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ACM REDUCIBLE CURVES

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Abstract: Here we prove the existence of arithmetically Cohen-Macaulay nodal curves $X = X_1 \cup X_2 \subset \mathbb{P}^r$, X_1 and X_2 connected, $p_a(X_1) = p_a(X_2) = 0$, $h^1(X, \mathcal{O}_X(2)) = 0$ for almost all triples $(p_a(X), \deg(X_1), \deg(X_2))$ compatible with these restrictions.

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1. Introduction

Here we prove the existence of many arithmetically Cohen-Macaulay embeddings into projective spaces of certain reducible curves. We recall that a *two-component curve* is a projective, connected and nodal curve with 2 irreducible components, bot of them smooth. Let $C = C_1 \cup C_2$ be a two-component curve. Its abstract topological type is uniquely determined (up to a choice of an ordering of its irreducible components) by the triple of integers $(p_a(C_1), p_a(C_2), \sharp(C_1 \cap C_2))$. We have $g = p_a(C_1) + p_a(C_2) + \sharp(C_1 \cap C_2)$. Any $L \in \mathrm{Pic}(C)$ has a bidegree $\underline{\deg}(L) := (\deg(L|C_1, \deg(L|C_2)) \in \mathbb{Z}^{\oplus 2}$ which uniquely determines the topological type of L. Obviously, $\deg(L) = \deg(L|C_1) + \deg(L|C_2))$. The two-component curve C is called *binary* if $p_a(C_1) = p_a(C_2) = 0$. In general we are not able to prove the existence of arithmetically Cohen-Macaulay embedding of binary curves. In the next result we allow that C_1 and/or C_2 may be reducible

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(but connected, nodal and of arithmetic genus 0).

We prove the following result.

Theorem 1. *Fix integers r, c, g, d_1, d_2 such that $r \geq 4$, $0 \leq c \leq r-2$, $d_1 \geq r$, $d_2 \geq r$, $d_1 + d_2 = g + r - c$, and*

$$c(2r - c + 3)/2 \leq g \leq r(r-1)/2 + 2c\,. \tag{1}$$

Then there is a nodal, non-degenerate and arithmetically Cohen-Macaulay curve $C = C_1 \cup C_2 \subset \mathbb{P}^r$ such that $p_a(C) = g$, $\deg(C_i) = d_i$, $i = 1, 2$, each C_i is connected, $p_a(C_1) = p_a(C_2) = 0$, $\langle C_1 \rangle = \langle C_2 \rangle = \mathbb{P}^r$, each C_i has at most $r-2$ irreducible components and every irreducible component of C is smooth.

In step (a) of the proof of Theorem 1 (i.e. the case $r = 3$) we will show that it may fail if we drop the assumption "$d_1 \geq r$ and $d_2 \geq r$". If $r \geq 4$, then this assumption may be weakened (see the inductive cases with D formed by one component in steps (c) and (d) of the proof of Theorem 1 and the case in which one of the component of D is a line in step (e)).

Obviously without this condition we must drop the condition "$\langle C_1 \rangle = \langle C_2 \rangle = \mathbb{P}^r$" for the curve C_i of degree $< r$.

2. Proofs and Another Existence Result

We need the following well-known lemma (the so-called Horace Lemma) ([2]).

Lemma 1. *Let $H \subset \mathbb{P}^r$ be a hyperplane. Fix any projective scheme $T \subset \mathbb{P}^r$. Let $\mathrm{Res}_H(T)$ be the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_T : \mathcal{I}_H$ as its ideal sheaf. Then*

$$h^i(\mathbb{P}^r, \mathcal{I}_T(t)) \leq h^i(\mathbb{P}^r, \mathcal{I}_{\mathrm{Res}_{H(T)}}(t-1)) + h^i(H, \mathcal{I}_{T \cap H, H}(t))$$

for all integers $i \geq 0$ and $t \geq 0$.

Remark 1. Let $C_1 \subset \mathbb{P}^3$ be any integral non-degenerate curve of degree 5 and $C_2 \subset \mathbb{P}^3$ any smooth conic. Since C_1 is non-degenerate, the plane $\langle C_2 \rangle$ contains at most 5 points of C_1. Hence $\mathbb{P}^3$ contains no binary curve of genus 5 and bidegree $(5, 2)$.

Remark 2. Let $C_1 \subset \mathbb{P}^3$ be any integral non-degenerate curve of degree 1 and $C_2 \subset \mathbb{P}^3$ any line. Fix a general $P \in C_1$, Since C_1 is non-degenerate, the plane $\langle C_2 \cup \{P\} \rangle$ contains at most 6 points of C_1. Hence $\sharp(C_1 \cap C_2) \leq 5$. Hence $\mathbb{P}^3$ contains no binary curve of genus 5 and bidegree $(5, 2)$.

Proof of Theorem 1. (a) Here we assume assume $r = 3$. Fix a smooth quadric surface $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. Since $r = 3$, $c \in \{0, 1\}$. First assume $c = 0$. We need to cover the following pairs (d, g): $(3, 0)$, $(4, 1)$, $(5, 2)$, $(6, 3)$. In each case need to cover all bidegrees (d_1, d_2) such that $d_1+d_2 = d$ and $1 \leq d_2 \leq \lfloor d/r \rfloor$. We list first the cases which are arithmetically Cohen-Macaulay for trivial reasons. For the pair $(3, 0)$ we only have the bidegree $(2, 1)$. In this case we take as C_1 a smooth plane conic and as C_2 any line intersecting C_1, but not in the plane $\langle C_1 \rangle$. For the pair $(4, 1)$ we have the bidegrees $(2, 2)$ and $(3, 1)$. In the first case we take as C_1 and C_2 two smooth conics lying in different planes, but with $\sharp(C_1 \cap C_2) = 2$; in this case $C_1 \cup C_2$ is the complete intersection of two quadrics, one of them being the union of the planes $\langle C_i \rangle$, $i = 1, 2$. For the bidegree $(3, 1)$ we are forced to take as C_1 a rational normal curve and as C_2 a secant line of C_1; all these configurations are projectively equivalent and we see that $C_1 \cup C_2$ is the complete intersection of two smooth quadrics, just constructing one such configuration on Q. For the pair $(5, 2)$ we have the bidegrees $(3, 2)$ and $(4, 1)$. Both bidegress are obtained as a reducible divisor of type $(3, 2)$ on Q (respectively as union of a curve of type $(2, 1)$ and a curve of type $(1, 1)$ and of a curve of type $(3, 1)$ and a curve of type $(1, 1)$; in both cases C is arithmetically Cohen-Macaulay, because $h^1(C, \mathcal{O}_C(1)) = 0$ and Q is the unique quadric surface containing C. For the pair $(6, 3)$ we have 3 bidegrees: $(3, 3)$, $(4, 2)$ and $(5, 1)$. By Castelnuovo-Mumford's Lemma in this case it is sufficient to prove the existence of a non-special binary curve $C \subset \mathbb{P}^3$ contained in no quadric surface. For the bidegree $(3, 3)$ and genus 2 notice that no such configuration may be contained in a smooth quadric surface or in a quadric cone, in the cone case notice that the image of $C_1 \cup C_2$ by the linear projection from a points O is not a conic, unless $O \in C_1 \cap C_2$ and both C_1 and C_2 are contained in a quadric cone Q' with vertex O. The blowing-up Q'' of the vertex O of Q' is isomorphic to the Hirzebruch surface F_2. Since C_i is smooth, its strict transform in Q'' is isomorphic to it. In this way we see that $C_1 \cup C_2$ is the complete intersection of Q' and a cubic surface. Thus $p_a(C) = 4$, contradiction. For the bidegree $(4, 2)$ and the genus 2 we take as C_1 a general smooth rational curve of degree 4. Thus C_1 is contained in a unique quadric surface A and A is smooth. Since A contains no genus 2 binary curve of bidegree $(4, 2)$, C is contained in no quadric and hence it is arithmetically Cohen-Macaulay. For the bidegree $(5, 1)$ we start with a general smooth rational curve $C_1 \subset \mathbb{P}^3$ of degree 5. No quadric surface contains C_1, because it has maximal rank (see [2]); C_2 has infinitely many trisecant lines which are not 4-secants and we take one of them as C_2.

Now assume $c = 1$. We need to cover the following pairs (d, g): $(6, 4)$ and

$(7,5)$. For the pair $(6,4)$ the bidegrees are $(3,3)$, $(4,2)$ and $(5,1)$. All of them are realized on Q with C complete intersection of Q and a cubic surface.

For the pair $(7,5)$ the bidegrees are $(4,3)$, $(5,2)$ and $(6,1)$. In all cases to check the vanishing of $h^1(\mathcal{I}_C(t))$ for all $t \geq 3$ is easy. Hence in each case we just check the vanishing of $h^1(\mathcal{I}_C(2))$, i.e. (Riemann-Roch) the vanishing of $h^0(\mathcal{I}_C(2))$. The bidegree $(4,3)$ is realized in the following way. Let $C_2 \subset \mathbb{P}^3$ be any smooth rational normal curve intersecting transversally Q. Take as C_1 any smooth curve of type $(3,1)$ of Q containing these 6 points; this is possible for general $C_2 \cap Q$, because $h^0(Q, \mathcal{O}_Q(3,1)) = 8$. Since Q is the unique quadric surface containing C_1 and Q does not contains C_2, $h^0(\mathcal{I}_C(2)) = 0$. The bidegree $(5,2)$ is realized in the following way. Let $C_1 \subset \mathbb{P}^3$ be a general smooth rational curve of degree 5. Since C_1 has maximal rank, no quadric surface contains C_1. The bidegrees $(5,2)$ and $(6,1)$ cannot be realized by Remarks 1 and 2.

(b) From now on we assume $r \geq 4$ and that Theorem 1 is true for the integer $r' := r-1$. Fix a bidegree (d_1, d_2) and a hyperplane $H \subset \mathbb{P}^r$.

(c) Here we assume $c \leq r-3$ and $g \leq (r-1)(r-2)/2 + 3c$. Set $c' := c$, $r' := r-1$ and $g' := g-c$. Since $c'(2r'-c'+3)/2 = c(2r-c+3)/2 - c$, we have $g' \geq c'(2r'-c'+3)/2$. By assumption we have $g' = g-c \leq (r-1)(r-2)/2+2c$. Hence we may apply the inductive assumption with the datum (r', g', c'). Fix an admissible bidegree (b_1, b_2) for the datum (r', g', c'). Hence $b_1 \geq r-1$ and $b_2 \geq r-1$. Let $Y \subset H$ be an arithmetically Cohen-Macalulay nodal curve $Y = Y_1 \cup Y_2$ such that $p_a(Y) = g'$, $h^1(Y, \mathcal{O}_Y(1)) = c$, $\deg(Y_i) = b_i$, each Y_i is connected, $p_a(Y_i) = 0$ and $\langle Y_1 \rangle = \langle Y_2 \rangle = H$. We will add a connected and nodal curve D of degree $c+1$ which is either a rational normal curve or a binary curve of genus 0, say $D = D_1 \cup D_2$. In the first case we write $D = D_1 \cup D_2$ with either $D_1 = \emptyset$ and $D_2 = D$ or $D_1 = D$ and $D_2 = \emptyset$. Set $a_i := \deg(D_i)$. Hence $a_1 + a_2 = c+1$. We first cover the bidegree (b_1+c+1, b_2) under the assumption $b_2 \geq c$. Fix $S_2 \in\subset (Y_2)_{reg}$ such that $\sharp(S_2) = c$ and $\dim(\langle S_2 \rangle) = c-1$; this is possible, because $\langle Y_2 \rangle = H$ and $c \leq r-1$. Fix a general $P \in Y_1$. Set $S := S_2 \cup \{P\}$. Since P is general, $\langle Y_1 \rangle = H$ and $c \leq r-1$, $\dim(\langle S \rangle) = c$. Let $V \subset \mathbb{P}^r$ be a linear space such that $V \cap H = \langle S \rangle$. Let D be a rational normal curve of V containing S. Since $\deg(D) = c+1$, S is the scheme-theoretic intersection of D and H. Here we used the case $A_2 = 0$. Notice that $Y_1 \cup D$ is a connected and nodalcurve with degree $b_1 + c + 1$ and with arithmetic genus 0. Since $(Y \cup D) \cap H = Y$ (scheme-theoretically, $h^1(\mathbb{P}^r, \mathcal{I}_D(t-1)) = 0$ for all $t \geq 2$ and $h^1(H, \mathcal{I}_{Y,H}(t)) = 0$ for all $t \geq 1$, Lemma 1 shows that $Y \cup D$ is arithmetically Cohen-Macaulay. Hence $Y_1 \cup D$ is a solution of Theorem 1 for these numerical data. In the same way taking $a_1 = 0$, we cover the bidegrees

$(b_1, b_2 + c + 1)$ if $b_1 \ge c$ and $\langle Y_2 \rangle$ is a linear space of dimension at least c; we even assumed $b_1 \ge r - 1$ and $\langle Y_2 \rangle = H$. Now assume $a_1 > 0$ and $a_2 > 0$. We have $a_1 + a_2 = c + 1$. We need $b_1 \ge a_2$, $b_2 \ge a_2$, a. We fix a general $Q_i \in Y_i$ and a general $B_i \subset Y_i$ such that $\sharp(B_i) = 2 - i$, $i = 1, 2$. Set $S_1 := B_2 \cup \{Q_1\}$ and $S_2 := B_1 \cup \{Q_2\}$. We assume $D_i \cap H = S_i$. Set $E_i := Y_i \cup D_i$. Each E_i is a nodal curve of arithmetic genus 0 and $\sharp(E_1 \cap E_2) = g + 1$.

(d) Here we assume $c \le r - 3$ and $(r-1)(r-2)/2 + 3c < g \le r(r-1)/2 + 2c$. Set $c' := c$, $r' := r - 1$ and $g' := (r-1)(r-2)/2 + 2c$. Notice that $g - g' \ge c$. Use the inductive assumption in H with respect to the datum (r', g', c'). Take a bidegree (b_1, b_2) obtained in this way. We do the same costruction with the curve $D = D_1 \cup D_2$ as in step (c), and the integers a_1, a_2 with $D \cap H \subset Y$. If $a_i = 0$ then we take D_{2-i} intersecting Y_{2-i} in one point and Y_i in $g - g'$ points. If $a_1 > 0$ and $a_2 > 0$, then we take $\sharp(D_1 \cap Y_1) = 1$, $\sharp(D_2 \cap Y_2) = 1$, $\sharp(D_1 \cap Y_2) = a_1 - 1$ and $\sharp(D_2 \cap Y_1) = a_2 - 1$.

(e) Here we assume $c = r - 2$. Set $c' := c - 1 = r - 3$, $r' := r - 1$ and $g' := g - r - 1$. We have $g' \ge c'(2r - c' + 3)/2$. Hence we may use the inductive assumption in H with respect to the datum (r', g', c'). Here D has always two irreducible components D_1, D_2, $\sharp(D_1 \cap D_2) = 2$, $\sharp(D_1 \cap Y_1) = 1$, $\sharp(D_2 \cap Y_2) = 1$, $\sharp(D_1 \cap Y_2) = a_1 - 1$ and $\sharp(D_2 \cap Y_1) = a_2 - 1$. We may apply Lemma 1. □

Apply the case $r' := r - 1$ of [1], Theorem 2, and then use the inductive step given in the proof of [1], Theorem 2. We get the following result.

Proposition 1. *Fix integers r, c, g such that $r \ge 4$, $0 \le c \le r - 2$ and*

$$c(2r - c + 3)/2 \le g \le r(r-1)/2 + 2c\,. \tag{2}$$

Then there is a nodal, non-degenerate and arithmetically Cohen-Macaulay curve $C \subset \mathbb{P}^r$ such that $p_a(C) = g$, $\deg(C) = g + r - c$, $h^1(C, \mathcal{O}_C(1)) = c$, $h^1(C, \mathcal{O}_C(2)) = 0$, $h^1(C, N_C) = 0$, and C has two irreducible components say $C = C_1 \cup C_2$, both of them being smooth. If $c \le r-3$ and $g \le (r-1)(r-2)/2 + 3c$, then we may take $C = C_1 \cup C_2$ such that $p_a(C_2) = 0$, $\deg(C_2) = c + 1$, $p_a(C_1) = g - c$ and hence $\sharp(C_1 \cap C_2) = c + 1$ and $\deg(C_2) = g + r - 2c - 1$. If $c \le r - 3$ and $(r-1)(r-2)/2 + 3c < g \le r(r-1)/2 + 2c$, then we may take $C = C_1 \cup C_2$ such that $p_a(C_2) = 0$, $\deg(C_2) = r$, $p_a(C_1) = g - c$ and hence $\sharp(C_1 \cap C_2) = r$ and $\deg(C_2) = g - c$. If $r = c - 2$, then we may take $C = C_1 \cup C_2$ such that $p_a(C_2) = 1$, $\deg(C_2) = r + 1$, $p_a(C_1) = g - r - 1$ and hence $\deg(C_1) = g - c - 1$ and $\sharp(C_1 \cap C_2) = r + 1$.

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