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PRE-PRIME AND PRE-SEMIPRIME IDEALS IN Γ-SEMIRINGS

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Abstract: The concepts of the pre-prime ideals and pre-semiprime ideals in Γ-semirings are introduced and some of their properties are studied here. Also the notions of pre-Noetherian and k-regular semiring (Γ-semiring) are introduced here.

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1. Introduction and Preliminaries

A semiring is a nonempty set S with two associative binary compositions '+' and '.' with an additive identity called zero of S such that '.' is distributive over '+' and $a.0 = 0.a = 0$ for every $a \in S$. The notion of Γ-semiring was introduced by M.M.K. Rao [8]. Dutta et al [2] studied it via operator semirings. Sardar et al [9] observed the Γ-semiring from a different point of view. Olson et al (see [6,7]), by using the notion of k-ideals, introduced pre-prime ideals and pre-semiprime ideals in semirings. In this paper we have extended these ideas to Γ-semirings. We have found some interplays of these ideals of a Γ-semiring and its operator

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semirings. These are then used to obtain inclusion preserving bijection between the set of each type of the above mentioned ideals in Γ-semirings and the set of corresponding types of ideals in the operator semirings. The notions of pre-Noeitherian, k-regular semirings and Γ-semirings are also introduced here.

For an ideal P of a semiring S, the k-closure or subtractive closure (Golan [5]) of I is denoted by $\overline{P}$ and defined by $\overline{P}$={x∈S: x+i∈P, for some i∈P }. P is said to be a pre-prime (Olson et al [6]) ideal of S if IJ⊆ $\overline{P}$ implies I⊆ $\overline{P}$ or J⊆ $\overline{P}$, where I, J are ideals of S. Any ideal P of a semiring S is pre-prime if and only if $\overline{P}$ is prime in S (Olson et al [6]). P is said to be a pre-semiprime (Olson et al [7]) ideal of S if $I^2 \subseteq \overline{P}$ implies I⊆ $\overline{P}$ where I is an ideal of S.

For more on preliminaries we refer to the references and their references.

2. Pre-Prime Ideals in Γ-Semiring

Definition 2.1. Let S be a Γ-semiring and I be any ideal of S. Then the k-closure of I is denoted by $\overline{I}$ and defined by $\overline{I}$={x∈S: x+i∈I, for some i∈I}.

Note. i) $\overline{\mathrm{I}}$ is the smallest k-ideal (Dutta et al [2]) containing I.

ii) If A and B are two ideals of the Γ-semiring S with A⊆B, then $\overline{A} \subseteq \overline{B}$.

Definition 2.2. An ideal P of a Γ-semiring S is said to be a pre-prime ideal of S if IΓJ⊆ $\overline{P}$ implies I⊆ $\overline{P}$ or J⊆ $\overline{P}$, where I and J are ideals of S, i.e., P is a pre-prime ideal of S if $\overline{P}$ is a prime ideal (Dutta et al [1]) of S.

Note. For k-ideals of Γ-semiring the notions of prime and pre-prime are equivalent.

In this section S will denote a Γ-semiring with the unities (Dutta et al [2]) and L and R will denote respectively the left and right operator semirings (Dutta et al [2]) of S if otherwise not mentioned.

Now we obtain the following lemmas which play crucial role to the development of this paper.

Lemma 2.3. *Let I be any ideal of S. Then* $\overline{I^{*'}} = \overline{I,}^{*'}$ *where* $I^{*'}$ *is defined in Dutta et al [2] by*

$$I^{*'} := \left\{ \sum_{k=1}^{m} [\alpha_k, x_k] \in R : S\left(\sum_{k=1}^{m} [\alpha_k, x_k]\right) \subseteq I, \text{ i.e., } \sum_{k=1}^{m} s\alpha_k x_k \in I \text{ for all } s \in S \right\}.$$

Proof. Let $\sum_{j=1}^{n}[\gamma_j, f_j]$ be the right unity of the Γ-semiring S.

Let x$\in \overline{\mathrm{I}^{*'}}$. Then x$+\sum_{k=1}^{m}[\alpha_k, y_k] \in \mathrm{I}^{*'}$, for some $\sum_{k=1}^{m}[\alpha_k, y_k] \in \mathrm{I}^{*'}$.

Then for all s∈S, sx$+\sum_k s\alpha_k y_k \in$I. Since $\sum_k[\alpha_k, y_k] \in \mathrm{I}^{*'}$, $\sum_k s\alpha_k y_k \in$I, for all s∈S. Hence sx$\in \overline{\mathrm{I}}$, for all s∈S. i.e., x$\in \overline{\mathrm{I}}^{*'}$. Thus $\overline{\mathrm{I}^{*'}} \subseteq \overline{\mathrm{I}}^{*'}$.

Next let $\sum_k[\alpha_k, y_k] \in \overline{\mathrm{I}}^{*'}$. Then $\sum_k s\alpha_k y_k \in \overline{\mathrm{I}}$, for all s∈S. Hence $\sum_k s\alpha_k y_k$ $+p \in I$, for some p∈I and for all s∈S. Then in particular, $\sum_k f_j\alpha_k y_k$+p∈I, for some p∈I and for all j. Hence $[\alpha, \sum_k f_j\alpha_k y_k$+p]$\in \mathrm{I}^{*'}$ for all $\alpha \in \Gamma$ and for some p∈I and for all j (cf. Theorem 6.6 in Dutta et al [2]). So $[\alpha, \sum_k f_j\alpha_k y_k] + [\alpha,$ p]$\in \mathrm{I}^{*'}$, for all $\alpha \in \Gamma$ and for some p∈I and for all j.

Since I is an ideal and p∈I; for all $s \in$S, sαp∈I, and so [α, p]$\in \mathrm{I}^{*'}$, for all $\alpha \in \Gamma$. It follows that $[\alpha, \sum_k f_j\alpha_k y_k] \in \overline{\mathrm{I}^{*'}}$, for all $\alpha \in \Gamma$ and for all j. In particular $[\gamma_j, \sum_k f_j\alpha_k y_k] \in \overline{\mathrm{I}^{*'}}$ for all j. So summing over j, $\sum_j[\gamma_j, \sum_k f_j\alpha_k y_k] \in \overline{\mathrm{I}^{*'}}$, i.e., $\left(\sum_j[\gamma_j, f_j]\right)\left(\sum_k[\alpha_k, y_k]\right) \in \overline{\mathrm{I}^{*'}}$, i.e., $\sum_k[\alpha_k, y_k] \in \overline{\mathrm{I}^{*'}}$. So $\overline{\mathrm{I}}^{*'} \subseteq \overline{\mathrm{I}^{*'}}$. Hence $\overline{\mathrm{I}^{*'}} = \overline{\mathrm{I}}^{*'}$. □

Applying similar argument as above and using Theorem 6.3 (Dutta et al [2]) instead of Theorem 6.6 (Dutta et al [2]), the left unity $\sum_{i=1}^{m}[e_i, \delta_i]$ instead of the right unity $\sum_{j=1}^{n}[\gamma_j, f_j]$ we can deduce the following lemma.

Lemma 2.4. *Let I be any ideal of S . Then* $\overline{I^{+'}} = \overline{I}^{+'}$, *where* $I^{+'}$ *is defined in Dutta et al [2] by* $I^{+'} := \left\{\sum_{k=1}^{m}[x_k, \alpha_k] \in L : \left(\sum_{k=1}^{m}[x_k, \alpha_k]\right) S \subseteq I\right\}$.

Lemma 2.5. *Let Q be any ideal of R. Then* $\overline{Q^*}=\overline{Q}^*$, *where* Q^* *is defined in Dutta et al [2] by* $Q^* = \{a \in S : [\Gamma, a] \subseteq Q$, *i.e.,* $[\alpha, a] \in Q \ for\ all\ \alpha \in \Gamma\}$.

Proof. Let $\sum_{i=1}^{m}[e_i, \delta_i]$ be the left unity of the Γ-semiring S.

Let x∈$\overline{Q^*}$. Then x+k∈Q^* for some k∈Q^*. So [α, k]∈Q and [α,x+k]∈Q for all $\alpha \in \Gamma$. So [α,x] + [α,k]∈Q for all $\alpha \in \Gamma$. It follows that [α,x]∈ $\overline{Q}$, for all $\alpha \in \Gamma$. Consequently, x∈$\overline{Q}^*$. So $\overline{Q^*} \subseteq \overline{Q}^*$. Next let x∈$\overline{Q}^*$. Then [α,x]∈ $\overline{Q}$, for all $\alpha \in \Gamma$. Then [α,x]+q∈Q, for some q∈Q and for all $\alpha \in \Gamma$. In particular, [δ_i, x]+q∈Q, for all i=1,2,...,m. So sδ_ix + sq∈Q^* for all s ∈S (cf. Theorem 6.6 Dutta et al [2]). Again sq∈Q^*, for all s∈ S, as q∈Q. Then sδ_ix∈ $\overline{Q^*}$, for all s∈ S and for all i. So in particular $e_i\delta_i$x∈ $\overline{Q^*}$, for all i=1,2,...,m. Now summing over i, we have $\sum_{i=1}^{m} e_i\delta_i$x∈ $\overline{Q^*}$. So $x \in \overline{Q^*}$. So $\overline{Q}^* \subseteq \overline{Q^*}$. Hence $\overline{Q^*}=\overline{Q}^*$. □

Application of similar argument as above and use of Theorem 6.3 (Dutta et al [2]) and the right unity $\sum_{j=1}^{n}[\gamma_j, f_j]$ instead of Theorem 6.6 (Dutta et al [2]) and the left unity $\sum_{i=1}^{m}[e_i, \delta_i]$ respectively give the following lemma.

Lemma 2.6. *Let P be any ideal L. Then* $\overline{P^+}=\overline{P}^+$, *where* P^+ *is defined in Dutta et al [2] by* $P^+ = \{a \in S : [a, \Gamma] \subseteq P\}$.

Definition 2.7. A Γ-semiring (semiring) S is said to be a pre-Noetherian Γ-semiring (respectively semiring) if for any ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of ideals of S the corresponding ascending chain $\overline{I_1} \subseteq \overline{I_2} \subseteq \cdots$ of ideals terminates.

Proposition 2.8. *If a Γ-semiring (semiring) S is Noetherian Dutta et al [4], then S is pre-Noetherian.*

Theorem 2.9. *S is pre-Noetherian if and only if L(R) is pre-Noetherian.*

Proof. Let S be pre-Noetherian and $P_1 \subseteq P_2 \subseteq \cdots$ be an ascending chain of ideals of L. Then $P_1^+ \subseteq P_2^+ \subseteq \cdots$ is ascending chain of ideals of S (cf. Theorem 6.3 Dutta et al [2]), i.e., $\overline{P_1^+} \subseteq \overline{P_2^+} \subseteq \cdots$ is an ascending chain of ideals of S. Since S is pre-Noetherian, there exists a positive integer m such that $\overline{P_n^+} = \overline{P_m^+}$, for all $n \geq m$, i.e., $\left(\overline{P_n^+}\right)^{+'} = \left(\overline{P_m^+}\right)^{+'}$, for all $n \geq m$. i.e., $\overline{\left(P_n^+\right)^{+'}} = \overline{\left(P_m^+\right)^{+'}}$ (cf. Lemma 2.4), for all $n \geq m$, i.e., $\overline{P_n} = \overline{P_m}$ (cf. Theorem 6.3 Dutta et al [2]), for all $n \geq m$. So L is pre-Noetherian. □

Similarly using Theorem 6.6 (Dutta et al [2]) and Lemma 2.3, we can show that R is pre-Noetherian.

Next let L be pre-Notherian and $A_1 \subseteq A_2 \subseteq \cdots$ be an ascending chain of ideals in S. Then $A_1^{+'} \subseteq A_2^{+'} \subseteq \cdots$ is an ascending chain of ideals in L (cf. Theorem 6.3 Dutta et al [2]). So $\overline{A_1^{+'}} \subseteq \overline{A_2^{+'}} \subseteq \cdots$ is an ascending chain of ideals in L. Since L is pre-Noetherian, there exists a positive integer m such that $\overline{A_i^{+'}} = \overline{A_m^{+'}}$, for all $i \geq m$. So $\left(\overline{A_i^{+'}}\right)^+ = \left(\overline{A_m^{+'}}\right)^+$, for all $i \geq m$. So $\overline{\left(A_i^{+'}\right)^+} = \overline{\left(A_m^{+'}\right)^+}$ (cf. Lemma 2.6), for all $i \geq m$, i.e., $\overline{A_i} = \overline{A_m}$ (cf. Theorem 6.3 Dutta et al [2]) for all $i \geq m$. Hence S is pre-Noeitherian.

Similarly supposing R is pre-Noetherian we can prove (using Lemma 2.5 and Theorem 6.6 Dutta et al [2]) that S is pre-Noetherian. □

Corollary 2.10. *L is a pre-Noetherian semiring if and only if R is a pre-Noetherian semiring.*

Proposition 2.11. *Let Q be a pre-prime ideal of R(L). Then Q^* (respectively Q^+) is a pre-prime ideal of S.*

Proof. Let Q be a prime ideal of R. Then by Proposition 6.4 (Dutta et al [2]), Q^* is an ideal of S. Let A, B be two ideals of S with AΓB⊆ $\overline{Q^*}$. Then AΓB⊆ $\overline{Q}^*$ (cf. Lemma 2.5). So [Γ, AΓB]⊆ $\overline{Q}$ (cf. Definition of Q^* in Dutta et al [2]), i.e., [Γ, A][Γ, B]⊆ $\overline{Q}$. Since Q is pre-prime and [Γ,A], [Γ,B] are ideals of R (Dutta et al [2]), [Γ, A]⊆ $\overline{Q}$ or[Γ, B]⊆ $\overline{Q}$ (Olson et al [6]). So A⊆ $\overline{Q}^*$ or B⊆$\overline{Q}^*$ which implies that A ⊆$\overline{Q^*}$ or B⊆ $\overline{Q^*}$ (cf. Lemma 2.5). So Q^* is a pre-prime ideal of S. Similarly using Lemma 2.6 and Proposition 6.1 (Dutta et al [2]) we can prove that if Q is a pre-prime ideal of L then Q^+ is a pre-prime ideal of S. □

Proposition 2.12. *Let I be a pre-prime ideal of S. Then $I^{*'}$ (respectively $I^{+'}$) is a pre-prime ideal of R (respectively L).*

Proof. By Proposition 6.5 (Dutta et al [2]), $I^{*'}$ is an ideal of R. Let A and B be two ideals of R such that AB⊆ $\overline{I^{*'}}$. Then AB⊆ $\overline{I}^{*'}$ (cf. Lemma 2.3). Then ARB⊆AB⊆ $\overline{I}^{*'}$. So A[Γ, S]B⊆$\overline{I}^{*'}$. So (SA)Γ(SB)⊆ $\overline{I}$ (cf. Definition of $\overline{I}^{*'}$ Dutta et al [2]). Since SA and SB are ideals of S(Dutta et al [2]) and I is a pre-prime ideal of S, SA⊆$\overline{I}$ or SB⊆$\overline{I}$ (cf. Definition 2.2). So A⊆$\overline{I}^{*'}$ or B⊆$\overline{I}^{*'}$, i.e., A⊆ $\overline{I^{*'}}$ or B⊆$\overline{I^{*'}}$ (cf. Lemma 2.3). So $I^{*'}$ is a pre-prime ideal R.

Similarly using Lemma 2.4 and Proposition 6.2 of Dutta et al [2] we can show that $I^{+'}$ is a pre-prime ideal of L. □

Now by Propositions 2.11 and 2.12 and Theorem 6.6 (Dutta et al [2]) we have the following theorem.

Theorem 2.13. *There exists an inclusion preserving bijection between the pre-prime ideals of S and that of L (respectively R) via the mapping* $I \mapsto I^{+'}$ *(respectively,* $I \mapsto I^{*'}$*).*

The following theorem characterizes pre-prime ideals of a Γ-semiring.

Theorem 2.14. *The following statements are equivalent for an ideal I of S.*

i) I is pre-prime.

ii) For a, b $\in S$ *such that* $a\Gamma S\Gamma b \subseteq \overline{I}$ *implies either* $a \in \overline{I}$ *or* $b \in \overline{I}$*.*

iii) If $<a>$, $<b>$ *are principal ideals of S such that* $<a>\Gamma<b> \subseteq \overline{I}$ *then either* $a \in \overline{I}$ *or* $b \in \overline{I}$*.*

iv) If U and V are two right (left) ideals of S such that $U\Gamma V \subseteq \overline{I}$*, then either* $U \subseteq \overline{I}$ *or* $V \subseteq \overline{I}$*.*

Proof. Since I is pre-prime, $\overline{I}$ is prime. So by Theorem 3.6 (Dutta et al [1]), the theorem follows. □

3. Pre-Semiprime Ideals in Γ-Semirings

Definition 3.1. An ideal P of a Γ-semiring S is said to be a pre-semiprime ideal of S if $I\Gamma I \subseteq \overline{P}$ implies $I \subseteq \overline{P}$ where I is an ideal of S.

Notes. i) For k-ideals of Γ-semiring the notions of semiprime and pre-semiprime are equivalent.

ii) If P is a pre-semiprime ideal of a semiring with identity then P is also a pre-semiprime ideal of the Γ-semiring S where Γ = S.

Proposition 3.2. *Let S be a Γ-semiring. Any ideal P of S is pre-semiprime if and only if* $\overline{P}$ *is semiprime in S.*

Note. Any pre-prime ideal P of a Γ-semiring S is pre-semiprime in S.

In this section S will denote a Γ-semiring with the unities and L and R will denote the left and right operator semirings respectively of S if otherwise not mentioned.

Proposition 3.3. *Let Q be a pre-semiprime ideal of R (respectively L). Then Q^* (respectively Q^+) is a pre-semiprime ideal of S.*

Proof. Similar to the proof of Proposition 2.11. □

Proposition 3.4. *Let S be a Γ-semiring with the unities and I be a pre-semiprime ideal of the Γ-semiring S. Then $I^{*'}$ (respectively $I^{+'}$) is a pre-semiprime ideal R (respectively L).*

Proof. Similar to the proof of Proposition 2.12. □

Theorem 3.5. *There exists an inclusion preserving bijection $I \longrightarrow I^{+'}$ ($I \longrightarrow I^{*'}$) between the pre-semiprime ideals of S and that of L (respectively R).*

Proof. The theorem follows from Propositions 3.3 and 3.4 and Theorem 6.3 of Dutta et al [2]. □

Theorem 3.6. *The following statements are equivalent for an ideal I of S.*

i) I is pre-semiprime.

ii) For $a \in S$ such that $a\Gamma S\Gamma a \subseteq \overline{I}$ implies $a \in \overline{I}$.

iii) If $<a>$ is the principal ideal of S such that $<a>\Gamma<a> \subseteq \overline{I}$ then $a \in \overline{I}$.

iv) If U is a right (left) ideal of S such that $U\Gamma U \subseteq \overline{I}$, then $U \subseteq \overline{I}$.

Proof. By Proposition 3.2 and Theorem 3.6 (Dutta et al [3]) the theorem follows. □

Definition 3.7. A Γ-semiring (resp. semiring) S is said to be k-regular if every k-ideal of S is semiprime.

Theorem 3.8. *A Γ-semiring (resp. semiring) S is k-regular if and only if every ideal of S is pre-semiprime.*

Proof. Let S be k-regular Γ-semiring (respectively semiring) and I be an ideal of S. Then $\overline{I}$ is a k-ideal of S. So by definition of k-regularity, $\overline{I}$ is semiprime. So I is pre-semiprime.

Next let every ideal of S be pre-semiprime and I be a k-ideal of S. Then by hypothesis, I is pre-semiprime. So $\overline{I}$ is semiprime, i.e., I is semiprime (since $\overline{I}$ $=$I). Hence S is k-regular. □

Theorem 3.9. *A Γ-semiring S is k-regular if and only if L (respectively*

R) is k-regular.

Proof. Let S be k-regular and I be any ideal of L. So I^+ is an ideal of S (cf. Proposition 6.1 Dutta et al [2]). So I^+ is pre-semiprime (cf. Theorem 3.8). Then $(I^+)^{+'}$ is pre-semiprime (cf. Proposition 3.4), i.e., I is pre-semiprime (cf. Theorem 6.3 Dutta et al [2]). Hence L is K-regular (cf. Theorem 3.8).

Conversely, if L is k-regular then reversing the above argument we can prove that S is k-regular.

Similarly we can prove the theorem for R. □

Corollary. *L is a k-regular semiring if and only if R is a k-regular semiring.*

Remark. The above theorem can also be proved using Theorem 3.5.

Concluding Remark. Ideal theory of semirings differs from that of rings. To fill in this gap k-ideals and h-ideals were introduced in semirings. The notions of pre-prime ideals and pre-semiprime ideals are based on k-ideals. So they can also be called k-prime and k-semiprime ideals respectively. Keeping this in mind we have introduced the notions of h-prime and h-semiprime ideals in our next paper.

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