $(x,t)=(x_1,x_2,\ldots,x_n,t)\in\mathbb{R}^n\times(0,\infty)$, and c is a positive constant. Then we obtain

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right]t + i(\xi,x)\right]d\xi, \tag{11}$$

as a elementary solution of (10), where $\sum_{j=p+1}^{p+q} \xi_j^2 > \sum_{i=1}^{p} \xi_i^2$.

Proof. Let

$$\mathrm{L}E(x,t) = \delta(x,t),$$

where $E(x,t)$ is the elementary solution of operator L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t}E(x,t) + c^2 \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^3 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right] E(x,t) = \delta(x)\delta(t).$$

Take the Fourier transform defined by (6) to both sides of the equation, we obtain

$$\frac{\partial}{\partial t}\widehat{E(\xi,t)} - c^2 \left[\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right] \widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\delta(t).$$

Thus

$$\widehat{E(\xi,t)} = \frac{H(t)}{(2\pi)^{n/2}} \exp\left[c^2 \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right) t \right],$$

where $H(t)$ is the Heaviside function. Since $H(t) = 1$ for $t > 0$. Therefore,

$$\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}} \exp\left[c^2 \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right) t \right]$$

which has been already defined by (9). Thus

$$\begin{aligned} E(x,t) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{E(\xi,t)}\, d\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\Omega} e^{i(\xi,x)} \widehat{E(\xi,t)}\, d\xi\,, \end{aligned}$$

where Ω is the spectrum of $E(x,t)$. Thus from (7)

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp\left[c^2 \left[\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right] t + i(\xi,x) \right] d\xi. \quad \square$$

3. Main Results

Theorem 3. *Given the equation*

$$\frac{\partial}{\partial t} u(x,t) + c^2 \otimes u(x,t) = 0 \tag{12}$$

with the initial condition

$$u(x,0) = f(x)\,. \tag{13}$$

The operator is

$$\begin{aligned} \otimes &= \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^3 \\ &= \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right) \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 + \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right) \cdot \right. \\ &\quad \left. \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right) + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 \right] \\ &= \frac{3}{4} \Diamond \triangle + \frac{1}{4} \square^3 , \end{aligned}$$

$p+q = n$ is the dimension of Euclidean space $\mathbb{R}^n$, k is a positive integer, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is the given generalized function, and c is a positive constant. Then we obtain

$$u(x,t) = E(x,t) * f(x)$$

as a solution of (12) which satisfies (13) where $E(x,t)$ is given by (11).

Proof. Taking the Fourier transform defined by (6) to both sides of (12), we obtain

$$\frac{\partial}{\partial t} \widehat{u}(\xi,t) - c^2 \left[\left(\sum_{i=1}^{p} \xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^3\right] \widehat{u}(\xi,t) = 0$$

(see Lemma 6). Thus

$$\widehat{u}(\xi,t) = K(\xi) \exp\left[c^2 \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^3\right) t\right] , \tag{14}$$

where $K(\xi)$ is constant and $\widehat{u}(\xi,0) = K(\xi)$.

Now, by (13) we have

$$K(\xi) = \widehat{u}(\xi, 0) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)\, dx \tag{15}$$

and by the inversion in (7), (14) and (15) we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{u}(\xi,t)\, d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\xi,x)} e^{-i(\xi,y)} f(y) \cdot \\ & \quad \exp\left[c^2 \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right) t \right] dy\, d\xi. \end{aligned}$$

Thus

$$\begin{aligned} u(x,t) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\xi,x-y)} \cdot \\ & \quad \exp\left[c^2 \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right) t \right] f(y)\, dy\, d\xi\,, \end{aligned}$$

or

$$\begin{aligned} u(x,t) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left[c^2 \left[\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right] t \right. \\ & \quad \left. + i(\xi, x-y) \right] f(y)\, dy\, d\xi. \end{aligned} \tag{16}$$

Set

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp\left[c^2 \left[\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right] t + i(\xi,x) \right] d\xi. \tag{17}$$

We choose $\Omega \subset \mathbb{R}^n$ be the spectrum of $E(x,t)$ and by (11), we have

$$\begin{aligned} E(x,t) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp\left[c^2 \left[\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right] t + i(\xi,x) \right] d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\Omega} \exp\left[c^2 \left[\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right] t + i(\xi,x) \right] d\xi. \end{aligned} \tag{18}$$

Thus (16) can be written in the convolution form

$$u(x,t) = E(x,t) * f(x).$$

Since $E(x,t)$ exists, then

$$\begin{aligned}\lim_{t\to 0} E(x,t) &= \frac{1}{(2\pi)^n}\int_\Omega e^{i(\xi,x)}\,d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(\xi,x)}\,d\xi \\ &= \delta(x), \quad \text{for} \quad x\in\mathbb{R}^n. \end{aligned} \tag{19}$$

See [1], p. 396, equation (10.2.19b).

Thus for the solution $u(x,t) = E(x,t) * f(x)$ of (12), then

$$\lim_{t\to 0} u(x,t) = u(x,0) = \delta * f(x) = f(x)$$

which satisfies (13). □

Theorem 4. *The kernel $E(x,t)$ defined by (18) has the following properties :*

(1) $E(x,t)\in\mathcal{C}^\infty$-the space of continuous function for $x\in\mathbb{R}^n$, $t>0$ with infinitely differentiable.

(2) $$\left(\frac{\partial}{\partial t} + c^2\left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]\right)E(x,t) = 0 \quad \text{for} \ t>0.$$

(3) $E(x,t) > 0$ for $t>0$.

(4) $$|E(x,t)| \le \frac{2^{2-n}}{\pi^{n/2}}\frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}, \quad \text{for} \quad t>0,$$

where $M(t)$ is a function of t in the spectrum Ω and Γ denote the Gamma function. Thus $E(x,t)$ is bounded for any fixed $t>0$.

(5) $\lim_{t\to 0} E(x,t) = \delta$.

Proof. (1) From (18), since

$$\begin{aligned}\frac{\partial^n}{\partial x^n}E(x,t) \;=\; &\frac{1}{(2\pi)^n}\int_\Omega \frac{\partial^n}{\partial x^n}\exp\left[c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right]t \right. \\ &\left. + i(\xi,x)\right]d\xi.\end{aligned}$$

Thus $E(x,t)\in\mathcal{C}^\infty$ for $x\in\mathbb{R}^n$, $t>0$.

(2) By computing directly, we obtain

$$\left(\frac{\partial}{\partial t} + c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right]\right)E(x,t) = 0.$$

(3) $E(x,t) > 0$ for $t > 0$ is obvious by (18).

(4) We have

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\Omega}\exp\left[c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right]t + i(\xi,x)\right]d\xi.$$

$$|E(x,t)| \leq \frac{1}{(2\pi)^n}\int_{\Omega}\exp\left[c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)\right]d\xi.$$

By changing to bipolar coordinates

$$\xi_1 = r\omega_1,\ \xi_2 = r\omega_2, \ldots,\ \xi_p = r\omega_p \quad \text{and}$$

$$\xi_{p+1} = s\omega_{p+1},\ \xi_{p+2} = s\omega_{p+2}, \ldots,\ \xi_{p+q} = s\omega_{p+q}\,,$$

where $\sum_{i=1}^{p}\omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q}\omega_j^2 = 1$. Thus

$$|E(x,t)| \leq \frac{1}{(2\pi)^n}\int_{\Omega}\exp\left[c^2\left(s^6 - r^6\right)t\right]r^{p-1}s^{q-1}\,dr\,ds\,d\Omega_p\,d\Omega_q\,,$$

where $d\xi = r^{p-1}s^{q-1}\,dr\,ds\,d\Omega_p\,d\Omega_q$, $d\Omega_p$ and Ω_q are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Since $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ and we suppose $0 \leq r \leq R$ and $0 \leq s \leq T$, where R and T are constants. Thus we obtain

$$\begin{aligned}|E(x,t)| &\leq \frac{\Omega_p\,\Omega_q}{(2\pi)^n}\int_0^R\int_0^T \exp\left[c^2\left(s^6 - r^6\right)t\right]r^{p-1}s^{q-1}\,ds\,dr \\ &= \frac{\Omega_p\,\Omega_q}{(2\pi)^n}M(t) \quad \text{for any fixed } t > 0 \text{ in the spectrum } \Omega \\ &= \frac{2^{2-n}}{\pi^{n/2}}\frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}\,,\end{aligned}$$

where

$$M(t) = \int_0^R\int_0^T \exp\left[c^2\left(s^6 - r^6\right)t\right]r^{p-1}s^{q-1}\,ds\,dr$$

is a function of t, $\Omega_p = \dfrac{2\pi^{p/2}}{\Gamma\left(\frac{p}{2}\right)}$ and $\Omega_q = \dfrac{2\pi^{p/2}}{\Gamma\left(\frac{q}{2}\right)}$. Thus, for any fixed $t > 0$,

$E(x,t)$ is bounded.

(5) Obvious by (19). □

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