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GENERAL DEGREE $g-1$ LINE BUNDLES ON REDUCIBLE CURVES

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Abstract: Let X be a reduced and connected projective curve. Here we show the existence of a degree $p_a(X)-1$ line bundle L on X such that $h^1(X,L)=0$. If X is semistable, then we may take L semibalanced in the sense of Caporaso.

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1. Introduction

Let X be a reduced and connected projective curve. Set $g := p_a(X) = 1-\chi(\mathcal{O}_X)$. For any integer d let $\mathrm{Pic}^d(X)$ denote the set of all degree d line bundles on X. The algebraic scheme $\mathrm{Pic}^d(X)$ has pure dimension g. If X is irreducible, then $\mathrm{Pic}^d(X)$ is irreducible and its general member, L, has the following properties: if $d \le g-1$, then $h^0(X,L)=0$. If X is reduced, but reducible, then $\mathrm{Pic}^d(X)$ is not connected. In this case we prove that the same occurs in at least one of these connected components, i.e. we prove the following result.

Proposition 1. *Let X be a reduced and connected projective curve. Fix an integer $d \ge p_a(X)-1$. Then there exists $L \in Pic^d(X)$ such that $h^1(X,L)=0$.*

If X is semistable, then it is essential to find L which is semibalanced in the sense of [2], [3], [4] and [6]. Here we prove the following result.

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Theorem 1. *Let X be a semistable curve of genus $g \geq 2$. Then there is a semibalanced line bundle L on X such that $\deg(L) = g - 1$ and $h^1(X, L) = 0$.*

2. The Proofs

For any reduced projective curve Y let $\mathcal{B}(Y)$ denote the set of its irreducible components.

Lemma 1. *Let Y be a reduced projective curve and A, B proper subcurves of Y such that $Y = A \cup B$ and $A \cap B$ is finite. Fix $L \in Pic(Y)$. The restriction map $\rho_{L,B} : H^0(Y, L) \to H^0(B, L|B)$ (resp. $\rho_{L,A} : H^0(Y, L) \to H^0(A, L|A)$) is surjective if the natural injective map $h^1(B, \mathcal{I}_{A\cap B} \otimes L|B)$ (resp. $h^1(A, \mathcal{I}_{A\cap B} \otimes L|A)$) is surjective. Hence $\rho_{L,B} : H^0(Y, L) \to H^0(B, L|B)$ (resp. $\rho_{L,A}$) is surjective if $h^1(B, \mathcal{I}_{A\cap B} \otimes L|B) = 0$ (resp. $h^1(A, \mathcal{I}_{A\cap B} \otimes L|A) = 0$).*

Proof. For the map $\rho_{L,B}$ use the exact sequence of sheaves on B:

$$0 \to \mathcal{I}_{A\cap B} \otimes L|B \to L|B \to L|A \cap B \to 0\,. \tag{1}$$

In the same way we get the surjectivity of $\rho_{L,A}$. The last assertion of the lemma is obvious. □

Lemma 2. *Let Y be a reduced projective curve and A, B proper closed subcurves of Y such that $Y = A \cup B$ and the scheme $A \cap B$ has dimension zero. Fix $L \in Pic(Y)$ such that $h^1(A, L|A) = 0$ and $h^1(B, \mathcal{I}_{A\cap B} \otimes (L|B)) = 0$. Then $h^1(Y, L) = 0$.*

Proof. Look at the Mayer-Vietoris exact sequence

$$0 \to L \to L|A \otimes L|B \to L|A \cap B \to 0\,. \tag{2}$$

By assumption we have $h^1(A, L|A) = 0$. Since $h^1(B, \mathcal{I}_{A\cap B} \otimes (L|B)) = 0$ and B is a curve, $h^1(B, L|B) = 0$. Since $h^1(B, \mathcal{I}_{A\cap B} \otimes (L|B)) = 0$, the restriction map $H^0(B, L|B) \to H^0(A \cap B, L|A \cap B)$ is surjective. Hence the long cohomology exact sequence of (2) gives $h^1(Y, L) = 0$. □

We recall the following well-known observation.

Remark 1. Let Y be a reduced projective curve. Then the natural map $\mathrm{Pic}(Y) \to \times_{T\in\mathcal{B}(Y)}\mathrm{Pic}(T)$ is surjective. If A and B are connected subcurves of Y and $Y = A \cup B$, then the natural map $\mathrm{Pic}(Y) \to \mathrm{Pic}(A) \times \mathrm{Pic}(B)$ is surjective.

Lemma 3. *Fix an integral projective curve T, a zero-dimensional scheme $Z \subset T$ and an integer $d \geq length(Z)$. Let L be the general element of $Pic^d(T)$. Then $h^1(T, \mathcal{I}_Z \otimes L) \leq \max\{0, d - length(Z) - p_a(T) + 1\}$.*

Proof. Since the case $T \cong \mathbb{P}^1$ is trivially true, we may assume $p_a(T) > 0$. First assume $d = \text{length}(Z)$. Fix a general $S \subset T_{reg}$ such that $\sharp(S) = \text{length}(Z)$. Any sheaf $\mathcal{I}_Z \otimes \mathcal{O}_T(S)$ is a degree 0 torsion free sheaf on T with rank 1. If Z is not a Cartier divisor, then $\mathcal{I}_Z \otimes \mathcal{O}_T(S)$ is not trivial (for any S) and hence $h^0(T, \mathcal{I}_Z \otimes \mathcal{O}_T(S)) = 0$. Riemann-Roch gives $h^1(T, \mathcal{I}_Z \otimes \mathcal{O}_T(S)) = p_a(T) - 1$. Hence the lemma is true in this case. Now assume that Z is a Cartier divisor. Thus each $\mathcal{I}_Z \otimes \mathcal{O}_T(S)$ is a line bundle. Since $p_a(T) > 0$ for general S this line bundle is not trivial. Hence $h^0(T, \mathcal{I}_Z \otimes \mathcal{O}_T(S)) = 0$. Thus $h^1(T, \mathcal{I}_Z \otimes \mathcal{O}_T(S)) = p_a(T) - 1$, proving the lemma in tis case. Now assume $d > \text{length}(Z)$. By induction on d we may assume $h^1(T, \mathcal{I}_Z \otimes M) \leq \max\{0, d - \text{length}(Z) - p_a(T) + 2\}$ for a general $M \in \text{Pic}^{d-1}(T)$. If either the previous inequality is strict or $h^1(T, \mathcal{I}_Z \otimes M) = 0$, then for any $P \in T_{reg}$ the line bundle $L := M(P)$ satisfies the lemma. Hence we may assume $h^1(T, \mathcal{I}_Z \otimes M) = d - \text{length}(Z) - p_a(T) + 2 > 0$. Fix a general $P \in T_{reg}$. Since M is a subsheaf of $M(P)$ and $\dim(T) = 1$, $h^1(T, \mathcal{I}_Z \otimes M(P)) \leq h^1(T, \mathcal{I}_Z \otimes M)$. Hence it is sufficient to prove that for general P the previous inequality is strict. By duality (see [1]) it is sufficient to prove $h^0(T, Hom(\mathcal{I}_Z \otimes M(P), \omega_T)) < h^0(T, Hom(\mathcal{I}_Z \otimes M, \omega_T))$ for general P, knowing that the latter integer is $Hom(\mathcal{I}_Z \otimes M, \omega_T))$ has a global section $\sigma \neq 0$. Since $Hom(\mathcal{I}_Z \otimes M, \omega_T))$ has no torsion, the evaluation of σ at a general $Q \in T_{reg}$ is non-zero. Tale $P := Q$. □

Proof of Proposition 1. By twisting with an effective Cartier divisor of degree $d - p_a(X) + 1$ we easily reduce to the case $d = p_a(X) - 1$. If X is irreducible, then the result is well-known (or take the case $Z = \emptyset$ of Lemma 3). Assume $r := \sharp(\mathcal{B}(X)) \geq 2$ and that the result is true for all reduced and connected curves with at most $r - 1$ irreducible components. There is an ordering $T_1, \dots, T_r$ of $\mathcal{B}(X)$ such that for every $i \in \{2, \dots, r-1\}$ the curve $\cup_{j=1}^{i} T_i$ is connected. Set $A := \cup_{j=1}^{r-1} T_i$ and $B := T_r$. Since A and B are connected, we have $p_a(X) = p_a(A) + p_a(B) + \text{length}(A \cap B) - 1$. By the inductive assumption there is $M \in \text{Pic}(A)$ such that $\deg(M) = p_a(A) - 1$ and $h^1(A, M) = 0$. Lemma 3 gives the existence of $R \in \text{Pic}(B)$ such that $h^1(B, \mathcal{I}_{A \cap B} \otimes R) = 0$ and $\deg(R) = p_a(A) + \text{length}(A \cap B) - 1$. Remark 1 gives the existence of $L \in \text{Pic}(Y)$ such that $L|A \cong M$ and $L|B \cong R$. Fix any such line bundle L. We have $\deg(L) = p_a(Y) - 1$. Lemma 2 gives $h^1(X, L) = 0$. □

Remark 2. Let X be a semistable curve of genus $g \geq 2$. Fix $L \in \text{Pic}(X)$. L is semibalanced if and only if $p_a(Z) - 1 \leq \deg(L|Z) \leq p_a(Z) - 1 + \delta_Z$ for every proper subcurve Z of X.

Proof of Theorem 1. If X is irreducible, then every line bundle on X is semibalanced. Hence we may assume that X is reducible. Assume $r :=$

$\sharp(\mathcal{B}(X)) \geq 2$. There is an ordering $T_1, \dots, T_r$ of $\mathcal{B}(X)$ such that for every $i \in \{2, \dots, r-1\}$ the curve $\cup_{j=1}^{i} T_i$ is connected. We take as $L|T_1$ a general line bundle on T_1 with degree $p_a(T_1)-1$. Then we construct step by step $L|(T_1 \cup T_2)$, $L|(T_1 \cup T_2 \cup T_3)$ and so on, with the multidegrees obtained by the inductive proof of Proposition 1. Remark 2 gives that each line bundle $L|(T_1 \cup \cdots \cup T_i)$, $2 \leq i \leq r$, is semibalanced. □

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References

[1] A. Altman, S. Kleiman, Introduction to Grothendieck duality theory, *Lect. Notes in Math.*, **146**, Springer, Berlin (1970).

[2] L. Caporaso, Néron models and compactified Picard schemes over the moduli stack of stable curves, *Amer. J. Math.*, **130**, No. 1 (2008), 1-47.

[3] L. Caporaso, Brill-Noether theory of binary curves, *ArXiv:* math/0807.1484.

[4] L. Caporaso, Linear series on semistable curves, *ArXiv:* math/0812.1682.

[5] D. Gieseker, Lectures on moduli of curves, *Tata Inst. Fund. Res. Lectures on Math. and Phys.*, **69**, Springer-Verlag, Berlin-New York (1982).

[6] M. Melo, Compactified Picard stacks over $\overline{\mathcal{M}}_g$, *ArXiv:* math/0710.3008; *Math. Z.*, published on line DOI 10.1007/s00209-008-0447-x.