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NUMERICAL SOLUTION OF TWO-POINT BOUNDARY VALUE PROBLEMS WITH MIXED BOUNDARY CONDITIONS USING WEIGHTED RESIDUAL METHOD

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Abstract: In this paper, we examined the two-point boundary value problems of the form

$$L(y) = f(x, y)$$

subject to the mixed boundary conditions

$$y'(a) - cy(a) = A\,,$$
$$y'(b) + dy(b) = B\,,$$

where L is the differential operator (linear or non-liear) involving spartial derivatives of y, $c \geq 0$, $d \geq 0$, $c+d > 0$, A and B are constants and and $x \in [a, b]$. These equations are solved using weighted residual method. An approximating function with some constants is assumed to satisfy the boundary conditions. These constants are determined using various methods such as Galerkin method, collocation method, partion method, moment method and least-squares method. The results obtained from each method were compared with the analytical solutions.

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1. Introduction

Boundary value problems occur in various aspects of engineering, physics and mathematics. Various methods, including finite difference methods [3] and spline methods [4] to mention a few, have been used to tackle this class of problems. Aregbesola [2] obtained a numerical solution to this class of problem using two-point Taylor series. The series, involving a set of parameters, is made to satisfy both the differential equation and the associated boundary conditions. By using the same number of points uniformly spread in the range of integration as the number of parameters to be determined, equal number of equations as the parameters to be determined can be obtained and these equations can then be solved iteratively.

The purpose of this paper is to obtain a numerical solution using the weighted residual method. The method presented will reduce the work in the numerical solution of the boundary value problem to solving a system of linear simultaneous equations.

2. Description of the Method

The method of weighted residuals is a general method of obtaining solutions to equations in form of trial functions, which are specified, but with adjustable constants (or functions), which are chosen to give the best solution to the differential equation.

Consider the boundary value problem

$$L(y) = f(x, y), \quad x \in D, \tag{2.1}$$

$$M(y) = \mu, \quad x \in \partial D, \tag{2.2}$$

where L denotes a general differential operator involving spatial derivatives of y, M represents the appropriate number of boundary conditions and D is domain with boundary ∂D.

Let the trial function be of the form

$$y_n = \phi_0 + \sum_{i=1}^{n} C_i \phi_i \,, \tag{2.3}$$

where ϕ_0 must satisfy all the specified boundary conditions of the problem and ϕ_i must satisfy the homogeneous form of the specified boundary conditions.

Substitution of the approximation (2.3) into the operator equation (2.1) results in a residual (an error in the equation)

$$R_n = L(y_n) - f(x, y_n) \neq 0\,. \tag{2.4}$$

If $R_n = 0$, then the trial function is the exact solution.

In weighted residual method the parameters $C_{i's}$ are dertermined by setting the integral (over the domain) of weighted residual of the approximation to zero.

$$\int_D w_i(x, y) R_n(x, y, C_i) dxdy = 0, \quad i = 1, 2, ..., n\,, \tag{2.5}$$

where $w_{i's}$ are weight functions. The weighting functions can be chosen in many ways and each choice corresponds to a different criterion of method of weighted residuals.

(i) The Galerkin Method (GM)

In this method, the weighting functions are given by

$$w_i = \phi_i, \quad i = 1, 2, ..n. \tag{2.6}$$

(ii) The Collocation Method (CM)

The collocation method seeks approximate solution y_n to (2.1) in the form (2.3) by requiring the residual in the equation to be identically zero at n selected points $x_i = (x^i, y^i)$ $i = 1, 2, ..., n$ in the domain D:

$$R_n(x^i, y^i, C_i) = 0, \quad i = 1, 2, ..., n\,. \tag{2.7}$$

(iii) The Moment Method (MM)

The weighting functions w_i are chosen as $1, x, x^2, ...$ for the differential equations.

(iv) The Partition Method (PM)

In PM we divide the domain D into n smaller subdomains, D_i and choose

$$w_i = \begin{cases} 1, & x \in D_i\,, \\ 0, & x \notin D_i\,. \end{cases} \tag{2.8}$$

The differential equation, integrated over the subdomain, is then zero.

(v) The Least-Squares Method (LSM)

In this method the weighting functions are given by

$$w_i = \frac{\partial R_n}{\partial C_i}, \quad i = 1, 2, ..., n\,. \tag{2.9}$$

3. Numerical Examples

Example 1. We consider the differential equation

$$y'' = y - 4xe^x , \tag{3.1}$$

subject to the mixed boundary conditions

$$y'(0) - y(0) = 1 , \qquad y'(1) + y(1) = -e. \tag{3.2}$$

The analytical solution is

$$y(x) = x(1-x)e^x . \tag{3.3}$$

Due to the nature of the boundary conditions, the trial functions are carefully chosen such that the boundary conditions are satisfied.

The first four trial functions considered are given below

$$\begin{aligned} y_1 &= -\frac{1}{3}(2+e) + \frac{1}{3}(1-e)x + C_1(x^2 - x - 1) , \\ y_2 &= -\frac{1}{3}(2+e) + \frac{1}{3}(1-e)x + C_1(x^2 - x - 1) + C_2(x^3 - \frac{4}{3}x - \frac{4}{3}) , \\ y_3 &= -\frac{1}{3}(2+e) + \frac{1}{3}(1-e)x + C_1(x^2 - x - 1) + C_2(x^3 - \frac{4}{3}x - \frac{4}{3}) \\ &\quad + C_3(x^4 - \frac{5}{3}x - \frac{5}{3}) , \\ y_4 &= -\frac{1}{3}(2+e) + \frac{1}{3}(1-e)x + C_1(x^2 - x - 1) + C_2(x^3 - \frac{4}{3}x - \frac{4}{3}) \\ &\quad + C_3(x^4 - \frac{5}{3}x - \frac{5}{3}) + C_4(x^5 - 2x - 2). \end{aligned} \tag{3.4}$$

Equations (3.4) can be gotten by making equation (2.3) to satisfy the boundary conditions (3.2).

Substituting (3.4) into (3.1), we obtain the residuals as

$$\begin{aligned} R_1 &= \frac{1}{3}(2+e) - \frac{1}{3}(1-e)x + 4xe^x + C_1(3 + x - x^2) , \\ R_2 &= \frac{1}{3}(2+e) - \frac{1}{3}(1-e)x + 4xe^x + C_1(3 + x - x^2) + C_2(\frac{4}{3} + \frac{22}{3}x - x^3) , \\ R_3 &= \frac{1}{3}(2+e) - \frac{1}{3}(1-e)x + 4xe^x + C_1(3 + x - x^2) + C_2(\frac{4}{3} + \frac{22}{3}x - x^3) \\ &\quad + C_3(\frac{5}{3} + \frac{5}{3}x + 12x^2 - x^4) , \\ R_4 &= \frac{1}{3}(2+e) - \frac{1}{3}(1-e)x + 4xe^x + C_1(3 + x - x^2) + C_2(\frac{4}{3} + \frac{22}{3}x - x^3) \\ &\quad + C_3(\frac{5}{3} + \frac{5}{3}x + 12x^2 - x^4) + C_4(2 + 2x + 20x^3 - x^5) . \end{aligned} \tag{3.5}$$

R_i	C_i	GM	CM	MM	PM	LSM
R_1	C_1	-1.84451776	-1.58664106	-1.85025503	-1.85025503	- 1.84451776
R_2	C_1	0.731096972	0.747377799	0.726237085	0.693505191	0.731096979
	C_2	-1.71707649	-1.61270592	-1.71766141	-1.69584014	-1.71707649
R_3	C_1	-0.12756918	-0.2093226	-0.12674977	-0.13318455	-0.12756941
	C_2	-0.02898731	-0.00570685	-0.02945825	-0.03592608	-0.02898690
	C_3	-0.84404459	-0.81810835	-0.84410158	-0.83599063	-0.84404478
R_4	C_1	0.01393255	0.03273546	0.01386963	0.0173900877	0.0139288165
	C_2	-0.58896791	-0.61666695	-0.58883399	-0.593929562	-0.588953372
	C_3	-0.14557532	-0.13887296	-0.14565736	-0.14606279	-0.145593292
	C_4	-0.27938774	-0.27423823	-0.27937769	-0.27738492	-0.27938060

Table 1: Coefficient $C_{i's}$ using different methods for $i = 1, 2, ..., 4$

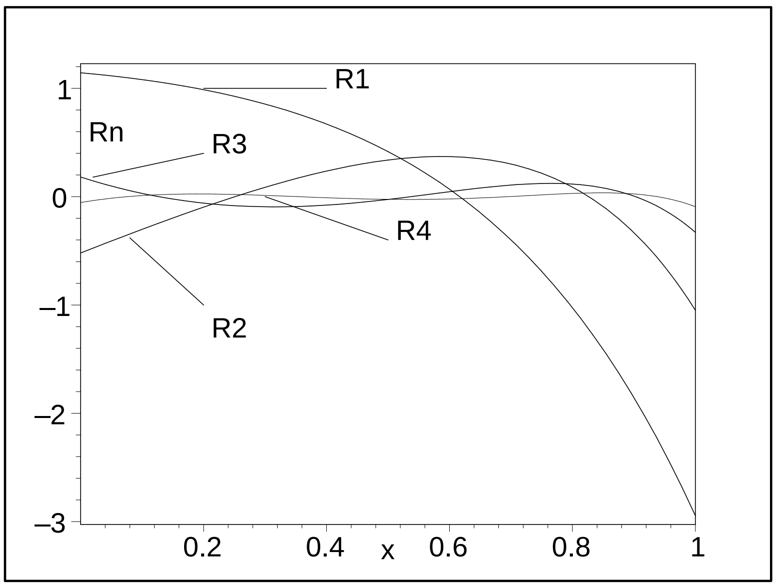

Figure 1: The residuals as a function of position

The constants $C_{i's}$ are determined using different methods.

The values of coefficients $C_{i's}$ obtained using different methods are as shown in Table 1.

These constants $C_{i's}$ are then substituted in equation (3.4) to obtain approximate solutions to the equation (3.1), Table 2.

y_i	x	GM	CM	MM	PM	LSM	Exact
y_1	0.0	0.2717572	0.0138805	0.2774944	0.2774944	0.2717572	0.0000000
	0.2	0.4523279	0.1531909	0.4589831	0.4589831	0.4523278	0.1954244
	0.4	0.4853372	0.1655701	0.4924514	0.4924514	0.4853372	0.3580379
	0.6	0.3707851	0.05101794	0.3778993	0.3778993	0.3707851	0.4373085
	0.8	0.1086715	-0.1904655	0.1153267	0.1153267	0.1086715	0.3560866
	1.0	-0.3010035	-0.5588802	-0.2952662	-0.2952662	-0.3010035	0.0000000
y_2	0.0	-0.0144223	-0.1698639	-0.0087825	-0.0051456	-0.0014422	0.0000000
	0.2	0.1982006	0.0131568	0.2047692	0.2079988	0.1982006	0.1954244
	0.4	0.3868915	0.1785579	0.3939721	0.3952232	0.3868915	0.3580379
	0.6	0.4692308	0.2489294	0.4763785	0.47512742	0.4692308	0.4373085
	0.8	0.3627988	0.1468614	0.3695406	0.3663111	0.3627988	0.3560866
	1.0	-0.0148241	-0.2050558	-0.0089893	-0.0126262	-0.0148241	0.0000000
y_3	0.0	0.0001999	0.0076850	0.0001028	0.0016431	0.0001993	0.0000000
	0.2	0.1935540	0.1994945	0.1934671	0.1950193	0.1935540	0.1954244
	0.4	0.3564052	0.3559586	0.3563696	0.3572902	0.3564052	0.3580379
	0.6	0.4387445	0.4296804	0.4387760	0.43857810	0.4387445	0.4373085
	0.8	0.3581523	0.34184751	0.3582385	0.3569036	0.3581523	0.3560866
	1.0	-0.0002025	-0.0177677	-0.0001040	-0.0018151	-0.0002025	0.0000000
y_4	0.0	-0.0000016	-0.0033421	-0.0000006	-0.0000368	-0.0000016	0.0000000
	0.2	0.1955213	0.1920556	0.1955209	0.1955776	0.1955213	0.1954244
	0.4	0.3579454	0.3547287	0.3579433	0.3581400	0.3579453	0.3580379
	0.6	0.4372044	0.4339147	0.4372024	0.4374136	0.4372044	0.4373085
	0.8	0.3561850	0.3524567	0.3561847	0.3562507	0.3561850	0.3560866
	1.0	-0.0000016	-0.0037268	-0.0000006	-0.0000608	-0.0000016	0.0000000

Table 2: Comparison of different methods with the exact solution

Example 2. We next consider

$$y'' = \frac{1}{2}(1+x+y)^3 \tag{3.6}$$

subject to the mixed boundary conditions

$$y'(0) - y(0) = -\frac{1}{2}, \qquad y'(1) + y(1) = 1\,. \tag{3.7}$$

The exact solution is given by

$$y = \frac{2}{2-x} - x - 1\,. \tag{3.8}$$

The first four trial functions considered were given below

$$y_1 = \frac{2}{3} + \frac{1}{6}x + C_1(x^2 - x - 1)\,,$$

$$y_2 = \frac{2}{3} + \frac{1}{6}x + C_1(x^2 - x - 1) + C_2(x^3 - \frac{4}{3}x - \frac{4}{3})\,,$$

R_i	C_i	GM	CM	MM	PM	LSM
R_1	C_1	0.75440210	0.68211713	0.75893118	0.75893118	0.81533976
R_2	C_1	-0.002413595	0.02968512	-0.005509397	0.02018136	-0.01554906
	C_2	0.49520334	0.44743618	0.49970958	0.48288384	0.50696013
R_3	C_1	0.34465206	0.36668423	0.34036808	0.33435710	0.34278475
	C_2	-0.17422815	-0.13755680	-0.17848706	-0.15190617	-0.18180874
	C_3	0.33326610	0.29204977	0.33859713	0.32162063	0.33993632
R_4	C_1	0.21940794	0.20267507	0.2231129	0.22098001	0.39057883
	C_2	-0.28744271	0.29025061	0.28470118	0.27490010	-0.264939027
	C_3	-0.23326121	-0.19112643	-0.23863363	-0.21187132	0.37907308
	C_4	0.22630795	0.19573259	0.23083642	0.21609201	-0.000111134

Table 3: Coefficient $C_{i's}$ using different methods for $i = 1, 2, ..., 4$

$$
\begin{aligned}
y_3 =& \frac{2}{3} + \frac{1}{6}x + C_1(x^2 - x - 1) + C_2(x^3 - \frac{4}{3}x - \frac{4}{3}) + C_3(x^4 - \frac{5}{3}x - \frac{5}{3}), \quad (3.9)\\
y_4 =& \frac{2}{3} + \frac{1}{6}x + C_1(x^2 - x - 1) + C_2(x^3 - \frac{4}{3}x - \frac{4}{3}) + C_3(x^4 - \frac{5}{3}x - \frac{5}{3})\\
&+ C_4(x^5 - 2x - 2).
\end{aligned}
$$

Substiyuting (3.9) into (3.6), we obtain the residuals as

$$
\begin{aligned}
R_1 =& 2C_1 - \frac{1}{2}(\frac{5}{3} + x - C_1 + (-C_1 + \frac{1}{6})x + C_1x^2)^3,\\
R_2 =& 2C_1 + 6C_2x - \frac{1}{2}(\frac{5}{3} + x - C_1 - \frac{4}{3}C_2 + (-C_1 - \frac{4}{3}C_2 + \frac{1}{6})x + C_1x^2\\
&+ C_2x^3)^3,\\
R_3, =& 2C_1 + 6C_2x + 12C_3x^2 - \frac{1}{2}(\frac{5}{3} + x - C_1 - \frac{4}{3}C_2 - \frac{5}{3}C_3 + (-C_1\\
&- \frac{4}{3}C_2 - \frac{5}{3}C_3 + \frac{1}{6})x + C_1x^2 + C_2x^3 + C_3x^4)^3, \quad (3.10)\\
R_4 =& 2C_1 + 6C_2x + 12C_3x^2 + 20C_4x^3 - \frac{1}{2}(\frac{5}{3} + x - C_1 - \frac{4}{3}C_2\\
&- \frac{5}{3}C_3 - 2C_4 + (-C_1 - \frac{4}{3}C_2 - \frac{5}{3}C_3 - 2C_4 + \frac{1}{6})x + C_1x^2\\
&+ C_2x^3 + C_3x^4 + C_4x^5)^3.
\end{aligned}
$$

The constants $C_{i's}$ are determined using different methods.

The values of $C_{i's}$ obtained using different methods are as shown in Table 3.

These constants $C_{i's}$ are then substituted into equation (3.9) to obtain approximate solutions to the equation (3.6).

y_i	x	GM	CM	MM	PM	LSM	Exact
y_1	0.0	-0.0877354	-0.0154505	-0.0922645	-0.0922645	-0.1486731	0.0000000
	0.2	-0.1751064	-0.09125587	-0.1803602	-0.1803602	-0.2457941	-0.0888889
	0.4	-0.2021253	-0.1124919	-0.2077413	-0.2077413	-0.2776880	-0.1500000
	0.6	-0.1687919	-0.0791586	-0.1744080	-0.1744080	-0.2443546	-0.171429
	0.8	-0.0751064	-0.00874414	-0.0803602	-0.0803602	-0.1457941	-0.1333333
	1.0	0.0789312	0.15121622	0.0744022	0.0744022	0.0179936	0.0000000
y_2	0.0	0.0088092	0.0404000	0.0058966	0.0026402	0.0062689	0.0000000
	0.2	-0.0855639	-0.0467531	-0.0891467	-0.0921615	-0.0890436	-0.0888889
	0.4	-0.1563604	-0.1100545	-0.1606448	-0.1621702	-0.161266	-0.1500000
	0.6	-0.1798103	-0.1280272	-0.1846115	-0.1842075	-0.1860641	-0.171429
	0.8	-0.1321441	-0.0791943	-0.1370608	-0.1350951	-0.1391038	-0.1333333
	1.0	0.0104080	0.0579213	0.0059934	0.0083456	0.0039489	0.0000000
y_3	0.0	-0.0011260	-0.0033581	-0.000047	-0.0011832	-0.0002670	0.0000000
	0.2	-0.0884258	-0.0899956	-0.0873281	-0.0887463	-0.0875196	-0.0888889
	0.4	-0.1490511	-0.1473591	-0.1483623	-0.1496479	-0.1484616	-0.1500000
	0.6	-0.1721688	-0.1652293	-0.1722141	-0.1726543	-0.1722396	-0.171429
	0.8	-0.1341482	-0.1221722	-0.1349454	-0.1341815	-0.1349465	-0.1333333
	1.0	0.0014388	0.0144610	0.0003837	0.0017051	0.0003784	0.0000000
y_4	0.0	0.0001546	0.0040697	0.0000021	0.0000881	-0.0022263	0.0000000
	0.2	-0.0890395	-0.0849306	-0.0891033	-0.0891258	-0.0885615	-0.0888889
	0.4	-0.1499361	-0.1461870	-0.1498204	-0.1501375	-0.1478772	-0.1500000
	0.6	-0.1713112	-0.1673812	-0.1711577	-0.1715831	-0.1710613	-0.171429
	0.8	-0.1335173	-0.12850120	-0.1335409	-0.1336389	-0.1344538	-0.1333333
	1.0	0.0002065	0.0056712	0.0000210	0.0002769	0.000149104	0.0000000

Table 4: Comparison of different methods with the exact solution

The approximate solutions are shown in Table 4.

4. Results and Conclusion

We have used five methods to determine the constants $C_{i's}$: the Galerkin method, the collocation method, the moment method, the partion method and the least-squares method. From Tables 2 and 4 we observed that the moment method gives the best results. It converges to the analytical solution more rapidly as more terms of the $C_{i's}$ are involved.

Figures 1 and 2 show the residuals as a function of position for the moment method for the two problems considered. As n increases the residual becomes smaller. This shows that better approximation can be obtained as n increases.

For $n = 4$, the residual is very small. This suggests that the residual can be used as an error analysis and the size of the residual gives a qualitative guide to

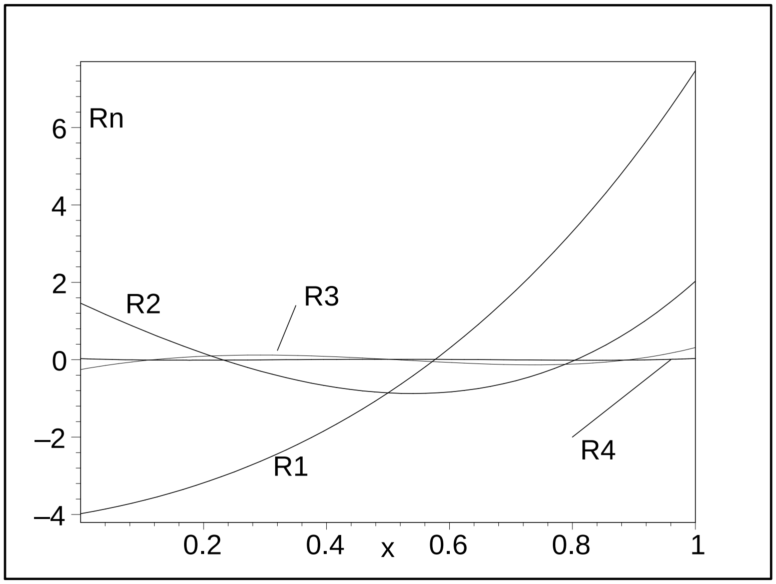

Figure 2: The residuals as a function of position

the accuracy of the approximation. Comparing the residuals in Figures 1 and 2, we would conclude that the approximate solution corresponding to the case $n = 4$ gives the best solution though higher values of n can also be considered.

The method of weighted residual discussed provided a numerical approach of solving both linear and nonlinear boundary value problems with mixed boundary conditions.

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