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OUTPUT FEEDBACK STABILIZATION OF NONLINEAR SYSTEMS WITH UNCERTAIN INTEGRATORS

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Abstract: We present a control design to solve the global output feedback stabilization problem of a class of uncertain nonlinear systems. Each system in this class consists of some dynamic uncertainties, linear parametric uncertainties and, most importantly, output-dependent uncertain integrator gains or virtual coefficients. To solve the problem, an observer design is proposed, and the robust adaptive backstepping and the nonlinear small gain techniques are applied to construct an output feedback dynamic controller. The controller guarantees that all the closed-loop signals are bounded and also the output of the systems can be made arbitrarily small. If extra conditions are imposed on the systems, regulation of the systems' states can also be achieved.

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1. Introduction

In dealing with the control design for uncertain strict feedback systems, there are two commonly used techniques; namely the adaptive backstepping (see, e.g., [8]), and the robust backstepping (see, e.g., [2]), techniques. In order to take advantage of each of the techniques, a natural approach is to merge them

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(see, e.g., [10] and [4]). Consequently, based on the combined approach, the control design developed by the adaptive backstepping is robustified and, at the same time, the information required for robust backstepping is reduced thanks to the adaptive capability.

The objective of this paper is to study the global output feedback stabilization problem of the class of uncertain nonlinear systems described by

$$\begin{aligned} \dot{\zeta} &= q(\zeta, y, t)\,, \\ \xi i &= a_i g_i \xi_{i+1} + \theta^T \phi_i(y) + w_i, \quad i = 1, 2, \cdots, n-1\,, \\ \xi n &= a_n g_n u + \theta^T \phi_n(y) + w_n\,, \\ y &= \xi_1\,, \end{aligned} \tag{1}$$

where $\xi = [\xi_1, \xi_2, \ \cdots, \xi_n]^T \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^q$, $u \in \mathbb{R}$ is the control, $y \in \mathbb{R}$ is the output, $\theta \in \mathbb{R}^m$ is an unknown constant parameter vector, and the vector function $\phi_i(y) = [\phi_{i,1}(y), \phi_{i,2}(y), \cdots, \phi_{i,m}(y)]^T$ $(i = 1, \cdots, n)$ is known and consists of smooth functions depending on y only. The uncertain system (1) contains not only the unknown parameters θ, the uncertain nonlinearity $w_i(\cdot)$ $(i = 1, 2, \cdots, n)$; but also the uncertain integrator gains $a_i g_i(y)$ $(i = 1, 2, \cdots, n)$. Each integrator gain $a_i g_i(y)$, for $i = 1, 2, \cdots, n$, consists of two components: the *unknown* scaling constant a_i and the *known* smooth function $g_i(y)$.

The systems previously studied in the works, e.g., [4, 3, 7, 6, 11] were assumed to be free of uncertainties in the integrator gains, i.e., the terms $a_i g_i(y)$, $i = 1, 2, \ldots, n$, in system (1) are *certain*. Therefore, the main contribution of this paper is that we allow the integrator gains in system (1) to be *uncertain* and solve its global stabilization problem using output feedback. This paper, in particular, extends the works [4, 3].

This paper is organized as follows: Section 2 contains some useful definitions, preliminary results and notation. In Section 3, the problem statement of this paper and some standing assumptions are stated. Since only the output of the system is measurable, we present an observer design for (1) in Section 4. In Section 5 a stabilizing controller is constructed for (1) to solve the proposed problem. Then, the main results of this paper are stated in Section 6. In order to demonstrate the usefulness of our design, an illustrative example is presented in Section 7. Finally, some concluding remarks are drawn in Section 8.

2. Preliminaries

Before proceeding, we provide some useful notation, definitions and results which are frequently employed throughout this paper.

Notation 2.1. The matrix N and vectors b, c are defined as: $N := \begin{bmatrix} \mathbf{0} & I \\ 0 & \mathbf{0}^T \end{bmatrix}$, $b := [0, \cdots, 0,\ 1]^T$, $c := [1,\ 0, \cdots, 0]^T$, respectively, where I is an identity matrix and $\mathbf{0}$ is a zero vector. The dimensions of N, b and c depend upon the context. $\operatorname{diag} x$ denotes a diagonal matrix with vector x on the diagonal. The usual Euclidean norm for vectors is denoted by $|\cdot|$. $|A|$ denotes the induced norm of matrix A, i.e., $|A| := \sup_{|x|=1}\{|Ax|\}$. A^T denotes the transpose of matrix A and $A_{i,j}$ represents the (i,j)-th component of matrix A. A partial vector $\vec{x}_i$ of $x \in \mathbb{R}^n$ is defined as $\vec{x}_i := [x_1, x_2, \cdots, x_i]^T$, where $i \le n$.

Notation 2.2. We say (see [1]): 1) A vector or a matrix function f of x has *lower triangular dependence (LTD) in* x if the i-th component or row is a function of $\vec{x}_i$ only. Then, f is said to be LTD in x. 2) A vector or a matrix function f of x has strictly *lower triangular dependence (SLTD) in* x if the i-th component or row is a function of $\vec{x}_{i-1}$ only[1]. Then, f is said to be SLTD in x. 3) A matrix is said to be *lower triangular* if its elements above the main diagonal are zero.

Definition 2.1. (Input-to-State Practically Stable, see [4])) A control system $\dot{x} = f(x,u)$ is *input-to-state practically stable (ISpS)* if there exist a function β of class[2] KL, a function γ of class K, and a nonnegative constant d such that, for any initial condition $x(0)$ and each measurable essentially bounded control $u(t)$ defined for all $t \ge 0$, the associated solution $x(t)$ exists for all $t \ge 0$ and satisfies

$$|x(t)| \le \beta(|x(0)|, t) + \gamma(||u_t||) + d, \tag{2}$$

where u_t is the truncated function of u at t and $||\cdot||$ stands for the L_∞ supremum norm.

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[1] That is, the first element or row is independent of x.

[2] A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to class K if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class ∞ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ is said to belong to class KL if, for each fixed s, the mapping $\beta(r,s)$ belongs to class K with respect to r and, for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$ (see [5, p. 144]).

3. Problem Statement

The purpose of this paper is to study the global output feedback stabilization problem of the systems described by (1). Throughout this paper, the following assumptions are imposed on (1).

Assumption 3.1. For $i = 1, \cdots, n$, the unknown scaling factor a_i in the integrator gain is assumed to be non-zero, and the sign of the product of them is assumed to be positive, namely $\prod_{i=1}^{n} a_i > 0$.

Assumption 3.2. For all $y \in \mathbb{R}$, the smooth output-dependent functions in the integrator gains satisfy the following conditions:

$$g_i(y) \geq \sigma > 0, \quad 1 \leq i \leq n; \quad g_{i+1}(y) \leq g_i(y), \quad 1 \leq i \leq n-2, \tag{3}$$

for some scalar $\sigma > 0$.

Assumption 3.3. For $i = 1, 2, \cdots, n$, each uncertain nonlinear function w_i satisfies

$$|w_i| \leq \nu_i^* \psi_{i1}(|y|) + \nu_i^* \psi_{i2}(|\zeta|), \quad 1 \leq i \leq n, \tag{4}$$

where ν_i^* is an unknown positive constant, $\psi_{i1}(\cdot)$ and $\psi_{i2}(\cdot)$, with $\psi_{i2}(0) = 0$, are known positive smooth functions.

Assumption 3.4. The ζ-dynamics in (1) is input-to-state practically stable (ISpS) and thus, it has an ISpS Lyapunov function V_ζ, i.e., for all $(\zeta, y, t) \in \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}^+$,

$$\begin{aligned} \alpha_l(|\zeta|) \leq V_\zeta \leq \alpha_u(|\zeta|)\,, \\ \frac{\partial V_\zeta}{\partial \zeta} q(\zeta, y, t) \leq -\alpha_\zeta(|\zeta|) + \gamma_\zeta(|y|) + d_\zeta, \end{aligned} \tag{5}$$

where $d_\zeta > 0$, α_ζ, α_l, α_u and γ_ζ are class K_∞ functions.

Under the above assumptions, our objective is to construct a robust adaptive output feedback controller

$$\dot{\chi} = \eta(\chi, y), \quad u = \gamma(\chi, y) \tag{6}$$

for system (1), so that the solutions of the closed-loop system $(\zeta,\ \xi,\ \chi)$ are globally uniformly bounded, and also the output y can be rendered arbitrarily small. Furthermore, with extra conditions imposed on (1), regulation of states ξ and ζ can also be achieved.

Remark 3.1. Assumption 3.1 can be relaxed so that only the sign of $\prod_{i=1}^{n} a_i$ is known. The conditions of $a_i > 0$ and $g_i(\cdot) > 0$ $(i = 1, 2, \cdots, n)$ in Assumptions 3.1 and 3.2 guarantee that (1) is always controllable. The second condition in Assumption 3.2 on g_i $(i = 1, 2, \cdots, n)$ allows us to construct an observer as shown later.

It is worthwhile to mention that the following example shows another reason why systems in the form of (1) are considered.

Example 3.1. Consider the nonlinear system

$$\begin{aligned}
\dot{\eta}_1 &= a_1\eta_2\,,\\
\dot{\eta}_2 &= a_2\eta_3 + a_1\psi_2(y)\eta_2^2 + \Psi_2^T(y)\theta\,,\\
\dot{\eta}_3 &= a_3\eta_4 + a_1\psi_3(y)\eta_2\eta_3 + \Psi_3^T(y)\theta\,,\\
&\vdots\\
\dot{\eta}_n &= a_n u + a_1\psi_n(y)\eta_2\eta_n + \Psi_n^T(y)\theta\,,\\
y &= \eta_1.
\end{aligned} \tag{7}$$

In system (7), the nonlinearities, which multiply with the unknown parameter a_1, i.e., $\psi_i(y)\eta_2\eta_i$ $(i = 2, 3, \cdots, n)$, depend not only on y, but also on the unmeasured state variables η_i $(i = 2, \cdots, n)$. Such nonlinearities violate a standard assumption of the existing adaptive output feedback designs (see, for instance, [8]). However, we will show that after a suitable transformation, the transformed system is a special case of (1).

We define a coordinate transformation as follows:

$$\xi_1 = \eta_1, \quad \xi_i = e^{-\int_0^y \psi_i(s)ds}\eta_i, \qquad i = 2, \cdots, n. \tag{8}$$

Therefore, the transformed system becomes

$$\begin{aligned}
\xi 1 &= a_1 e^{\int_0^y \psi_2(s)ds}\xi_2 &&=: a_1 g_1(y)\xi_2,\\
\xi 2 &= a_2 e^{-\int_0^y \psi_2(s)ds} e^{\int_0^y \psi_3(s)ds}\xi_3 + e^{-\int_0^y \psi_2(s)ds}\Psi_2^T(y)\theta &&=: a_2 g_2(y)\xi_3 + \phi_2^T(y)\theta,\\
\xi 3 &= a_3 e^{-\int_0^y \psi_3(s)ds} e^{\int_0^y \psi_4(s)ds}\xi_4 + e^{-\int_0^y \psi_3(s)ds}\Psi_3^T(y)\theta &&=: a_3 g_3(y)\xi_4 + \phi_3^T(y)\theta,\\
&\vdots &&\vdots\\
\xi n &= a_n e^{-\int_0^y \psi_n(s)ds} u + e^{-\int_0^y \psi_n(s)ds}\Psi_n^T(y)\theta &&=: a_n g_n(y)u + \phi_n^T(y)\theta,\\
y &= \xi_1.
\end{aligned} \tag{9}$$

System (9) turns out to be a special case of (1); in fact, it has a cascade of integrators with uncertain gains $a_i g_i(y)$ $(i = 1, 2, \cdots, n)$ and with the parametric uncertainties $\phi_i^T(y)\theta$ $(i = 2, \ldots, n)$. Furthermore, if, for all $y \in \mathbb{R}$, $g_1(y) \geq \cdots \geq g_{n-1}(y)$, then Assumption 3.2 holds.

4. An Observer Design

In this section, we propose an observer design for system (1). By utilizing this observer, a dynamic controller can then be constructed via the robust adaptive backstepping technique described in the following sections and, consequently, it solves the problem as stated in Section 3.

First, we define the scaling transformation

$$X_i := \xi_i / \prod_{j=i}^{n} a_j, \quad i = 1, \cdots n, \tag{10}$$

to eliminate the unknown constants in the integrator gains. By using transformation (10), and writing $X := [X_1, \cdots, X_n]^T$, system (1) becomes

$$\begin{aligned} \dot{\zeta} &= q(\zeta, y, t)\,, \\ \dot{X} &= G(y)(NX + bu) + \Delta(y, \zeta, t)\,, \\ y &= \gamma X_1, \end{aligned} \tag{11}$$

where $b = [0, \cdots, 0,\ 1]^T$, and $\alpha_i := \prod_{j=i}^{n} a_j$, $\gamma := \prod_{j=1}^{n} a_j$,
$\Delta(y, \xi, t) := [\alpha_1^{-1}(\theta^T \phi_1(y) + w_1), \alpha_2^{-1}(\theta^T \phi_2(y) + w_2), \cdots, \alpha_n^{-1}(\theta^T \phi_n(y) + w_n)]^T$.

Lemma 4.1. *Consider the matrix*

$$G(y) = \operatorname{diag}[g_1(y), g_2(y), \cdots, g_n(y)], \tag{12}$$

and suppose that the elements of matrix $G(y)$ satisfy Assumption 3.2. Then, there exist a positive scalar $\epsilon > 0$, a vector $l(y) = [l_1(y) \ \cdots l_n(y)]^T$, and a constant matrix $P = P^T > 0$ such that the inequality

$$P\left(G(y)N - l(y)c^T\right) + \left(G(y)N - l(y)c^T\right)^T P \leq -\epsilon I \tag{13}$$

holds for all $y \in \mathbb{R}$, where $c^T = [1,\ 0,\ \cdots,\ 0]$.

Proof. See Appendix A.1. □

By utilizing Lemma 4.1, we are able to construct an observer to estimate X, and it is defined as

$$\dot{\hat{X}} = (G(y)N - l(y)c^T)\hat{X} + G(y)bu, \quad \hat{X}(0) = 0, \tag{14}$$

where $\hat{X} := [\hat{X}_1, \cdots, \hat{X}_n]^T$ and $l(y) := [l_1(y), \cdots, l_n(y)]^T$. As a matter of fact, this observer is simply a generalization of the well-known Luenberger's observer (see, for instance, [9]) with time-varying gain $l(y)$ and matrix $G(y)N$.

Next, the state estimation errors are defined as

$$e_i := \frac{1}{k^*}(\hat{X}_i - X_i), \quad i = 1, 2, \cdots, n, \tag{15}$$

where $k^* > 0$ is an unknown scalar. This unknown scalar is defined as:

$$k^* := \max\left\{\alpha_1^{-1},\ \left(\frac{1+\pi}{\epsilon}\right)\alpha_i^{-1}|P||\theta|,\ \left(\frac{1+\pi}{\epsilon}\right)\alpha_i^{-1}|P|\nu_i^*,\ i=1,2,\cdots,n\right\}, \quad (16)$$

where ϵ and P are obtained from (13), and $\pi > 0$ is an arbitrary constant which will be defined later. As a consequence, the estimation error dynamics become:

$$\dot{e} = (G(y)N - l(y)c^T)e - \frac{1}{\gamma k^*}l(y)y - \frac{1}{k^*}\Delta(y,\zeta,t), \quad (17)$$

where $e := [e_1,\ e_2, \cdots, e_n]^T$.

By applying the scaling transformation (10) and the filtered transformation (15), the original system (1) becomes the following augmented system:

$$\begin{aligned}
\dot{\zeta} &= q(\zeta, y, t)\,,\\
\dot{e} &= (G(y)N - l(y)c^T)e - \frac{1}{\gamma k^*}l(y)y - \frac{1}{k^*}\Delta(y,\zeta,t)\,,\\
\dot{x} &= AG(y)(Nx + bu) + \Phi^T(y)\bar{\theta} + w(y,\zeta,t) + v(y,e)\,,\\
y &= x_1,
\end{aligned} \quad (18)$$

where $\bar{\theta} := [\theta,\ \gamma^{-1}]^T$, $A := \mathrm{diag}\,[\gamma,\ 1, \cdots, 1]$, $x := [y, \hat{X}_2, \hat{X}_3, \cdots, \hat{X}_n]^T$, $w(y,\zeta,t) := [w_1(y,\zeta,t),\ 0, \cdots, 0]^T$,

$$\Phi^T(y) := \begin{bmatrix} \phi_1(y)^T & 0 \\ 0 & -l_2(y)y \\ 0 & -l_3(y)y \\ \vdots & \vdots \\ 0 & -l_n(y)y \end{bmatrix}, \quad v(y,e) := \begin{bmatrix} -\gamma k^* g_1(y) e_2 \\ -k^* l_2(y) e_1 \\ -k^* l_3(y) e_1 \\ \vdots \\ -k^* l_n(y) e_1 \end{bmatrix}. \quad (19)$$

When defining the new state vector x, we have used y instead of using the estimate $\hat{X}_1$, since y can be related to $\hat{X}_1$ via $y = \gamma\hat{X}_1 - \gamma k^* e_1$. This is preferable because the output of the x-dynamics, i.e., y, becomes the input to the (ζ,e)-dynamics. In turns, the signals ζ and e can be treated as external dynamic disturbances acting on the x-subsystem.

By Assumption 3.4, $\zeta-$dynamics are ISpS. On the other hand, the stability of the error dynamics can be analyzed by defining a positive definite function

$$V_e(e) := \left(\frac{1+\pi}{\epsilon}\right) e^T P e, \quad (20)$$

where $\pi > 0$ is an arbitrary constant, which will be defined later, ϵ and P are obtained directly from (13).

After some manipulation (see Appendix A.2), the time derivative of V_e

in (20) can be over-bounded as follows:

$$\dot{V}_e \le -\pi|e|^2 + \delta_1(y)y^2 + \delta_2(|\zeta|^2) + d_1, \tag{21}$$

where the functions $\delta_1(\cdot)$ and $\delta_2(\cdot)$ are some smooth nonnegative functions both vanishing at zero, and additionally, $\delta_2(\cdot)$ is of class ∞.

5. Controller Design

In this section, we design a dynamic controller, using the robust adaptive backstepping technique, to render the $(e,\ x)$-dynamics in (18) ISpS with respect to the ζ dynamic disturbance input.

First, by using (20), we define a positive definite function

$$\begin{aligned} V_1 := V + \left(\frac{1+\pi}{\epsilon}\right) V_e = \frac{1}{2}\sum_{i=1}^{n} \eta_i(z_i^2) \\ + \frac{1}{2}\tilde{\theta}^T \Gamma_1^{-1}\tilde{\theta} + \frac{1}{2\lambda_1}\tilde{\gamma}^2 + \frac{1}{2\lambda_2}\tilde{p}^2 + \frac{1}{2\lambda_3}\gamma\tilde{r}^2 + \left(\frac{1+\pi}{\epsilon}\right) e^T P e\,, \end{aligned} \tag{22}$$

where $\eta_i = z_i^2$ for $2 \le i \le n$, $\eta_1(z_1^2)$ is a smooth class ∞ function, Γ_1 is a positive definite diagonal matrix, and λ_1, λ_2 and λ_3 are strictly positive scalars. At the moment, we do not specify what η_1 is, but it will be chosen at the later stage.

To perform backstepping, we define a nonlinear transformation as

$$\begin{aligned} z &= x + N^T \hat{R} f(x, \hat{\theta}, \hat{\gamma}, \hat{r}, \hat{p}), \\ 0 &= u + b^T \hat{R} f(x, \hat{\theta}, \hat{\gamma}, \hat{r}, \hat{p}), \end{aligned} \tag{23}$$

where $\hat{\gamma}$ is an estimate of γ, $\hat{r}$ is an estimate of $r := \gamma^{-1}$, whereas $\hat{p}$ is an estimate of the unknown positive constant $p := \max\{(\gamma k^*)^2,\ (k^*)^2,\ (\nu_1^*)^2\}$. Here, the diagonal matrix $\hat{R}$ is defined as $\hat{R} := \mathrm{diag}\,[\hat{r}, 1, \cdots, 1]$. The vector $f = [f_1, \cdots, f_n]^T$ is LTD in x and $\hat{r}$, and is SLTD in $\hat{\gamma}$. We also estimate the inverse of γ, i.e., r with an estimate of $\hat{r}$. This is because we will utilize $\hat{r}$, instead of $\hat{\gamma}^{-1}$, in the controller in order to preventing the division of zero. We call the nonlinear transformation (23) as the *backstepping transformation*.

As a result, by using transformation (23) and $\dot{V}_e$ (21), $\dot{V}_1$ (22) becomes

$$\begin{aligned} \dot{V}_1 = z^T SG\hat{A}Nz - z^T SGf + z^T SW\Phi^T\hat{\theta} + z^T SN^T\hat{R}\nabla_x f G \mathrm{diag}\, Nxc\hat{\gamma} \\ + z^T SW\Phi^T\tilde{\theta} + z^T S(cg_1 z_2 + N^T\hat{R}\nabla_x f G\mathrm{diag}\, Nxc)\tilde{\gamma} + z^T SG\gamma c f_1 \tilde{r} \\ + z^T SN^T\hat{R}(\nabla_{\hat{\theta}} f\dot{\hat{\theta}} + \nabla_{\hat{\gamma}} f\dot{\hat{\gamma}} + \nabla_{\hat{r}} f\dot{\hat{r}} + \nabla_{\hat{p}} f\dot{\hat{p}}) + z^T SN^T c f_1 \dot{\hat{r}} + z^T SW(w+v) \end{aligned}$$

$$-\tilde{\theta}^T\Gamma_1^{-1}\dot{\hat{\theta}} - \frac{1}{\lambda_1}\tilde{\gamma}\dot{\hat{\gamma}} - \frac{1}{\lambda_2}\tilde{p}\dot{\hat{p}} - \frac{1}{\lambda_3}\gamma\tilde{r}\dot{\hat{r}} - \pi|e|^2 + z^T\delta_1(y)cc^Tz + \delta_2(|\zeta|^2) + d_1\,, \quad (24)$$

where

$$S := \mathrm{diag}\,[\eta_1'(z_1^2), 1, \dots, 1], \quad \hat{A} := \mathrm{diag}\,[\hat{\gamma}, 1, \cdots, 1],$$
$$c := [1,\ 0, \cdots, 0]^T, \quad W := (I + N^T\nabla_x f).$$

The function $\eta_1'(z_1^2)$ is the value of derivative of η at z_1^2. We will later choose η_1 so that $\eta_1'(\cdot)$ is strictly positive.

Before choosing the update laws for the uncertain parameter estimates, the term $z^TSW(w+v)$ has to be analyzed first. By following the procedure presented in Appendix A.3, this term can be over-bounded by

$$z^TSW(w+v) \le z^TSDSzp + \rho_1 z^Tcc^Tz + \rho_2|e|^2 + \rho_3\psi_{12}^2(|\zeta|) + \rho_4\psi_{11}^2(0)\,, \quad (25)$$

where ρ_i $(i = 1,2,3,4)$ are some strictly positive constants and the diagonal matrix $D(\cdot)$ is LTD in x and SLTD in f.

Now, we are in a position to choose the update laws. The update laws, $\dot{\hat{\gamma}}$, $\dot{\hat{r}}$, $\dot{\hat{p}}$ and $\dot{\hat{\theta}}$, are chosen such that the terms involving $\tilde{\gamma}$, $\tilde{r}$, $\tilde{p}$ and $\tilde{\theta}$ in (24) are eliminated. In other words, we choose the update laws as

$$\begin{aligned}
\dot{\hat{\theta}} &= \Gamma_1(W\Phi^T)^TSz - \Gamma_1\sigma_\theta(\hat{\theta} - \theta_0)\,,\\
\dot{\hat{\gamma}} &= \lambda_1(g_1z_2c^T + (N^T\hat{R}\nabla_x fGx_2c)^T)Sz - \lambda_1\sigma_\gamma(\hat{\gamma} - \gamma_0)\,,\\
\dot{\hat{p}} &= \lambda_2 z^TSDSz - \lambda_2\sigma_p(\hat{p} - p_0)\,,\\
\dot{\hat{r}} &= \lambda_3 f_1c^TGSz - \lambda_3\sigma_r(\hat{r} - r_0),
\end{aligned} \quad (26)$$

where θ_0, a_0, p_0 and r_0 are some design parameters.

Essentially, by using $\dot{V}$ (24), the backstepping technique is to set up a vector $\mathcal{H}(x,f) := [\mathcal{H}_1(\vec{x}_1)\ \mathcal{H}_2(\vec{x}_2,\vec{f}_1)\dots\mathcal{H}_n(\vec{x}_n,\vec{f}_{n-1})]^T$, that is LTD in x and SLTD in f, such that the closed-loop system has a desired property. Due to the structure of $\mathcal{H}(x,f)$, each element of f can be evaluated recursively, via $f = \mathcal{H}(x,f)$, in the following way:

$$f_1 = \mathcal{H}_1(\vec{x}_1), \quad f_2 = \mathcal{H}_2(\vec{x}_2,\vec{f}_1), \quad \dots, \quad f_n = \mathcal{H}_n(\vec{x}_n,\vec{f}_{n-1}), \quad (27)$$

and hence a control u can be obtained via the transformation (23). From now on, if the vector $\mathcal{H}(x,f)$ is LTD in x and SLTD in f, we will say $\mathcal{H}(x,f)$ has the right structure. When substituting the update laws (23) into $\dot{V}_1$ (24), we have:

$$z^TSN^T\hat{R}\nabla_{\hat{\gamma}}f\dot{\hat{\gamma}} = z^TSN^T\hat{R}\nabla_{\hat{\gamma}}f\lambda_1((N^T\hat{R}\nabla_x fGx_2c)^T)Sz$$

$$
\begin{aligned}
&+ z^T S N^T \hat{R} \nabla_{\hat{\gamma}} f (\lambda_1 g_1 z_2 c^T S z - \lambda_1 \sigma_\gamma (\hat{\gamma} - \gamma_0)) \\
&=: z^T S U S z + z^T S N^T \hat{R} \nabla_{\hat{\gamma}} f (\lambda_1 g_1 z_2 c^T S z - \lambda_1 \sigma_\gamma (\hat{\gamma} - \gamma_0));
\end{aligned}
$$

$$
\begin{aligned}
z^T S N^T \hat{R} \nabla_{\hat{\theta}} f \dot{\hat{\theta}} &= z^T S N^T \hat{R} \nabla_{\hat{\theta}} f (W \Phi^T)^T S z - z^T S N^T \hat{R} \nabla_{\hat{\theta}} f \Gamma_1 \sigma_\theta (\hat{\theta} - \theta_0) \\
&=: z^T S Y S z - z^T S N^T \hat{R} \nabla_{\hat{\theta}} f \Gamma_1 \sigma_\theta (\hat{\theta} - \theta_0); \quad (28) \\
z^T S N^T \hat{R} \nabla_{\hat{p}} f \dot{\hat{p}} &= z^T S N^T \hat{R} \nabla_{\hat{p}} f \lambda_2 z^T S D S z - z^T S N^T \hat{R} \mathrm{grad}\, p \lambda_2 \sigma_p (\hat{p} - p_0) \\
&=: z^T S M S z - z^T S N^T \hat{R} \mathrm{grad}\, p \lambda_2 \sigma_p (\hat{p} - p_0),
\end{aligned}
$$

where

$$
\begin{aligned}
U &:= N^T \hat{R} \nabla_{\hat{\gamma}} f \lambda_1 ((N^T \hat{R} \nabla_x f G x_2 c)^T), \\
Y &:= N^T \hat{R} \nabla_{\hat{\theta}} f (W \Phi^T)^T, \\
M &:= N^T \hat{R} \nabla_{\hat{p}} f \lambda_2 z^T S D.
\end{aligned}
$$

However, the matrices Y, M and U in (28) do not have the right structure, i.e., they are not LTD in x and SLTD in f. In other words, these matrices cannot be used to construct $\mathcal{H}(x, f)$ directly. Fortunately, they can be decomposed in the following manner:

$$
U = U_l + U_u, \; Y = Y_l + Y_u, \; M = M_l + M_u, \quad (29)
$$

where U_l, Y_l and M_l are the lower triangular matrix parts, and U_u, Y_u and M_u are the strictly upper triangular matrix parts, of the matrices U, Y, and M, respectively. Due to the structure of U, Y and M, the matrices U_l, Y_l, M_l, U_u^T, Y_u^T, and M_u^T are in fact LTD in x and SLTD in f.

Consequently, by using (26)–(29), $\dot{V}_1$ (24) becomes

$$
\begin{aligned}
\dot{V}_1 \le z^T S \Big(& G \hat{A} N z - G f + D S z \hat{p} + \rho_1 S^{-1} c c^T z + S^{-1} \delta_1(y) c c^T z \\
& + W \Phi^T \hat{\theta} + N^T \hat{R} \nabla_x G \mathrm{diag}\, N x c \hat{\gamma} + N^T \hat{R} \nabla_{\hat{\gamma}} f \lambda_1 g_1 z_2 c^T S z \\
& + (N^T \hat{R} \nabla_{\bar{r}} f r + N^T \mathrm{diag}\, f c) \lambda_3 f_1 c^T G S z \\
& + (U_l + U_u^T) S z + (Y_l + Y_u^T) S z + (M_l + M_u^T) S z \\
& - (N^T \hat{R} \nabla_{\bar{r}} f r + N^T \mathrm{diag}\, f c) \lambda_3 \sigma_r (\hat{r} - r_0) \\
& - N^T \hat{R} \nabla_{\hat{\theta}} f \Gamma_1 \sigma_\theta (\hat{\theta} - \theta_0) - N^T \hat{R} \nabla_{\hat{\gamma}} f \lambda_1 \sigma_\gamma (\hat{\gamma} - \gamma_0) \\
& - N^T \hat{R} \nabla_{\hat{p}} f \lambda \sigma_p (\hat{p} - p_0) \Big) - \frac{\sigma_r}{2} \gamma \tilde{r}^2 + \frac{\sigma_r}{2} \gamma (r - r_0)^2 \\
- \frac{\sigma_\theta}{2} \tilde{\theta}^T \tilde{\theta} + \frac{\sigma_\theta}{2} |\theta - \theta_0|^2 & - \frac{\sigma_\gamma}{2} \tilde{\gamma}^2 + \frac{\sigma_\gamma}{2} (\gamma - \gamma_0)^2 - \frac{\sigma_p}{2} \tilde{p}^2 + \frac{\sigma_p}{2} (p - p_0)^2 \\
& - \pi |e|^2 + \delta_2(|\zeta|^2) + d_1 + \rho_2 |e|^2 + \rho_3 \psi_{12}^2(|\zeta|) + \rho_4 \psi_{11}^2(0). \quad (30)
\end{aligned}
$$

Next, we define a positive definite matrix

$$C(y) := \mathrm{diag}\,[c_1 v(y), c_2, \cdots, c_n], \tag{31}$$

where $c_i > 0$ $(i = 1, 2, \cdots, n)$ and $v(y) > 0$ for all $y \in \mathbb{R}$. With this positive definite matrix, we choose $f = \mathcal{H}(x, f)$, where

$$\begin{aligned}\mathcal{H}(x,f) := G^{-1}\Big(& C(y)z + S^{-1}N^T\hat{A}GSz + DSz\hat{p} + \rho_1 S^{-1}cc^Tz + S^{-1}\delta_1(y)cc^Tz \\ & + W\Phi^T\hat{\theta} + N^T\hat{R}\nabla_x fG\mathrm{diag}\,Nxc\hat{\gamma} + N^T\hat{R}\nabla_{\hat{\gamma}}f\lambda_1 g_1 z_2 c^T Sz \\ & + (N^T\hat{R}\nabla_{\bar{r}}fr + N^T\mathrm{diag}\,fc)\lambda_3 f_1 c^T GSz \\ & + (U_l + U_u^T)Sz + (Y_l + Y_u^T)Sz + (M_l + M_u^T)Sz \\ & - (N^T\hat{R}\nabla_{\bar{r}}fr + N^T\mathrm{diag}\,fc)\lambda_3\sigma_r(\hat{r} - r_0) \\ & - N^T\hat{R}\nabla_{\hat{\theta}}f\Gamma_1\sigma_\theta(\hat{\theta} - \theta_0) - N^T\hat{R}\nabla_{\hat{\gamma}}f\lambda_1\sigma_\gamma(\hat{\gamma} - \gamma_0) \\ & - N^T\hat{R}\nabla_{\hat{p}}f\lambda\sigma_p(\hat{p} - p_0)\Big).\end{aligned} \tag{32}$$

Remark 5.1. In fact, the decompositions in (29), and using the transpose of the upper diagonal parts of U, Y and M in the construction of vector $\mathcal{H}(x, f)$, can be associated with the tuning functions technique (see, e.g., [8]).

For a given $\eta_1(y^2)$, the smooth strictly positive function $v(y)$ is chosen such that

$$\eta_1(y^2) \le \eta_1'(y^2)v(y)y^2, \qquad \forall y \in \mathbb{R}. \tag{33}$$

We then pick $\pi > \rho_2$ in (22) and define $\tau := \pi - \rho_2$. Therefore, by using (32) and substituting $f = \mathcal{H}(x, f)$ (32) into (30), we have

$$\begin{aligned}\dot{V}_1 \le & -z^TSC(y)z - (\pi - \rho_2)|e|^2 - \frac{\sigma_\theta}{2}\tilde{\theta}^T\tilde{\theta} - \frac{\sigma_\gamma}{2}\tilde{\gamma}^2 - \frac{\sigma_p}{2}\tilde{p}^2 - \frac{\sigma_r}{2}\gamma\tilde{r}^2 \\ & + \frac{\sigma_r}{2}\gamma(r - r_0)^2 + \frac{\sigma_\theta}{2}|\theta - \theta_0|^2 + \frac{\sigma_\gamma}{2}(\gamma - \gamma_0)^2 + \frac{\sigma_p}{2}(p - p_0)^2 + d_1 + \rho_4\psi_{11}^2(0) \\ & + \delta_2(|\zeta|^2) + \rho_3\psi_{12}^2(|\zeta|) \le -\epsilon_2 V_1 + \delta_3(|\zeta|^2) + \kappa,\end{aligned} \tag{34}$$

where

$$\begin{aligned}\epsilon_2 := & \\ & \min\left\{2c_i,\ \frac{\tau}{(\frac{1+\pi}{\epsilon})\lambda_{\max}(P)},\ \frac{\sigma_\theta}{\lambda_{\max}(\Gamma_1^{-1})},\ \lambda_1\sigma_\gamma,\ \lambda_2\sigma_p, \lambda_3\sigma_r;\ 1 \le i \le n\right\},\end{aligned} \tag{35}$$

$\kappa > 0$ is some constant. The smooth function $\delta_3(|\zeta|^2)$ can be chosen to be class ∞ function and, at the same time, to satisfy $\delta_3(|\zeta|^2) \ge \delta_2(|\zeta|^2) + \rho_2\psi_{12}^2(|\zeta|)$, for all $\zeta \in \mathbb{R}^q$, since $\delta_2(0) = \psi_{12}(0) = 0$.

Finally, it is straightforward to show that $\mathcal{H}(x, f)$ is LTD in x and SLTD in f, and hence f_n can be evaluated recursively from $f = \mathcal{H}(x, f)$. Therefore, by applying the control

$$u = -f_n(x, \hat{\theta}, \hat{\gamma}, \hat{r}, \hat{p}), \tag{36}$$

and together with the inverse transformation of (23), the (e, x)-subsystem in (18) is rendered ISpS with respect to the disturbance ζ (cf. inequality (34)).

6. Main Results

From the previous section, we have designed a dynamic controller (given by (36), (26) and (14)) so that the (e, x)-subsystem in (18) is rendered ISpS with respect to the disturbance ζ. On the other hand, by Assumption 3.4, the uncertain ζ-dynamics in (18) are ISpS with respect to the input y. Therefore, we have an interconnected system with two ISpS subsystems, namely, (e, x)-subsystem and ζ-dynamics. Our next task is to robustify the control design, which presented in the previous section, so that the interconnected system satisfies a small-gain condition. In fact, the main idea is to choose an appropriate function η_1 in the ISpS Lyapunov function of (22) based on the nonlinear small-gain theory [3]. In order to determine the required η_1, we follow [3]. Consequently, by following the procedure presented in Sections 4–5, we have the following result.

Theorem 6.1. *If Assumptions 3.1–3.4 hold, the dynamic controller given by (36), (26) and (14) guarantees the global uniform boundedness of all closed-loop signals. Furthermore, if the bounds on the unknown parameters, ν_i^*, θ, a_i $(i = 1, \cdots, n)$ are known, the output y of system (18) can be steered to an arbitrarily small neighborhood of the origin.*

Proof. First of all, we focus on the (e, x)-subsystem in (18) by observing its $\dot{V}_1$ as shown in (34). For any given λ_1 with $0 < \lambda_1 < \epsilon_2$, we have $\dot{V}_1 \le -\lambda_1 V_1$ if

$$V_1 \ge \max\left\{ \frac{2}{\epsilon_2 - \lambda_1}\delta_5([\alpha_l^{-1}(V_\zeta(\zeta))]^2), \;\; \frac{2\kappa}{\epsilon_2 - \lambda_1} \right\}, \tag{37}$$

where α_l and V_ζ are obtained from (5). Next, for any given β of class ∞, we choose η_1 in (22) to satisfy

$$\beta^{-1} \circ \gamma_0(|y|) \le \frac{1}{4}\eta_1(y^2) + \epsilon_3, \quad \forall y \in \mathbb{R}, \tag{38}$$

where $\epsilon_3 > 0$ is an arbitrary scalar. At this stage, we have not yet defined the function β, but it will be chosen so that a simple contraction holds for the gains of the two subsystems of the interconnected system. We then move to the ζ-

dynamics. Similarly, for any given $\lambda_2 \in (0,1)$ and $\epsilon_4 > 0$, we have $\dot{\tilde{V}}_\zeta \leq -a(\tilde{V}_\zeta)$ if

$$\tilde{V}_\zeta \geq \max\left\{2\epsilon_4\alpha_u \circ \alpha_\zeta^{-1}\left(\frac{2\beta(V_1)}{1-\lambda_2}\right),\ 2\epsilon_4\alpha_u \circ \alpha_\zeta^{-1}\left(\frac{2d_\zeta + 2\beta(2\epsilon_3)}{1-\lambda_2}\right)\right\}, \quad (39)$$

where $\tilde{V}_\zeta := \epsilon_4 V_\zeta$, and a is a class ∞ function.

Let χ_1 and χ_2 be the gain functions of the (e,x)-subsystem and the ζ-dynamics, respectively. Therefore, by observing (37) and (39), these gain functions are defined by:

$$\chi_1(s) := \frac{2}{\epsilon_2 - \lambda_1}\delta_3([\alpha_l^{-1}(\frac{1}{\epsilon_4}s)]^2), \quad \chi_2(s) := 2\epsilon_4\alpha_u \circ \alpha_\zeta^{-1}\left(\frac{2}{1-\lambda_2}\beta(s)\right). \quad (40)$$

Therefore, if we choose the smooth function $\beta(\cdot)$ as

$$\beta(s) < \frac{1-\epsilon_4}{2} \circ \alpha_\zeta \circ \alpha_u^{-1} \circ \frac{1}{2}\alpha_l\left(\sqrt{\delta_3^{-1}(\frac{\epsilon_2-\lambda_1}{2})s}\right), \qquad \forall s > 0, \qquad (41)$$

then the composition of the gain functions in (40) is a contraction mapping, i.e.,

$$\chi_1 \circ \chi_2(s) = \chi_1(\chi_2(s)) < s, \qquad \forall s > 0. \qquad (42)$$

Thus, the small-gain condition is satisfied and the solutions of the interconnected system are uniformly bounded. From (37) and (39), the residual set depends on the constants $\frac{2\kappa}{\epsilon_2-\lambda_1}$ and $2\epsilon_4\alpha_u \circ \alpha_\zeta^{-1}(\frac{2d_\zeta+2\beta(2\epsilon_3)}{1-\lambda_2})$, and these constants can be rendered small if the control parameters are chosen appropriately. This completes the proof of Theorem 6.1. □

Similar to [3], if certain additional conditions hold for the original system (1), regulation of the states ξ and ζ in (1) can also be achieved. This result is stated as follows:

Theorem 6.2. *If system (1) satisfies Assumptions 3.1–3.4 together with the following extra conditions:*

$$\limsup_{s\to 0^+}\frac{\gamma_\zeta(s)}{s^2} < +\infty; \quad \limsup_{s\to 0^+}\frac{\psi_{i2}^2(s)}{\alpha_\zeta(s)} < +\infty; \qquad (43)$$

and

$$\phi_i(0) = 0; \psi_{i1}(0) = \psi_{i2}(0) = d_\zeta = 0, \quad \text{for } i = 1, \cdots, n, \qquad (44)$$

then the dynamic controller given by (36), (26) and (14) can be modified so that it guarantees the states ξ and ζ in (1) asymptotically converging to 0, while maintaining the boundedness of the rest of the signals in the resulting closed-loop system.

Proof. With $\phi_i(0) = 0$ and $\psi_{i1}(0) = 0$ $(i = 1, 2, \cdots, n)$, the positive constant d_1 in $\dot{V}_e$ (21) becomes 0. Also, if the control parameters: σ_θ, σ_γ, σ_p and σ_r are set to zero, then $\dot{V}_1$ (34) turns out to be

$$\dot{V}_1 \le -c_1\eta_1(y^2) - \sum_{i=2}^{n} c_i z_i^2 - \tau|e|^2 + 6\sum_{i=1}^{n}\psi_{i2}^2(|\zeta|) + \rho_3\psi_{12}^2(|\zeta|). \tag{45}$$

We follow the lines of [3] to show the regulation property. First, we define a new positive definite and proper function for the ζ-dynamics in the following form:

$$U(\zeta) = \int_0^{V_\zeta(\zeta)} q(t)dt, \tag{46}$$

where $q : [0, \infty) \to [0 : \infty)$ is a smooth non-decreasing function, which also satisfies $q(t) > 0$ for all $t > 0$. With this new function U, we have

$$\begin{aligned} \dot{U} &\le q(V_\zeta(\zeta)(-\alpha_\zeta(|\zeta|) + \gamma_\zeta(|y|)) \\ &\le -\frac{1}{2} q \circ \alpha_l(|\zeta|)\alpha_\zeta(|\zeta|) + q \circ \alpha_u \circ \alpha_\zeta^{-1} \circ 2\gamma_\zeta(|y|\gamma_\zeta(|y|). \end{aligned} \tag{47}$$

Next, we define a new positive definite function

$$V_2 = V_1 + U \tag{48}$$

for the interconnected system and, by using (45) and (47), its time derivatives can be written as

$$\begin{aligned} \dot{V}_2 \le -c_1\eta_1(y^2) - \sum_{i=2}^{n} c_i z_i^2 - \tau|e|^2 + 6\sum_{i=1}^{n}\psi_{i2}^2(|\zeta|) + \rho_3\psi_{12}^2(|\zeta|) \\ - \frac{1}{2} q \circ \alpha_l(|\zeta|)\alpha_\zeta(|\zeta|) + q \circ \alpha_u \circ \alpha_\zeta^{-1} \circ 2\gamma_\zeta(|y|\gamma_\zeta(|y|). \end{aligned} \tag{49}$$

By using the fact that $\limsup_{s\to 0^+} \psi_{i2}^2(s)/\alpha_\zeta(s) < +\infty$, we can choose the function $q(t)$ in (46) such that

$$\frac{1}{4} q \circ \alpha_l(|\zeta|)\alpha_\zeta(|\zeta|) \ge 6\sum_{i=1}^{n}\psi_{i2}^2(|\zeta|) + \rho_3\psi_{12}^2(|\zeta|), \quad \forall\zeta \in \mathbb{R}^q. \tag{50}$$

Once the function $q(t)$ is chosen, we will then use it to choose η_1 in V_1 (22) as it has not yet been defined. By observing $\dot{V}_2$ (49) and using the condition that $\limsup_{s\to 0^+} \gamma_\zeta(s)/s^2 < +\infty$, we choose η_1 such that

$$\frac{1}{2} c_1\eta_1(y^2) \ge q \circ \alpha_u \circ \alpha_\zeta^{-1} \circ 2\gamma_\zeta(|y|\gamma_\zeta(|y|), \quad \forall y \in \mathbb{R}. \tag{51}$$

With the chosen q and η_1 functions, $\dot{V}_2$ (49) becomes

$$\dot{V}_2 \le -\left(\frac{1}{2}c_1\eta_1(y^2) + \sum_{i=2}^{n} c_i z_i^2 + \tau|e|^2 + \frac{1}{4}q\circ\alpha_l(|\zeta|)\alpha_\zeta(|\zeta|)\right)$$
$$=: -W_0(y, z_2, \cdots, z_n, e, \zeta), \quad (52)$$

where $W_0(\cdot)$ is a positive definite function. We can then conclude that all the closed-loop signals are bounded, and also $\lim_{t\to\infty}\int_0^t W_0(s)ds < +\infty$.

Through the use of transformation (23), it is not hard to show that the state vector $x = [y, \hat{X}_2, \cdots, \hat{X}_n]^T$ is bounded. Since the signals x, z, u, e and ζ are bounded, the boundedness of $\dot{x}$ and $\dot{z}$ can be established via (18), (23) and (26). Furthermore, boundedness of $\dot{\zeta}$ and $\dot{e}$ can be shown from (18). Thus, by applying Barbalat's Lemma (see, e.g., [5]), we have $\lim_{t\to\infty} W_0(t) = 0$ which then gives $\lim_{t\to\infty}(|y(t)| + |z(t)| + |e(t)| + |\zeta(t)|) = 0$. Hence, by using transformation (23) and observing (18), $x(t)$ and $\hat{X}(t)$ converge to the origin as $t \to \infty$. Finally, based on the above argument together with transformations (10) and (15), states ξ and ζ can be regulated, namely $\lim_{t\to\infty}|\xi(t)| = 0$ and $\lim_{t\to\infty}|\zeta(t)| = 0$. This completes the proof of Theorem 6.2. □

7. An Illustrative Example

In this section, we provide an illustrative example to demonstrate the usefulness of our design presented in this paper. Consider the following second-order uncertain nonlinear system:

$$\dot{\eta}_1 = a_1\eta_2; \quad \dot{\eta}_2 = a_2 u + a_1\psi_2(y)\eta_2^2 + \Psi_2(y)\theta, \quad y = \eta_1, \quad (53)$$

where a_1, a_2, and θ are unknown constant parameters with a_1 and a_2 are strictly positive. The functions $\psi_2(y) = \cos y$ and $\Psi(y) = y$ are known. We assume that the only measurable variable is the output y. It is clear that the uncertain nonlinear term $a_1\psi_2(y)\eta_2^2$ in the second equation of (53) consists of not only the output y, but also the unmeasured state η_2. Such nonlinearity violates the assumption imposed by [8] for their adaptive output feedback control design. However, as mentioned in Example 3.1, system (53) can be transformed into the form of (1) as shown in Example 3.1. Furthermore, the transformed system satisfies Assumptions 3.1–3.2. Therefore, by applying Theorem 6.2, we can construct an output feedback adaptive controller, presented in Sections 4 and 5, to regulate the states η_1 and η_2 to zero and, at the same time, all the closed-loop internal signals remain bounded for all time.

In the process of constructing the output feedback controller, we construct an observer (14) with $g_1(y) = e^{\sin y}$, $g_2(y) = e^{-\sin y}$, $l_1(y) = 2.5e^{\sin y} + 0.1e^{-2}$,

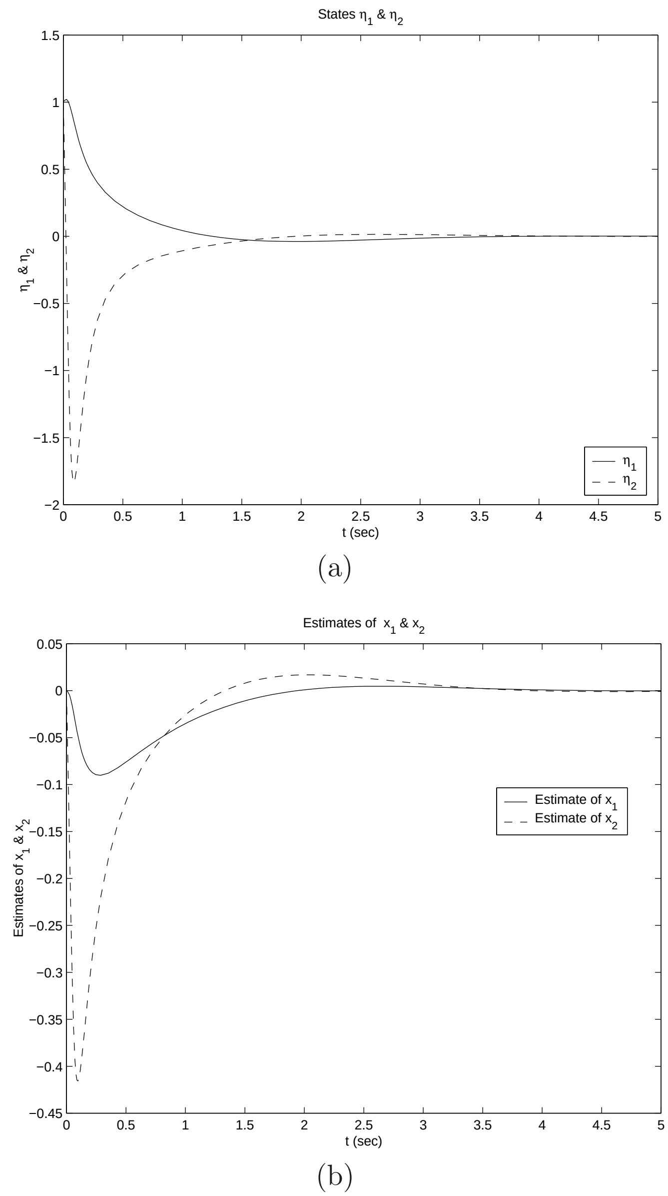
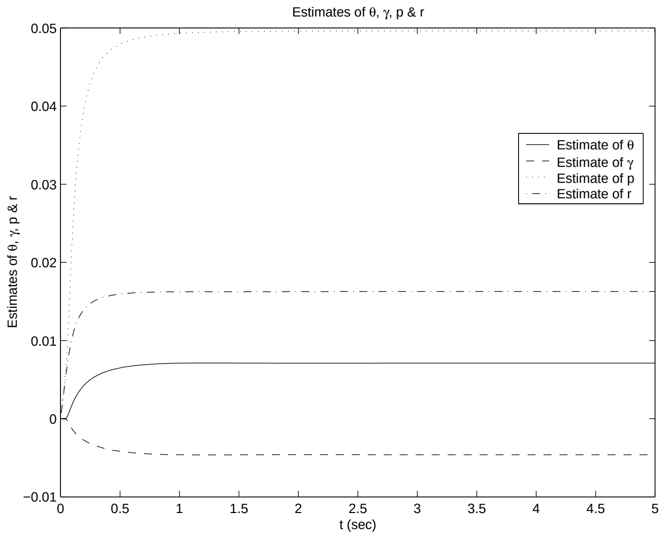
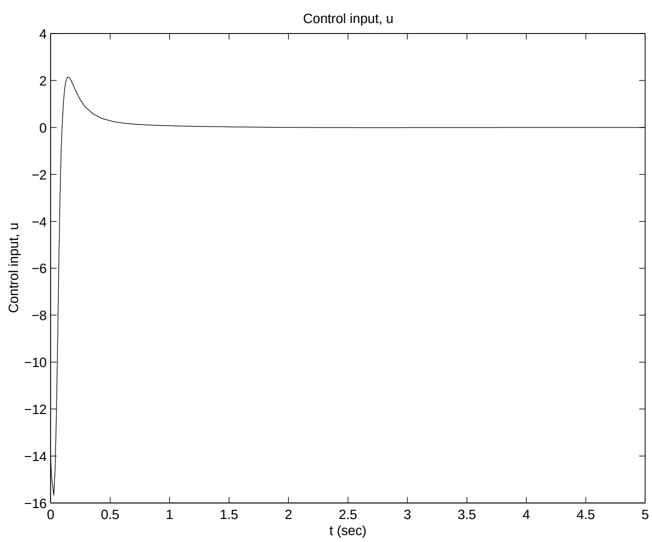


Figure 1: (a) Time responses of states η_1 and η_2, (b) time responses of observer states $\hat{X}_1$ and $\hat{X}_2$



(a)



(b)

Figure 2: (a) Time responses of $\hat{\theta}$, $\hat{\gamma}$, $\hat{p}$ and $\hat{r}$, (b) time responses of control input u

$l_1(y) = e^{\sin y} + 0.3e^{-3}$. Next, an output feedback adaptive controller is constructed by using this observer and following the design procedure in Section 5. For simulation purposes, we set the system, and the control, parameters as follows: $a_1 = 2, a_2 = 3, \theta = -1$, $c_1 = c_2 = 0.001$, $\Gamma = \text{diag}\,[0, 1]$, $\lambda_1 = 1$, $\lambda_2 = 0.1$, $\lambda_3 = 0.001$. Furthermore, the initial values of the closed-loop system are defined as: $\eta_1(0) = \eta_2(0) = 1$, $\hat{X}_1(0) = \hat{X}_2(0) = 0$, $\hat{\theta} = 0$, $\hat{\gamma}(0) = \hat{p}(0) = \hat{r}(0) = 0$. Finally, the time response of each closed-loop signal as well as the control input are shown in Figures 1–2. It is clear that all the signals are bounded, and also the regulation of states η_1 and η_2 is achieved.

8. Conclusion

We have presented a control design to solve the global output feedback stabilization problem of a class of uncertain nonlinear systems consisting of uncertain integrator gains. This class of systems includes, as special cases, various other systems considered previously in the literature. To solve the stabilization problem, an observer design has been proposed, and then a controller has been constructed by utilizing such an observer together with the robust adaptive backstepping technique. In this paper, the uncertain output-dependent integrator gains are assumed to be linearly parametrized, and this assumption allows us to construct an observer for the systems via a scaling procedure. However, if the uncertain integrator gains are nonlinearly parametrized, the construction of an observer becomes challenging. Therefore, it remains to develop a control design for systems with nonlinearly parametrized, output-dependent uncertain integrator gains.

Acknowledgments

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A. Appendices

A.1. Proof of Lemma 4.1

Proof. First of all, we define a system

$$\dot{\tilde{x}} = (G(y)N - l(y)c^T)\tilde{x}. \tag{54}$$

Next, define the following state-space transformation for system (54):

$$z = \tilde{x} - N^T S\tilde{x} =: T^{-1}(s)\tilde{x}, \tag{55}$$

where S is a diagonal matrix defined as $S := \mathrm{diag}\,[s_1, s_2, \cdots, s_{n-1}, 0]$, $s_i > 0$ $(i = 1, 2, \cdots, n-1)$, and

$$T(s) := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ s_1 & 1 & 0 & \cdots & 0 \\ s_1 s_2 & s_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ s_1 s_2 \cdots s_{n-1} & s_2 \cdots s_{n-1} & s_3 \cdots s_{n-1} & \cdots & 1 \end{bmatrix}. \tag{56}$$

It is clear that the constant matrix $T(s)$ is SLTD in s. Here, we replace $f(\tilde{x})$ with $-S\tilde{x}$. By observing the matrix T in (55), we have, for $i, j = 1, 2, \cdots, n$

$$T_{i,j} \begin{cases} = 1, & \text{when } i = j\,, \\ = 0, & \text{when } i < j\,, \\ = T_{i,j}(s_j, \cdots, s_{i-1}), & \text{otherwise.} \end{cases} \tag{57}$$

This observation is very useful when evaluating s_i, $i = 1, 2, \cdots, n$. Then, we define a positive definite function $V := z^T z/2$, and its time derivative becomes

$$\dot{V} = z^T G(y) N T(s) z - z^T N^T S G(y) N T(s) z - z^T (I - N^T S) l(y) z_1. \tag{58}$$

The first two terms on the right-hand side of (58) can be written as

$$z^T G(y) N T(s) z = z^T \begin{bmatrix} g_1(y) T_{2,1} \\ g_2(y) T_{3,1} \\ \vdots \\ g_{n-1}(y) T_{n,1} \\ 0 \end{bmatrix} z_1 + z^T \begin{bmatrix} g_1(y) T_{2,2} z_2 \\ g_2(y)(T_{3,2} z_2 + T_{3,3} z_3) \\ \vdots \\ g_{n-1}(y)(T_{n,2} z_2 + \cdots + T_{n,n} z_n) \\ 0 \end{bmatrix}$$

$$=: z^T U_1(y, s) z_1 + z^T U_2(y, s, z), \tag{59}$$

and

$$\begin{aligned} &- z^T N^T S G(y) N T(s) z \\ &= -z^T \begin{bmatrix} 0 \\ s_1 g_1(y) T_{2,1} \\ s_2 g_2(y) T_{3,1} \\ s_3 g_3(y) T_{4,1} \\ \vdots \\ s_{n-1} g_{n-1}(y) T_{n,1} \end{bmatrix} z_1 - z^T \begin{bmatrix} 0 \\ 0 \\ s_2 g_2(y) T_{3,2} z_2 \\ s_3 g_3(y)(T_{4,2} z_2 + T_{4,3} z_3) \\ \vdots \\ s_{n-1} g_{n-1}(y)(T_{n,2} z_2 + \cdots + T_{n,n-1} z_{n-1}) \end{bmatrix} \end{aligned}$$

$$-z^T \begin{bmatrix} 0 \\ s_1 g_1(y) z_2 \\ s_2 g_2(y) z_3 \\ s_3 g_3(y) z_4 \\ \vdots \\ s_{n-1} g_{n-1}(y) z_n \end{bmatrix} =: -z^T V_1(y,s) z_1 - z^T V_2(y,s,z) - z^T D(y,s) z. \quad (60)$$

The terms U_2 and V_2 in (59) and (60), respectively, can be over-bounded as follows:

$$z^T U_2(y,s,z) \leq \sum_{i=1}^{n-1} \sum_{j=2}^{i+1} \left(\frac{g_i(y) z_i^2}{2} + \frac{g_i(y) T_{i+1,j}^2 z_j^2}{2} \right), \quad \text{since } g_i(\cdot) > 0,$$
$$=: z^T \Delta_3(y) z + z^T \Delta_4(y,s) z, \quad (61)$$

where

$$\Delta_3(y) := \mathrm{diag}\left[g_1(y)/2,\ 2g_2(y)/2, \cdots, (n-1) g_{n-1}(y)/2,\ 0 \right],$$
$$\Delta_4(y,s) := \mathrm{diag}\left[0,\ \frac{1}{2} \sum_{j=1}^{n-1} g_j(y) T_{j+1,2}^2, \cdots, \frac{1}{2} \sum_{j=n-1}^{n-1} g_j(y) T_{j+1,n}^2 \right]. \quad (62)$$

Similarly,

$$-z^T V_2(y,s,z) \leq \sum_{i=2}^{n-1} \sum_{j=2}^{i} \left(\frac{g_i(y) z_{i+1}^2}{2} + \frac{g_i(y) s_i^2 T_{i+1,j}^2 z_j^2}{2} \right), \quad \text{since } g_i(\cdot) > 0$$
$$=: z^T \Delta_1(y) z + z^T \Delta_2(y,s) z, \quad (63)$$

where

$$\Delta_1(y) := \mathrm{diag}\left[0,\ 0,\ g_2(y)/2,\ 2g_3(y)/2, \cdots, (n-2) g_{n-1}(y)/2 \right],$$
$$\Delta_2(y,s) := \mathrm{diag}\left\{ [0,\ \frac{1}{2} \sum_{j=2}^{n-1} g_j(y) s_j^2 T_{j+1,2}^2, \cdots, \frac{1}{2} \sum_{j=n-1}^{n-1} g_j(y) s_j^2 T_{j+1,n-1}^2,\ 0] \right\}. \quad (64)$$

By Assumption 3.2 (i.e., $0 < \sigma \leq g_{i+1}(y) \leq g_i(y)$, for all y and $i = 1, \cdots, n-2$), there exists a diagonal matrix $\Delta(y,s)$ such that

$$z^T (\Delta_1(y) + \Delta_3(y)) z + z^T (\Delta_2(y,s) + \Delta_4(y,s)) z \leq z^T \Delta(y,s) z, \quad (65)$$

where

$$\Delta(s,y) := \mathrm{diag}\left[g_1(y)\beta_1,\ g_1(y)\beta_2,\ g_2(y)\beta_3, \cdots, g_{n-2}(y)\beta_{n-1},\ g_{n-1}(y)\beta_n \right] \quad (66)$$

with $\beta_i = \beta_i(s_i, \cdots, s_{n-1})$ $(i = 2, \cdots, n-1)$, β_1 and β_n are some positive constants which are independent of s. Specifically, the scalars $\beta_1, \cdots, \beta_n$ are defined as follows:

$$\beta_1 = 2, \ \ \beta_n = \frac{n-1}{2}, \\ \beta_i(s_i, \cdots, s_{n-1}) = (i-1) + \frac{1}{2}\sum_{j=i-1}^{n-1} T_{j+1,i}^2 + \frac{1}{2}\sum_{j=i}^{n-1} s_j^2 T_{j+1,i}^2, \tag{67}$$

where $i = 2, \cdots, n-1$.

As a consequence, equation (58) is in the form of

$$\dot{V} \le z^T \Delta(y,s) z - z^T D(s) z \\ - z^T \left((I - N^T S) l(y) + V_1(y,s) - U_1(y,s) \right) z_1. \tag{68}$$

Therefore, each s_i $(i = 1, \cdots, n)$ is chosen in reverse order, i.e., starting with s_{n-1} which is followed by s_{n-2}, until we get s_1. In other words, s_i $(i = 1, \cdots, n)$ is chosen as follows:

$$\begin{aligned} s_{n-1} &= k_n + \beta_n\,, \\ s_{n-2} &= k_{n-1} + \beta_{n-1}(s_{n-1})\,, \\ s_{n-3} &= k_{n-2} + \beta_{n-2}(s_{n-2}, s_{n-1})\,, \\ &\vdots \\ s_1 &= k_2 + \beta_2(s_2, \cdots, s_{n-1}), \end{aligned} \tag{69}$$

where, for $i = 2, \cdots, n$, k_i is an arbitrary positive constant.

Next, the vector $l(y)$ is chosen as

$$l(y) = N^T S l(y) - V_1(y,s) + U_1(y,s) + ck_1 + cg_1(y)\beta_1, \tag{70}$$

where $k_1 > 0$ and $c = [1, \ 0, \cdots, 0]^T$. Since the right-hand side of (70) is LTD in l, therefore each l_i, for $i = 1, \cdots, n$, can be chosen recursively. Then, equation (68) becomes

$$\dot{V} \le -k_1 z_1^2 - \sum_{i=2}^{n} k_i g_{i-1}(y) z_i^2 \le -k_1 z_1^2 - \sum_{i=2}^{n} k_i \sigma z_i^2 \le -k_1 z^T z, \tag{71}$$

if we choose $k_1 = \min_{2 \le i \le n}\{\sigma k_i\}$, where σ is from Assumption 3.2.

Finally, from the transformation (55), we can define

$$P := \frac{1}{2} T^{-T} T^{-1}, \ \ \epsilon := k_1 \lambda_{\min}(2P). \tag{72}$$

Therefore, by defining a Lyapunov function $V = \tilde{x}^T P \tilde{x}$ for (54), the vector $l(y)$ (70) and the constant $\epsilon > 0$ (72) satisfy the inequality (13). This completes the proof of Lemma 4.1. □

When comparing our proof with the one presented in [7], our approach is clearer and more explicit in demonstrating the procedure for the construction of the vector $l(y)$.

A.2. Derivation of Inequality (21)

By using equations (20), (18) and (13), and the inequality, $|ab| \leq 2|a|^2 + 2|b|^2$ for all a and b, it can be shown that

$$\begin{aligned}
\dot{V}_e &\leq -\pi|e|^2 + 3\Big(\sum_{i=1}^{n} l_i^2(y)y^2 + \sum_{i=1}^{n}\sum_{j=1}^{m} \phi_{i,j}^2(y) + 2\sum_{i=1}^{n} \psi_{i1}^2(|y|)\Big) + 6\sum_{i=1}^{n} \psi_{i2}^2(|\zeta|) \\
&\leq -\pi|e|^2 + 3\Big(\sum_{i=1}^{n} l_i^2(y)y^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{m} \tilde{\phi}_{i,j}^2(y)y^2 + 4\sum_{i=1}^{n} \tilde{\psi}_{i1}^2(y)y^2\Big) \\
&\quad + 6\sum_{i=1}^{n} \psi_{i2}^2(|\zeta|) + 6\Big(\sum_{i=1}^{n}\sum_{j=1}^{m} \phi_{i,j}^2(0) + \sum_{i=1}^{n} \psi_{i1}^2(0)\Big) \\
&\leq -\pi|e|^2 + \delta_1(y)y^2 + \delta_2(|\zeta|^2) + d_1, \quad (73)
\end{aligned}$$

where $\tilde{\phi}_{i,j}$ and $\tilde{\psi}_{i1}$ are some smooth functions chosen to be vanishing at zero, and also to satisfy $\phi_{i,j}^2(y) \leq 2\tilde{\phi}_{i,j}^2(y)y^2 + 2\phi_{i,j}^2(0)$ and $\psi_{i1}^2(|y|) \leq 2\tilde{\psi}_{i1}^2(y)y^2 + 2\psi_{i1}^2(0)$ for all $y \in \mathbb{R}$. Additionally, $\delta_2(\cdot)$ is chosen to be of class ∞. Both functions δ_1 and δ_2 can be computed explicitly thanks to the introduction of the scalar k^* when defining the errors (15).

A.3. Derivation of Inequality (25)

First, since ψ_{11} is smooth, we can construct a smooth positive function $\hat{\psi}_{11}$ such that $\psi_{11}(|y|) \leq |y|\hat{\psi}_{11}(y) + \psi_{11}(0)$. By using such $\hat{\psi}_{11}(y)$, it can be shown that

$$\begin{aligned}
z^T SWw \leq p\sum_{i=1}^{n}\Big(\frac{\hat{\psi}_{11}^2(y)}{4\epsilon_1} + \frac{1}{4\epsilon_2} + \frac{1}{4\epsilon_3}\Big)S_{i,i}^2 W_{i,1}^2 z_i^2& \\
+ n\epsilon_1 y^2 + n\epsilon_2\psi_{11}(0) + n\epsilon_3\psi_{12}^2(|\zeta|)\,,& \quad (74)
\end{aligned}$$

where $p := \max\{(\gamma k^*)^2,\ (k^*)^2,\ (\nu_1^*)^2\}$, and $\epsilon_i > 0$ $(i = 1, 2, 3)$ are arbitrary constants. Similarly, we can show that

$$z^T SWv \leq \frac{p}{4\epsilon_4}\sum_{i=1}^{n} z_i^2 S_{i,i}^2 \sum_{j=1}^{i} W_{i,j}^2 \psi_{j4}^2(y) + \epsilon_4 \sum_{i=1}^{n}\sum_{j=1}^{i} |e|^2, \quad (75)$$

where $\psi_{14} := g_1(y)$, $\psi_{i4} := l_i(y)$ for $i = 2, \cdots, n$. Therefore, we combine (74) and (75), and hence inequality (25) is obtained. Specifically, each element of the diagonal matrix D is defined by $D_{i,i} := \left(\frac{\hat{\psi}_{11}^2(y)}{4\epsilon_1} + \frac{1}{4\epsilon_2} + \frac{1}{4\epsilon_3}\right) W_i^T c c^T W_i + \frac{1}{4\epsilon_4} W_i^T (\operatorname{diag}[\psi_{14}^2, \cdots, \psi_{i4}^2, 0 \cdots, 0]) W_i$ for $i = 1, \cdots, n$, where W_i^T is the i-th row of the matrix W and W_i is the i-th column of the matrix W^T.