AN EXTENSION OF METHOD OF QUASILINEARIZATION 
FOR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract: This paper is devoted to existence and monotone approximation of the unique solution for integro-differential equation including a variety of possible cases by employing the method of generalized quasilinearization. Further, quadratic convergence of the monotone approximation is also discussed.

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1. Introduction

It is well known that the method of quasilinearization provides a powerful tool for obtaining a sequence of approximate solutions of nonlinear problems involving convex/concave functions. In some cases, it offers a constructive procedure for the solutions, and the iterates serve as upper and lower bounds for those solutions. In recent years, the method of quasilinearization has been general-
ized, refined and extended in diverse problems. We can refer the reader to the monographs [6], [9], papers [7], [10], [3], [11], [4], [5], [1] and references cited therein.

Integro-differential equations arise frequently as mathematical models in diverse disciplines, for example, the mathematical modeling of biological sciences such as spreading of disease by dispersal of infectious individuals. The theory and the applications of integro-differential equations is now an important area of investigation [2]. In [3], Deo and Knoll investigated the integro-differential equation by the method of quasilinearization developed in [2]. Here, to enlarge the class of functions, we should follow up the idea presented in [5] and attempt to extend the technique of quasilinearization to the following initial value problem

\[ x'(t) = f(t, x(t)) + \int_{t_0}^{t} q(t, s, x(s)) ds, \quad x(t_0) = x_0, \] (1.1)

where \( f \in C(J \times R, R) \), \( q \in C(J \times J \times R, R) \), \( t \in J = [t_0, t_0 + T] \), \( t_0 \geq 0, \ T > 0 \). Consequently, in this paper, the investigations of quasilinearized approximations to (1.1), as well as rapid convergence, become the goals of our investigation.

2. Preliminaries

Firstly, for \( y_0(t), z_0(t) \in C^1(J, R) \) with \( y_0 \leq z_0 \), we denote the following sets

\[ \Omega_1 = \{(t, u) : y_0 \leq u \leq z_0, \ t \in J \}, \]
\[ \Omega_1 = \{(t, u, v) : y_0 \leq u, v \leq z_0, \ t \in J \}, \]
\[ \Omega_2 = \{(t, s, u) : y_0 \leq u \leq z_0, \ (t, s) \in J \times J \}, \]
\[ \Omega_2 = \{(t, s, u, v) : y_0 \leq u, v \leq z_0, \ (t, s) \in J \times J \}. \]

We say \( u \in C^1(J, R) \) is a lower solution of problem (1.1), if

\[ u'(t) \leq f(t, u) + \int_{t_0}^{t} q(t, s, u(s)) ds, \quad t \in J, \quad u(0) \leq x_0. \]

Similarly \( v \in C^1(J, R) \) is an upper solution of problem (1.1), if the reverse inequalities hold.

Before we proceed further, we need to list the following known result.

**Lemma 2.1.** (see [3]) For problem (1.1), suppose that:

(A0) \( u, v \) are lower and upper solutions of problem (1.1) respectively;

(A1) \( f \in C(\Omega_1, R) \), and \( f(t, x) - f(t, y) \leq L(x - y) \), where \( t \in J, \ u(t) \leq \)

\[ f(t, u) + \int_{t_0}^{t} q(t, s, u(s)) ds, \quad t \in J, \quad u(0) \leq x_0. \]
For convenience, we denote these property by (A).

\[ q \in C(\Omega_2, R) \] is monotone nondecreasing in \( x \) for each \( (t, s) \in J \times J \), and \( q(t, s, x) - q(t, s, y) \leq N(x - y) \), where \( (t, s) \in J \times J \), \( u(t) \leq y \leq x \leq v(t) \), and \( N \geq 0 \).

Then \( u(t) \leq v(t) \) for \( t \in J \), provided that \( u(t_0) \leq v(t_0) \).

### 3. Main Results

Suppose that \( f \) and \( q \) have the splitting \( f(t, x) = F(t, x, x) \), and \( q(t, s, x) = Q(t, s, x, x) \), where \( F \in C(\Omega_1, R) \), \( Q \in C(\Omega_2, R) \). Then problem (1.1) shall take the form

\[
x'(t) = F(t, x(t), x(t)) + \int_{t_0}^{t} Q(t, s(x(s), x(s)))ds, \quad x(t_0) = x_0.
\] (3.1)

**Theorem 3.1.** For problem (3.1), suppose that:

\( y_0 \), \( z_0 \in C^1(J, R) \) are lower and upper solutions of problem (3.1), respectively, such that, \( y_0(t) \leq z_0(t) \) on \( J \);

\( F, F_x, F_y, F_{xx}, F_{xy}, F_{yy} \in C(\Omega_1, R) \) and

\( F_{xx}(t, x, y) \geq 0, \ F_{xy}(t, x, y) \leq 0, \ F_{yy}(t, x, y) \leq 0 \), for \( (t, x, y) \in \Omega_1 \);

\( Q_x, Q_y, Q_{xx}, Q_{xy}, Q_{yy} \in C(\Omega_2, R) \) and

\( Q_{xx}(t, s, x, y) \geq 0, \ Q_{xy}(t, s, x, y) \leq 0, \ Q_{yy}(t, x, y) \leq 0 \), for \( (t, s, x, y) \in \Omega_2 \);

\( Q_x(t, s, \eta(s), \nu(s)) - Q_y(t, s, \nu(s), \nu(s)) \geq 0 \), for \( y_0(t) \leq \eta(t) \leq \nu(t) \leq z_0(t) \), \( (t, s) \in J \times J \).

Then there exist monotone sequences \( \{y_n\}, \{z_n\} \) which converge uniformly to the unique solution of problem (3.1) on \( J \), and the convergence is quadratic.

**Proof.** Assumptions \( (B_0)-(B_2) \) guarantee that problem (3.1) has a unique solution on \( \Omega_1 \).

Obviously, \( (B_1) \) implies that \( F_x \) is nondecreasing in the second variable, \( F_x \) is nonincreasing in the third variable and \( F_y \) is nonincreasing in the last two variables; \( (B_2) \) leads to that \( Q_x \) is nondecreasing in the third variable, \( Q_x \) is nonincreasing in the last variable; \( Q_y \) is nonincreasing in the last two variables. For convenience, we denote these property by (A).

Consider the following linear problems for each \( n = 0, 1, 2, 3, \ldots \),

\[
y_{n+1} = F(t, y_n, y_n) + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)](y_{n+1} - y_n)
\]
By Lemma 2.1, in view of (F)rom the mean value theorem and property (A), we have

\[ y_{n+1}(t_0) = x_0, \]
\[ z'_{n+1} = F(t, z_n, z_n) + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)](z_{n+1} - z_n) \]
\[ + \int_{t_0}^t \{Q(t, s, y_n, y_n) + [Q_x(t, s, y_n, z_n) + Q_y(t, s, z_n, z_n)](y_{n+1} - y_n)\} ds, \]
\[ z_{n+1}(t_0) = x_0. \]

Clearly, each linear problem has a unique solution on \( J \). We wish to show that

\[ y_0 \leq y_1 \leq \cdots \leq y_n \leq z_n \leq \cdots \leq z_1 \leq z_0 \quad \text{on } J. \quad (3.2) \]

We claim first that \( y_0 \leq y_1 \) on \( J \). For this purpose, let \( p = y_0 - y_1 \), and note that \( p(t_0) = y_0(t_0) - y_1(t_0) \leq 0 \). Then

\[
p'(t) = F(t, y_0, y_0) + \int_{t_0}^t Q(t, s, y_0, y_0) ds - F(t, y_0, y_0) - [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)](y_1 - y_0) \]
\[ - \int_{t_0}^t \{Q(t, s, y_0, y_0) + [Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0)](y_1 - y_0)\} ds \]
\[ = [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)]p(t) + \int_{t_0}^t [Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0)]p(s) ds, \]
\[ t \in J. \]

By Lemma 2.1, in view of (B3), this implies \( y_0 \leq y_1 \) on \( J \).

Similarly, we can show that \( z_1 \leq z_0 \) on \( J \).

Next we prove that \( y_1 \leq z_1 \) on \( J \). Setting \( p = y_1 - z_1 \), and \( p(t_0) = 0 \). We obtain

\[
p'(t) = F(t, y_0, y_0) + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)](y_1 - y_0) \]
\[ + \int_{t_0}^t \{Q(t, s, y_0, y_0) + [Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0)](y_1 - y_0)\} ds \]
\[ - F(t, z_0, z_0) - [F_x(t, y_0, z_0) - F_y(t, z_0, z_0)](z_1 - z_0) \]
\[ - \int_{t_0}^t \{Q(t, s, z_0, z_0) + [Q_x(t, s, z_0, z_0)](z_1 - z_0)\} ds. \]

From the mean value theorem and property (A), we have
\[ p'(t) \leq F(t, y_0, y_0) - F(t, z_0, y_0) + F(t, z_0, y_0) - F(t, z_0, z_0) + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)](y_1 - y_0 - z_1 + z_0) \]
\[ + \int_{t_0}^{t} [Q(t, s, y_0, y_0) - Q(t, s, z_0, y_0) + Q(t, s, z_0, y_0) - Q(t, s, z_0, z_0)]ds \]
\[ + \int_{t_0}^{t} [Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0)](y_1 - y_0 - z_1 + z_0)ds \]
\[ = [F_x(t, \xi_1, y_0) + F_y(t, \xi_2, y_0)] - [F_x(t, \xi_3, \xi_2) - F_y(t, \xi_4, \xi_2)](y_0 - z_0) \]
\[ + \int_{t_0}^{t} [Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0)](y_1 - z_1)ds \]
\[ + \int_{t_0}^{t} [Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0)](y_0 - z_0)ds \]
\[ \leq [F_x(t, y_0, z_0) - F_x(t, y_0, y_0)](z_0 - y_0) + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)]p(t) \]
\[ + \int_{t_0}^{t} [Q_x(t, s, y_0, z_0) - Q_x(t, s, y_0, y_0)](z_0 - y_0)ds \]
\[ + \int_{t_0}^{t} [Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0)]p(s)ds \leq [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)]p(t) \]
\[ + \int_{t_0}^{t} [Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0)]p(s)ds, \ t \in J, \]
where \( y_0 \leq \xi_1, \xi_2, \xi_3, \xi_4 \leq z_0 \). Then, we get, by using Lemma 2.1, \( p(t) \leq 0 \) on \( J \) and consequently, \( y_1 \leq z_1 \) on \( J \).

As a result, it follows that
\[ y_0 \leq y_1 \leq z_1 \leq z_0, \ t \in J. \]

Now we need to show that \( y_1 \) and \( z_1 \) are lower and upper solutions of problem (3.1), respectively. The mean value theorem and property (A) yield
\[ y'_1(t) = F(t, y_1, y_1) + F(t, y_0, y_0) - F(t, y_1, y_0) + F(t, y_1, y_0) \]
\[ - F(t, y_1, y_1) + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)](y_1 - y_0) \]
\[ + \int_{t_0}^{t} [Q(t, s, y_1, y_1) + Q(t, s, y_0, y_0) - Q(t, s, y_1, y_0) \]
\[ + Q(t, s, y_1, y_0) - Q(t, s, y_1, y_1)]ds \]
\[ + \int_{t_0}^{t} [Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0)](y_1 - y_0)ds \]
\[ = F(t, y_1, y_1) + [F_x(t, \xi_3, y_0) + F_y(t, y_1, \xi_2)](y_0 - y_1) \]
+ \left[ F_x(t, y_0, z_0) + F_y(t, z_0, z_0) \right] (y_1 - y_0) \\
+ \int_{t_0}^{t} Q(t, s, y_1, y_1) ds + \int_{t_0}^{t} \left[ Q_x(t, s, y_0) + Q_y(t, s, y_1) \right] (y_0 - y_1) ds \\
+ \int_{t_0}^{t} \left[ Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0) \right] (y_1 - y_0) ds \\
\leq F(t, y_1, y_1) + \int_{t_0}^{t} Q(t, s, y_1, y_1) ds \\
+ \left[ F_x(t, y_0, z_0) - F_x(t, y_0, y_0) + F_y(t, z_0, z_0) - F_y(t, y_1, y_1) \right] (y_1 - y_0) \\
+ \int_{t_0}^{t} Q_x(t, s, y_0, z_0) - Q_x(t, s, y_0, y_0) \\
+ Q_y(t, s, z_0, z_0) - Q_y(t, s, y_1, y_1) \right] (y_1 - y_0) ds \leq F(t, y_1, y_1) + \int_{t_0}^{t} Q(t, s, y_1, y_1) ds, \\
t \in J,
\]

where \( y_0 \leq \xi_1, \xi_2, \xi_3, \xi_4 \leq z_0 \).

Similarly, proceeding as before, one can obtain that
\[
z_1'(t) \geq F(t, z_1, z_1) + \int_{t_0}^{t} Q(t, s, z_1, z_1) ds, \quad t \in J.
\]

The above proves that \( y_1 \) and \( z_1 \) are lower and upper solutions of problem (3.1).

Suppose that for some \( k \geq 1 \), we have
\[
y_{k-1} \leq y_k \leq z_k \leq z_{k-1} \quad \text{on} \quad J. \tag{3.3}
\]

Let \( y_k, z_k \) be lower and upper solutions of problem (3.1). We shall show that
\[
y_k \leq y_{k+1} \leq z_{k+1} \leq z_k \quad \text{on} \quad J. \tag{3.4}
\]

To do this, consider \( p = y_k - y_{k+1} \) on \( J \) so that \( p(t_0) = 0 \). From the fact that \( y_k \) is a lower solution of problem (3.1), it is clear that
\[
p'(t) \leq F(t, y_k, y_k) + \int_{t_0}^{t} Q(t, s, y_k, y_k) ds \\
- F(t, y_k, y_k) - [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)](y_{k+1} - y_k) \\
- \int_{t_0}^{t} \left\{ Q(t, s, y_k, y_k) + [Q_x(t, s, y_k, z_k) + Q_y(t, s, z_k, z_k)](y_{k+1} - y_k) \right\} ds \\
= \left[ F_x(t, y_k, z_k) + F_y(t, z_k, z_k) \right] p(t) \\
+ \int_{t_0}^{t} [Q_x(t, s, y_k, z_k) + Q_y(t, s, z_k, z_k)] p(s) ds, \quad t \in J.
\]

This leads to by Lemma 2.1, because of (B4), that \( y_k \leq y_{k+1} \) on \( J \). A similar
argument holds for $z_{k+1} \leq z_k$ on $J$.

Letting $p = y_{k+1} - z_{k+1}$, again $p(t_0) = 0$, and arguing as before, one can show that

$$p'(t) \leq [F_x(t, y_k, z_k) + F_y(t, z_k, y_{k+1})]p(t) + \int_{t_0}^{t} [Q_x(t, s, y_k, z_k) + Q_y(t, s, z_k, y_{k+1})]p(s)ds, \quad t \in J,$$

which yields $y_{k+1} \leq z_{k+1}$, on $J$. Thus we have (3.4) and by induction, we see that (3.2) is valid on $J$. By the standard arguments [6], it can be shown that the sequences $\{y_n(t)\}$ and $\{z_n(t)\}$ converge uniformly and monotonically to $x(t)$, the unique solution of problem (3.1) on $J$.

Finally, we shall show that the convergence is quadratic. For this purpose, put $p_n = x - y_n$ and $h_n = z_n - x$, where $x$ denotes the unique solution of problems (3.1). Note that $p_n(t_0) = h_n(t_0) = 0$, and $p_n \geq 0, h_n \geq 0$. Applying the mean value theorem and property (A), there exist $\xi_i, i = 1, 2, \cdots, n$, such that $y_n \leq \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6 \leq x, x \leq \xi_6, \xi_10 \leq z_n, y_n \leq \xi_7, \xi_8, \xi_11, \xi_12 \leq z_n$, and

$$p_{n+1}'(t) = F(t, x, x) + \int_{t_0}^{t} Q(t, s, x, x)ds$$

$$- F(t, y_n, y_n) - [F_x(t, y_n, z_n) + F_y(t, z_n, y_{k+1})](y_{n+1} - y_n)$$

$$- \int_{t_0}^{t} [Q_x(t, s, y_n, y_n) + Q_y(t, s, z_n, y_{k+1})](y_{n+1} - y_n)ds$$

$$= F(t, x, x) - F(t, y_n, x) + F(t, y_n, x) - F(t, y_n, y_n)$$

$$- [F_x(t, y_n, z_n) + F_y(t, z_n, y_{k+1})](y_{n+1} - x + y_n - y_n)$$

$$+ \int_{t_0}^{t} [Q_x(t, s, x, x) - Q(t, s, y_n, x) + Q(t, s, y_n, x) - Q(t, s, y_n, z_n)]ds$$

$$- \int_{t_0}^{t} [Q_x(t, s, y_n, z_n) + Q_y(t, s, z_n, z_n)](y_{n+1} - x + y_n - y_n)ds$$

$$\leq \int_{t_0}^{t} [F_x(t, x, x) - F(t, y_n, x) + F_x(t, y_n, x) - F(t, y_n, z_n)$$

$$+ F_y(t, y_n, y_n) - F_y(t, z_n, y_n) + F_y(t, z_n, y_n) - F_y(t, z_n, z_n)]p_n(t)$$

$$+ [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)]p_{n+1}(t)$$

$$+ \int_{t_0}^{t} [Q_x(t, s, x, x) - Q_x(t, s, y_n, x) + Q_x(t, s, y_n, x) - Q_x(t, s, y_n, z_n)$$

$$+ Q_y(t, s, y_n, y_n) - Q_y(t, s, z_n, y_n) + Q_y(t, s, z_n, y_n) - Q_y(t, s, z_n, z_n)]p_n(s)ds$$

$$+ \int_{t_0}^{t} [Q_x(t, s, y_n, z_n) + Q_y(t, s, z_n, z_n)]p_{n+1}(s)ds$$
where proof is complete.

ceeding of Theorem 3.1, as follows.

that is, which means

\[ |x - y_{n+1}| \leq \frac{2e^{MT}}{\sqrt{M^2 + 4N}} \left( (L_1 + L_3T) \max_{t \in J} p_n^2(t) + (L_2 + L_4T) \max_{t \in J} h_n^2(t) \right), \quad t \in J. \]

Analogous to the discussion of \( \{z_n\} \), we have

\[ \max_{t \in J} h_{n+1} \leq \frac{2e^{MT}}{\sqrt{M^2 + 4N}} \left( (K_1 + K_3T) \max_{t \in J} p_n^2(t) + (K_2 + K_4T) \max_{t \in J} h_n^2(t) \right), \quad t \in J, \]

which means

\[ \max_{t \in J} |z_{n+1} - x| \leq \frac{2e^{MT}}{\sqrt{M^2 + 4N}} \left( (K_1 + K_3T) \max_{t \in J} |x - y_n|^2(t) + (K_2 + K_4T) \max_{t \in J} |z_n - x|^2(t) \right), \quad t \in J, \]

where \( K_1 = \frac{1}{2}A_1, \quad K_2 = \frac{3}{2}A_1 + A_2 + A_3, \quad K_3 = \frac{1}{2}B_1, \quad K_4 = \frac{3}{2}B_1 + B_2 + B_3 \). The proof is complete.

Similarly, we can give various monotone sequences according to the proceeding of Theorem 3.1, as follows.

**Theorem 3.2.** For problem (3.1), suppose that:

(\( C_0 \)) \( y_0, z_0 \in C^1(J, R) \) are lower and upper solutions of problem (3.1), respectively, with \( y_0(t) \leq z_0(t) \) on \( J; \)
For problem (3.1), assume that \( \text{conditions } (C_2)-(C_3) \) hold. Then there exist monotone sequences \( \{y_n\}, \{z_n\} \), such that

\[
y_{n+1} = F(t, y_n, y_n) + [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)](y_{n+1} - y_n)
\]
\[
+ \int_{t_0}^{t} \{Q(t, s, y_n, y_n) + [Q_x(t, s, y_n, y_n) + Q_y(t, s, y_n, z_n)](y_{n+1} - y_n)\} ds,
\]

\[
y_{n+1}(t_0) = x_0,
\]

\[
z_{n+1} = F(t, z_n, z_n) + [F_x(t, z_n, z_n) + F_y(t, z_n, z_n)](z_{n+1} - z_n)
\]
\[
+ \int_{t_0}^{t} \{Q(t, s, z_n, z_n) + [Q_x(t, s, z_n, z_n) + Q_y(t, s, y_n, z_n)](z_{n+1} - z_n)\} ds,
\]

\[
z_{n+1}(t_0) = x_0,
\]

which converge uniformly to the unique solution of problem (3.1) on \( J \), and the convergence is quadratic.

**Theorem 3.3.** For problem (3.1), assume that \( (B_0)-(B_1) \) hold. Then conditions \( (C_2)-(C_3) \) imply that there exist monotone sequences \( \{y_n\}, \{z_n\} \), such that

\[
y_{n+1} = F(t, y_n, y_n) + [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)](y_{n+1} - y_n)
\]
\[
+ \int_{t_0}^{t} \{Q(t, s, y_n, y_n) + [Q_x(t, s, y_n, y_n) + Q_y(t, s, y_n, z_n)](y_{n+1} - y_n)\} ds,
\]

\[
y_{n+1}(t_0) = x_0,
\]

\[
z_{n+1} = F(t, z_n, z_n) + [F_x(t, z_n, z_n) + F_y(t, z_n, z_n)](z_{n+1} - z_n)
\]
\[
+ \int_{t_0}^{t} \{Q(t, s, z_n, z_n) + [Q_x(t, s, z_n, z_n) + Q_y(t, s, y_n, z_n)](z_{n+1} - z_n)\} ds,
\]

\[
z_{n+1}(t_0) = x_0,
\]

converge uniformly to the unique solution of problem (3.1) on \( J \), and the convergence is quadratic.

**Theorem 3.4.** For problem (3.1), assume that \( (C_0)-(C_1) \) hold. Then
conditions $(B_2)$-$(B_3)$ imply that there exist monotone sequences \(\{y_n\}, \{z_n\}\), such that

\[
y_{n+1}' = F(t, y_n, y_n) + [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)](y_{n+1} - y_n) \\
\quad + \int_{t_0}^{t} \{Q(t, s, y_n, y_n) + [Q_x(t, s, y_n, z_n) + Q_y(t, s, z_n, z_n)](y_{n+1} - y_n)\} ds,
\]

\[y_{n+1}(t_0) = x_0,\]

\[
z_{n+1}' = F(t, z_n, z_n) + [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)](z_{n+1} - z_n) \\
\quad + \int_{t_0}^{t} \{Q(t, s, z_n, z_n) + [Q_x(t, s, y_n, z_n) + Q_y(t, s, z_n, z_n)](z_{n+1} - z_n)\} ds,
\]

\[z_{n+1}(t_0) = x_0,\]

converge uniformly to the unique solution of problem (3.1) on \(J\), and the convergence is quadratic.

**Remark 3.1.** If \(q \in C[J \times J \times \mathbb{R}, \mathbb{R}]\) is monotone nonincreasing in \(x\) for each \((t, s) \in J \times J\), then one can also extend the method of quasilinearization to problem (1.1).

**Remark 3.2.** Assume that \(q = 0\), then Theorem 3.1 and Theorem 3.2 would include the results of [9], [5] as special cases.

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**References**


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