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NON-LINEARLY NORMAL CURVES WITH MAXIMAL RANK IN $\mathbb{P}^r$

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Abstract: Fix integers r, b such that $r \geq 3$ and $b > 0$. Here for most integers d, g such that $(r-2)g/(r-1) < d < g+r-b$ we prove the existence of a nodal, smoothable and connected $C \subset \mathbb{P}^r$ with maximal rank such that $\deg(C) = d$, $p_a(C) = g$, $h^1(C, N_C) = 0$, $h^0(C, \mathcal{O}_C(1)) = r+1+b$.

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1. The Statement

Let $C \subset \mathbb{P}^r$ be a curve. The curve C is said to have *maximal rank* if for every integer t the restriction map $\rho_{r,C,t} : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(C, \mathcal{O}_C(t))$ has maximal rank, i.e. it is either injective or surjective. C has maximal rank if and only if for every integer $t > 0$ either $h^0(\mathbb{P}^r, \mathcal{I}_C(t)) = 0$ (i.e. $\rho_{r,C,t}$ is injective) or $h^1(\mathbb{P}^r, \mathcal{I}_C(t)) = 0$ (i.e. $\rho_{r,C,t}$ is surjective).

Fix integers r, d, g such that $r \geq 3$, $g \geq 0$ and either $d \geq g+r$ or $d-r < g \leq d-r+\lfloor (d-r-2)/(r-2) \rfloor$. There is an irreducible component $W(d, g; r)$ of the Hilbert scheme of $\mathbb{P}^r$ which is generically smooth and of dimension $(r+1)d - (r-3)(g-1)$ such that a general $C \in W(d, r; g)$ has the following properties (see [1] for the case $r = 3$, [4] for the case $r \geq 4$):

(a) C is a smooth and connected non-degenerate curve with degree d, genus g and $h^1(C, N_C) = 0$;

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(b) if $d \geq g + r$, then $h^1(C, \mathcal{O}_C(1)) = 0$;

(c) if $d < g + r$, then C is linearly normal and $h^1(C, \mathcal{O}_C(2)) = 0$;

(d) if $\rho(g, r, d) \geq 0$, then C has general moduli;

(e) if $\rho(g, r, d) < 0$, then the general fiber of the natural rational map $\gamma_{d,g,r} : W(d, g; r) \dashrightarrow \mathcal{M}_g$ has dimension $\dim(\mathrm{Aut}(\mathbb{P}^r)) = r^2 + 2r$, i.e. $W(d, g; r)$ has the right number of moduli in the sense of [6].

Here (following [1] and [4]) we prove the following existence theorem for smoothable curves with maximal rank, but not linearly normal.

Theorem 1. *Fix integer r, a such that $r \geq 3$ and $a \geq 0$. Then there exists a function $u_{r,a} : \mathbb{N} \to \mathbb{N}$ such that*

$$\lim_{d \to +\infty} u_{r,a} = (r-2)/(r-1) \tag{1}$$

and for every integer b such that $0 \leq b \leq a$, $gu_{r,a} \leq d \leq g + r - a$ there is a nodal and connected $C \in W(d, g; r)$ with $h^0(C, \mathcal{O}_C(1)) = r + 1 + b$, $h^1(C, \mathcal{O}_C(2)) = 0$, $h^1(C, N_C) = 0$ and C with maximal rank.

The case $a = 0$, $r \geq 4$, of Theorem 1 is the main result of [4]. The case $a = 0$ and $r = 3$ of Theorem 1 was proved in [1]. Only the case $a > 0$ is new, although its proof follows the path of [1] and [4]. If $r = 3$ we will find C union of a smooth $A \in W(d - b, g; 3)$ and b disjoint lines, each of them intersecting quasi-transversally A and at a unique point. If $r \geq 4$

As in [4] we work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

2. The Proof

Fix integers g, d, r such that $W(d, g; r)$ is defined, i.e. such that $r \geq 3$, $g \geq 0$ and either $d \geq g + r$ or $g \geq 2$ and $d - r < g \leq d - r + \lfloor (d - r - 2)/(r - 2) \rfloor$. If $(d, g, r) \neq (r, 0, r)$, then the *critical value* of the triple (d, g, r) or of any element of $W(d, g; r)$ is the minmal integer $k \geq 2$ such that $kd + 1 - g \leq \binom{r+k}{r}$. We say that $(r, 0, r)$ has critical value 1. Take any $C \in W(d, g; r)$ such that $h^1(C, \mathcal{O}_C(2)) = 0$ (e.g. a sufficiently general element of $W(d, g; r)$). C has maximal rank if and only if $h^1(\mathbb{P}^r, \mathcal{I}_C(k)) = 0$ (i.e. the restriction map $\rho_{C,r,k}$ is surjective) and $h^1(\mathbb{P}^r, \mathcal{I}_C(k-1)) = 0$ (i.e. $\rho_{C,r,k-1}$ is injective).

Proof of Theorem 1. Fix a triple (d, g, r) for which we want to prove Theorem 1 for a certain integer b. Let k be the critical value of the triple (d, g, r). We need to find some $C \in W(d, g; r)$ with $h^0(C, \mathcal{O}_C(1)) = r + 1 + b$ and maximal rank, i.e. such that $h^1(\mathbb{P}^r, \mathcal{I}_C(k)) = 0$ and $h^0(\mathbb{P}^r, \mathcal{I}_C(k-1)) = 0$ (plus satisfying a few easy conditions, like nodal, $h^1(C, N_C) = 0$ and $h^1(C, \mathcal{O}_C(2)) = 0$). Since $d < g + r$ a general $X \in W(d, g; r)$ is linearly normal. Sincce $b > 0$, the condition "$h^0(C, \mathcal{O}_C(1)) = r + 1 + b$" is not an open condition. Hence it is not sufficient to find curves $C_1, C_2 \in W(d, g; r)$ such that $h^1(\mathbb{P}^r, \mathcal{I}_{C_1}(k)) = 0$ and $h^1(\mathbb{P}^r, \mathcal{I}_{C_2}(k-1)) = 0$ (and satisfying the other conditions of the statement of Theorem 1). We will overcame this problenm in a different way in the case $r = 3$ and in the case $r \geq 4$. Fix integers d, g, r, b such that $r \geq 3$, $b > 0$, $d - b \leq g + r$, and $W(d-b, g; r)$ is defined. Let $W(d, g; r, b)$ denote the set of all nodal curves $C \subset \mathbb{P}^r$ such that $C = X \cup T$ with X smooth, $X \in W(d-b, g; b)$, $h^0(X, \mathcal{O}_X(1)) = r + 13$, and T disjoint union of b lines, each of them intersecting quasi-transversally X at exactly one point. We used the condition $d - b \leq g + r$ to know that $h^0(X, \mathcal{O}_X(1)) = r + 1$ for a general $X \in W(d-b, g; r)$. A Mayer-Vietoris exact sequence gives $h^0(X \cup T, \mathcal{O}_{X \cup T}(1)) = r + 1 + b$ and $h^1(X \cup T, \mathcal{O}_{X \cup T}(1)) = h^1(X, \mathcal{O}_X(1)) = r + g - d + b$. A Mayer-Vietoris exact sequence gives that if $h^1(X, \mathcal{O}_X(2)) = 0$, then $h^1(X \cup T, \mathcal{O}_{X \cup T}(2)) = 0$. Using [5] we immediately get $h^1(X \cup T, N_{X \cup T}) = h^1(X, N_X)$. An easy modification of [1], Lemma 1.5, gives $W(d, g; 3, b) \subset W(d, g; 3)$. For every $r \geq 4$, [4], Lemma 2.2, gives $W(d, g; r, b) \subset W(d, g; r)$. Since $W(d, g; r, b)$ is irreducible, to prove Theorem 1 for the quadruple (d, g, r, b) it is sufficient to find $C_1, C_2 \in W(d, g; r, b)$ such that $h^1(\mathbb{P}^r, \mathcal{I}_{C_1}(k)) = 0$ and $h^1(\mathbb{P}^r, \mathcal{I}_{C_2}(k-1)) = 0$ (and satisfying some easy properties).

(a) Here we assume $r = 3$. Fix an integer $k \gg 0$ for which [1], Theorem 2, gives $h^1(\mathbb{P}^r, \mathcal{I}_Y(k)) = 0$ and $h^0(\mathbb{P}^3, \mathcal{I}_Z(k-1)) = 0$ for general $Y, Z \in W(d, g+b; 3)$. Taking b suitable nilpotents in the quadric surface used to prove the surjectivity part (resp. injectivity part) from R_{k-2} (resp. R_{k-3}) we get a specialization of some element of $W(d, g; 3, b)$ (here $C = X \cup T \cup \chi$ with $X \in W(d-b, g; 3)$, X smooth, T disjoint union of b lines, each line of T intersects transversally X at eactly two point and χ is the union of the first infinitesimal neigborhood of b points of $\mathbb{P}^3$ (more precisely, b of the $2b$ points of $X \cap T$), exactly one of these b points for each of the lines of T).

(b) Here we assume $r \geq 4$. Let $H \subset \mathbb{P}^r$ be a hyperplane. Fix integers b, d, g such that $b > 0$ and $W(d, g; r)$ is defined and for which [4] gives that a general $Y \in W(d, g; r)$. Let k be the critical value of (d, g, r). The proof in [4] (except a few easier cases) used a reducible curve $A \cup B \in W(d, g; r)$ with $h^i(\mathbb{P}^r, \mathcal{I}_A(k-1)) = 0$, $i = 0, 1$, and B a subcurve of H such that $h^1(B, \mathcal{O}_B(1)) = 0$ and with maximal rank. For the existence of B with maximal rank it was used [2] if $r - 1 = 3$, and [3] if $r - 1 \geq 4$. We only need to modify the step in H (modifying [2] as in part (a) below and then using [3] and induction on the integer $\dim(H)$) to get instead of B a curve $B' = B'' \cup T$ with B'' smooth, $h^1(B'', \mathcal{O}_{B''}(1)) = 0$,

T disjoint union of b lines, each of them intersecting transversally B'' at exactly one point and $A \cap T = \emptyset$. As in [4] we have $A \cup B'' \in W(d-b, g; r)$. Hence $A \cup B' \in W(d, g; r, b)$ (see [4], Lemma 2.2). □

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