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WEAK AND STRONG CONVERGENCE THEOREMS FOR COMMON FIXED POINTS OF RELATIVELY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract: In this paper, we prove weak and strong convergence theorems for the modified Mann type iteration process to fixed points of relatively asymptotically nonexpansive mappings. Our results improve various celebrated results of fixed point theory in the context of relatively asymptotically nonexpansive mappings.

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1. Introduction

Let C be a nonempty subset of a normed space X and T : C → C a mapping. The mapping T is said to be *Lipschitzian* if for each $n \in N$, there exists a positive number k_n such that

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$$||T^n x - T^n y|| \leq k_n ||x - y|| \text{ for all } x, y \in C.$$

A Lipschitzian mapping T is said to be *uniformly k-Lipschitzian* if $k_n = k$ for all $n \in N$ and asymptotically nonexpansive (cf. [5]) if $k_n \geq k$ for all $n \in N$ with $\lim_{n\to\infty} k_n = 1$.

Clearly every nonexpansive mapping T (i.e. $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in C$) is asymptotically nonexpansive with sequence $\{1\}$ and every asymptotically nonexpansive mapping is uniformly k-Lipschitzian with $k = \sup_{n\in N} k_n$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] as an important generalization of the class of nonexpansive mappings. The existence of fixed points of asymptotically nonexpansive mappings was proved by Goebel and Kirk [5] as below:

Theorem 1.1. (see [5]) *If C is a nonempty closed convex bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive mapping $T : C \to C$ has a fixed point in C.*

An iterative method for computing fixed points of asymptotically nonexpansive mappings was developed by Schu [12]. In paper [13], he proved an interesting result:

Theorem 1.2. (see [13]) *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X satisfying Opial's condition and $T : C \to C$ an asymptotically nonexpansive with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in [0, 1] satisfying the condition $\epsilon \leq \alpha_n \leq 1 - \epsilon$ for all $n \in N$ and for some $\epsilon > 0$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \in N. \tag{1.1}$$

converges weakly to a fixed point of T.

Iterative methods for computing fixed points of asymptotically nonexpansive mappings have been further studied by authors (see, e.g., [1, 2, 3, 4, 6, 7, 8, 10, 11, 15, 17, 18] and references therein).

In [16], Vijayaraju introduced the concept of asymptotically S-nonexpansive mappings as a natural generalization of concept of asymptotically nonexpansive mappings.

The purpose of this paper is to introduce an iteration process to approximate fixed points of asymptotically S-nonexpansive mappings. More precisely, we study weak and strong convergence of the proposed iteration process to approximate fixed points of relatively asymptotically nonexpansive mappings. Our result extends various known results of fixed point theory in the context of asymptotically S-nonexpansive mappings.

2. Preliminaries

First we recall following definition and examples:

Definition 2.1. (see Vijayaraju [16]) Let C be a nonempty subset of a normed linear space X. A mapping T : C → C is said to be an asymptotically nonexpansive mapping with respect to an another mapping S : C → C if, there is a sequence $\{k_n\}$ of real numbers with $k_n \geq 1, k_n \geq k_{n+1}$ and $k_n \to 1$ as $n \to \infty$ such that

$$||T^n x - T^n y|| \leq k_n ||Sx - Sy|| \text{ for all } \text{ x, y in C and n=1,2,....}$$

We note that if S is the identity mapping, then T is an asymptotically nonexpansive mapping (Goebel and Kirk [5]) of C into itself.

Definition 2.2. Let C be a nonempty subset of a normed linear space X and let S, T : C → C be two mappings. We say that T is uniform L-Lipschitzian with respect to S if there exists an $L > 0$ such that

$$||T^n x - T^n y|| \leq L||Sx - Sy|| \ \forall \text{ x, y} \in \text{C and n} \in \text{N}.$$

The following example shows that the class of asymptotically nonexpansive mappings with respect to another mapping is wider than the class of asymptotically nonexpansive mappings.

Example 2.1. (see [16]) Let X = R, the set of real numbers. Define a mapping T : X → X by $Tx = 2x$ for all $x \in X$. If $a = 1, b = 2 \in X$ then $|Ta - Tb| = 2 > 1 = |a - b|$ and hence T is not nonexpansive. Suppose that $x = 1/2, y = 3/2 \in X$. Then we have

$$Tx = 1,\ T^2 x = 2, ...\ T^m x = 2^{m-1}$$

and

$$Ty = 3,\ T^2 y = 6, ...\ T^m y = 2^{m-1}3\,.$$

Therefore,

$$|T^m x - T^m y| = 2^m$$

and

$$|x - y| = 1\,.$$

Thus, T is not an asymptotically nonexpansive self-mapping of X. Define a mapping S : X → X by Sy = 3y for all y∈X. Let u,v∈X. Then, we have

$$|Tu - Tv| \leq |Su - Sv| \,.$$

Hence T is nonexpansive with respect to S and thus, T is an S- asymptotically nonexpansive.

The following example shows that the class of asymptotically nonexpansive mappings with respect to another mapping includes properly the class of nonexpansive mappings with respect to another mapping.

Example 2.2. (see [16]) Let X = l^2 with the usual norm and C be the closed unit ball in X. We define a mapping T : C → C by

$$Tx = \left(0, \xi_1^2, A_2\xi_2, A_3\xi_3, ..., A_n\xi_n..\right)$$

for all $x = (\xi_1, \xi_2, \xi_3, ..., \xi_n..) \in C$, where $\{A_i\}$ is a sequence of real numbers such that $0 < A_i < 1$ for all i and

$$\prod_{i=2}^{\infty} A_i = \frac{1}{2} \,.$$

We define a mapping S : C → C by

$$Sy = (0, \eta_1, \eta_2, ...) \quad \text{for all} \quad \text{y} = (\eta_1, \eta_2, ...) \in C \;.$$

Let $a = (1, 0, ...), b = (i, 0, ...) \in C$ where $i^2 = -1$. Then $||Ta - Tb|| = 2$ and $||Sa - Sb|| = |1 - i| = 2^{\frac{1}{2}}$. Therefore

$$||Ta - Tb|| > ||Sa - Sb|| \,.$$

Hence T is not S-nonexpansive.

Now, let $x = (\xi_1, \xi_2, ...) \,, y = (\eta_1, \eta_2, ...) \in C$. Then we have

$$||Tx - Ty^2|| \leq 2^2|\xi_1 - \eta_1|^2 + \sum_{i=2}^{\infty} A_i^2|\xi_i - \eta_i|^2 \leq 2^2||x - y||^2,$$

since $0 < A_i < 1$ for all i. Therefore

$$||Tx - Ty|| \leq 2||x - y|| = 2||Sx - Sy|| \,.$$

Let $x \in C$. Then we have

$$T^n x = \left(0, ..., \prod_{i=2}^{n} A_i\xi_1^2, \prod_{i=2}^{n+1} A_i\xi_2, \prod_{i=3}^{n+2} A_i\xi_3, ..., \prod_{i=k}^{n+k-1} A_i\xi_k, ...\right) .$$

It follows by induction that

$$||T^n - T^n y|| \leq k_n||x - y|| = ||Sx - Sy||, \text{where } k_n = \prod_{i=2}^{n} A_i.$$

Therefore, T is asymptotically S-nonexpansive.

A Banach space X satisfies the Opial's condition [9] if for each sequence $\{x_n\}$ in X weakly convergent to a point x and for all $y \neq x$, we have

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||.$$

The example of Banach spaces which satisfy Opial's condition are Hilbert spaces and all $L_p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition [9].

Let C be a nonempty subset of a Banach space X, then a mapping S : C $\rightarrow$ C is said to be *demiclosed* provided whenever $\{x_n\} \subseteq C,\ x_n \rightarrow x$ weakly and $Sx_n \rightarrow y \in X$ then $Sx = y$.

A mapping S is said to be *weakly continuous* if $x_n \rightarrow x$ weakly, then $Sx_n \rightarrow Sx$ weakly.

We need the following lemmas:

Lemma 2.1. (see Schu [13]) *Let X be a uniformly convex Banach space and let $0 < a \leq \alpha_n \leq b < 1$ for all $n \in N$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that*

$$\limsup_{n\to\infty} ||x_n|| \leq r, \ \ \limsup_{n\to\infty} ||y_n|| \leq r$$

and

$$\lim_{n\to\infty} ||(1-\alpha_n)x_n + \alpha_n y_n|| = r$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Lemma 2.2. (see Osilike and Aniagbosor [10]) *Let $\{\delta_n\}$ be a sequence of nonnegative numbers satisfying:*

$$\delta_{n+1} \leq \beta_n \delta_n + \gamma_n \ \ \textit{for all } n \in N,$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of nonnegative numbers such that

$$\{\beta_n\} \subseteq [1, \infty)\,, \quad \sum_{n=1}^{\infty}(\beta_n - 1) < \infty, \quad \sum_{n=1}^{\infty}\gamma_n < \infty.$$

Then $\lim_{n\to\infty} \delta_n$ exists. If $\liminf_{n\to\infty} \delta_n = 0$, then $\lim_{n\to\infty} \delta_n = 0$.

3. Main Results

First we introduce the Mann type iteration process for two mappings:

Let C be a nonempty convex subset of a linear space X and let S, T : C $\rightarrow$ C be two mappings such that $(1-t)Sx + tT^n x \in S(C)$ for all $x \in C, t \in (0,1), n \in N$. Let $\{\alpha_n\}$ be a sequence in [a, b], where $0 < a < b < 1$. Then the sequence $\{Sx_n\}$ generated from arbitrary $x_1 \in C$ by

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n T^n x_n, \;\; n \in N\,, \tag{S}$$

will be called the modified Mann type iteration associated with mappings S and T.

If S=I, the identity mapping, then iteration process (S) reduces to the modified Mann type iteration process defined by (1.1) which was introduced by Schu [12].

We begin with the following lemma:

Lemma 3.1. *Let C be a nonempty subset of a normed space X and Let C be a nonempty convex subset of a linear space X and let S, $T : C \to C$ be two mappings such that:*

(i) $(1 - t)Sx + tT^n x \in S(C)$ *for all* $x \in C, t \in (0, 1), n \in N$*;*

(ii) T is uniformly L-Lipschitzian with respect to S;

(iii) $ST = TS$ *and* $S^2 = S$*.*

Let $\{, \alpha_n\}$ *be a sequence in [a, b], where* $0 < a < b < 1$*. Suppose* $\{x_n\}$ *is a sequence in C defined by such that* $\lim_{n\to\infty} ||Sx_n - T^n x_n|| = 0$*. Then* $\lim_{n\to\infty} ||Sx_n - Tx_n|| = 0$

Proof. For each $n \in N$, set $d_n := ||Sx_n - T^n x_n||$ and $L = \sup_{k\in N} k_n$. Then, we have

$$\begin{aligned} ||Sx_{n+1} - Tx_{n+1}|| &\le ||Sx_{n+1} - T^{n+1}x_{n+1}|| + ||T^{n+1}x_{n+1} - Tx_{n+1}|| \\ &\le d_{n+1} + L||Sx_{n+1} - ST^n x_{n+1}|| \le d_{n+1} + L(||Sx_{n+1} - Sx_n|| \\ &+ ||Sx_n - T^n x_n|| + ||T^n x_n - T^n Sx_{n+1}||) \le d_{n+1} + L(\alpha_n d_n + d_n \\ &+ L||Sx_n - S^2 x_{n+1}||) \le d_n + L\,(2 + L)\,d_n \to 0 \;\; \text{as } n \to \infty\,. \end{aligned}$$ □

Theorem 3.1. *Let C be a nonempty convex subset of a Banach space X and let S,* $T : C \to C$ *be two mappings such that:*

(i) T is S-asymptotically nonexpansive with sequence $\{k_n\}$ *such that* $\sum_{n=1}^{\infty} (k_n - 1) < \infty$*;*

(ii) $F(S) \cap F(T) = \phi$*;*

(iii) $(1 - t)Sx + tT^n x \in S(C)$ *for all* $x \in C, t \in (0, 1), n \in N$*.*

Suppose that $\{Sx_n\}$ *is the modified Mann type iteration process associated with S and T defined by*

$$\left.\begin{aligned} &x_1 \in C, \\ &Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n T^n x_n, \; n \in N, \end{aligned}\right\} \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $[a, b]$ with $0 < a < b < 1$. Then we have the following:

(a) $\lim_{n\to\infty} ||Sx_n - p||$ *exist if* $p \in F(T) \cap F(T)$*;*

(b) If X is uniformly convex, then $\lim_{n\to\infty} ||Sx_n - T^n x_n|| = 0$.

Proof. (a) Let $p \in F(S) \cap F(T)$. Then from (3.1), we have

$$||Sx_{n+1} - p|| = ||(1 - \alpha_n)(Sx_n - p) + \alpha_n(T^n x_n - p)||$$
$$\leq (1 - \alpha_n)(Sx_n - p) + \alpha_n||T^n x_n - p|| \leq k_n||Sx_n - p|| \text{ for all } \quad n \in \mathrm{N} \; .$$

Lemma 2.2 implies that $\{||Sx_n - p||\}$ is convergent.

(b) Let $p \in F(S) \cap F(T)$. By part (a), $\lim_{n\to\infty} ||Sx_n - p||$ exists. Since

$$\limsup_{n\to\infty} ||T^n x_n - p|| \leq \limsup_{n\to\infty} (k_n||Sx_n - p||) \leq \lim_{n\to\infty} ||Sx_n - p||$$

and

$$\lim_{n\to\infty} ||(1 - \alpha_n)(Sx_n - p) + \alpha_n(T^n x_n - p)|| = \lim_{n\to\infty} ||Sx_{n+1} - p||,$$

it follows from Lemma 2.1 that $\lim_{n\to\infty} ||Sx_n - T^n x_n|| = 0$. □

Now we are in the position to prove main results of this paper.

Theorem 3.2. *Let X be a uniformly convex Banach space satisfying the Opial condition and C a nonempty closed convex bounded subset of X. Let S,* $T : C \to C$ *be two mappings such that:*

(i) T is an S-asymptotically nonexpansive with sequence $\{k_n\}$ *such that* $\sum_{n=1}^{\infty} (k_n - 1) < \infty$*;*

(ii) $F(S) \cap F(T) \neq \phi$, $\quad ST = TS$ *and S is one-one;*

(iii) $(1 - t)Sx + tT^n x \in S(C)$ *for all* $x \in C, t \in (0, 1), n \in N$*;*

(iv) $||Tx - T^2x|| \neq \max\{||Tx - STx||, ||T^2x - STx||\}$*, whenever right-hand side is nonzero.*

Suppose that S is weakly continuous and S-T is demiclosed at zero. Then the sequence $\{Sx_n\}$ *in C defined by (3.1) converges weakly to an element of* $F(S) \cap F(T)$*.*

Proof. Since $\{x_n\} \subset C$ is bounded and X is reflexive, there exists a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to z$ weakly. Note that C is a closed and convex set of normed space X, it follows that C is weakly closed. Thus, $z \in C$. Since S is weakly continuous, it follows that $Sx_{n_i} \to Sz$ weakly. By Theorem 3.1 and Lemma 3.1, we have $\lim_{n\to\infty} ||Sx_n - Tx_n|| = 0$. Since S-T is demiclosed at zero, $Sz = Tz$. We assert that $TTz = Tz$. If not, from condition (iv), we have

$$||Tz - T^2z|| \neq \max\{||Tz - STz||, ||T^2z - STz||\} = ||Tz - T^2z||$$

a contradiction. So, $TTz = Tz = p$ (say). Thus, $Sp = STz = TSz = TTz = Tp = p$. Therefore, $Sx_{n_i} \to Sz$ weakly and $Sz \in F(S) \cap F(T)$.

To show that $\{x_n\}$ converges weakly, it suffices to show that $\omega_w(\{x_n\})$ consists of exactly one point, namely, z, where $\omega_w(\{x_n\})$ denotes the set of all weak subsequential limits of $\{x_n\}$. To this end, we suppose that $u \in \omega_w(\{x_n\})$ with $z \neq u$ So, $Sz \neq Su$. Then there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ such that $x_{n_j} \to u$ weakly. It is easy to see that $Sx_{n_j} \to Su$ weakly with $Su = Tu$. Suppose $Su = Tu = q$. It is easy to show that $Sq = Tq = q$. By the Opial condition, we have

$$\lim_{i\to\infty} ||Sx_n - Sz|| < \lim_{i\to\infty} ||Sx_{n_i} - Su||, \quad \lim_{j\to\infty} ||Sx_n - Su|| < \lim_{j\to\infty} ||Sx_{n_j} - Sz||,$$

a contradiction. Then $Sz = Su$. Therefore, $\{Sx_n\}$ converges weakly to a common fixed point of S and T. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let S, $T : C \to C$ be two mappings such that:*

(i) T is S-asymptotically nonexpansive with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$;

(ii) $F(S) \cap F(T) \neq \phi$, $ST = TS$ and $\overline{T(C)} \subseteq \overline{S(C)}$;

(iii) $(1-t)Sx + tT^n x \in S(C)$, and for all $x \in C, t \in (0,1), n \in N$;

(iv) $||Tx - T^2x|| \neq max\left\{||Tx - STx||,\ ||T^2x - STx||\right\}$, whenever right-hand side is nonzero.

Then the sequence $\{Sx_n\}$ defined by (3.1) converges strongly to a common fixed point of S and T if $\liminf_{n\to\infty} d\left(Sx_n, F(s) \cap (T)\right) = 0$, where $d(x, F(S) \cap (T)) = \inf\left\{||x - z|| : z \in F(S) \cap F(T)\right\}$.

Proof. From (3.2), we have

$$d\left(Sx_{n+1}, F(S) \cap F(T)\right) \leq k_n d\left(Sx_n, F(S) \cap (T)\right),$$

it follows from Lemma 2.2 that $\lim_{n\to\infty} d\left(Sx_n, F(s) \cap F(T)\right)$ exists. Since

$$\liminf_{n\to\infty} d\left(Sx_n, F(s) \cap F(T)\right) = 0,$$

it follows that $\lim_{n\to\infty} d\left(Sx_n, F(s) \cap F(T)\right) = 0$. Given $\epsilon > 0$ there exists a positive integer n_0 such that $d\left(Sx_n, F(S) \cap F(T)\right) < \epsilon/2$ for all $n \geq n_0$. Hence for $n, m \geq n_0$, we have

$$||Sx_n - Sx_m|| \leq ||Sx_n - p|| + ||Sx_m - p||, \ \forall\ p \in F(S) \cap F(T).$$

By taking infimum over $p \in F(S) \cap F(T)$ in above inequality, we have

$$||Sx_n - Sx_m|| \leq d\left(Sx_n, F(S) \cap F(T)\right) + d\left(Sx_m, F(S) \cap F(T)\right)$$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon \text{ for all } n, m \geq n_0 .$$

It follows that $\{Sx_n\}$ is Cauchy sequence in C and hence it converges strongly to a point $v \in C$, i.e., $\lim_{n\to\infty} Sx_n = v$. By Theorem 3.1(b) and Lemma 3.1, $\lim_{n\to\infty} ||Sx_n - Tx_n|| = 0$, hence $\lim_{n\to\infty} Tx_n = v$.

We now prove that v is common fixed point of S and T. Since $v \in \overline{T(C)}$ and $\overline{T(C)} \subset S(C)$, there exists u in C such that $v = Su$. Also T is asymptotically S-nonexpansive, we have

$$||Tx_n - Tu|| \leq k_1||Sx_n - Su||.$$

On letting $n \to \infty$ we get $Tu = v = Su$. Since $TSu = STu$ and $TTu = TSu = SSu$.

We claim that $v = Tu$ is common fixed point of S and T. If not, then by virtue of (iv) we get

$$||Tu - TTu|| \neq \max\left\{||Tu - STu||, ||T^2u - STu||\right\} = ||Tu - TTu|| ,$$

a contradiction. Hence $Tu = TTu$. Thus, $TTu = Tu = STu$, i.e., $v = Tu$ is a common fixed point of S and T. □

Remark 3.1. Theorems 3.2 and 3.3 extend several known results (see [1, 2, 3, 4, 6, 7, 8, 10, 11, 14, 15, 17, 18] and references therein) from asymptotically nonexpansive to asymptotically S-nonexpansive mappings.

References

[1] R.P. Agarwal, Donal O'Regan, D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, *J. Nonlinear Convex Analysis*, Accepted.

[2] S.S. Chang, Y.J. Cho, H. Zhou, Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings, *J. Korean Math. Soc.*, **38** (2001), 1245-1260.

[3] C.E. Chidume, B. Ali, Approximation of common fixed points for finite families of nonself asymptotically nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.*, In press.

[4] C.E. Chidume, E.U. Ofoedu, H. Zegeye, Strong and weak convergence theorems for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.*, **280** (2003), 364-374.

[5] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.*, **35**, No. 1 (1972), 171-174.

[6] S.H. Khan, H. Fukharuddin, Weak and strongconvergence of a scheme with errors for two nonexpansive mappings, *Nonlinear Analysis*, **61** (2005), 1295-1301.

[7] Z.Q. Liu, S.M. Kang, Weak and strong convergence for fixed points of asymptotically nonexpansive mappings, *Acta Math. Sinica* (2004), 1-10.

[8] K. Nammanee, M.A. Noor, S. Suantai, Convergence criteria of modified Noor iterations with errors for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.*, **314** (2006), 320-334.

[9] Z. Opial, Weak convergence of successive approximations for nonexpansive mappings, *Bul. Amer. Math. Soc.*, **73** (1967), 591-597.

[10] M.O. Osilike, S.C. Aniagbosor, Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings, *Mathematical and Computer Modelling*, **32** (2000), 1181-1191.

[11] B.E. Rhoades, Fixed point iterations for certain nonlinear mappings, *J. Math. Anal. Appl.*, **183** (1994), 118-120.

[12] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mapping, *J. Math. Anal.*, **159** (1991), 407-413.

[13] J. Schu, Weak and strong convergence of fixed points of asymptotically nonexpansive maps, *Bull. Austral. Math. Soc.*, **43** (1991), 153-159.

[14] H.F. Senter, W.G. Dotson, Jr., Approximating fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc.*, **44**, No. 2 (2974), 375-380.

[15] K.K. Tan, H.K. Xu, Fixed point iteration process for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.*, **122** (1994), 733-739.

[16] P. Vijayaraju, Applications of fixed point theorems to best simultaneous approximations, *Indian J. Pure Appl. Math.*, **24**, No. 1 (1993), 21-26.

[17] L. Wang, Strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings, *J. Math. Anal. Appl.*, In press.

[18] B.L. Xu, M.A. Noor, Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.*, **267** (2002), 444-453.