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INTEGRAL REPRESENTATIONS OF UNBOUNDED OPERATORS BY INFINITELY SMOOTH BI-CARLEMAN KERNELS

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Abstract: In this paper, we establish that if a closed linear operator in a separable Hilbert space $\mathcal{H}$ is unitarily equivalent to a bi-Carleman integral operator in an appropriate $L^2(Y,\mu)$, then that operator is unitarily equivalent to a bi-Carleman integral operator in $L^2(\mathbb{R})$, whose kernel $\boldsymbol{T}:\mathbb{R}^2\to\mathbb{C}$ and two Carleman functions $\boldsymbol{t}(s)=\overline{\boldsymbol{T}(s,\cdot)}$, $\boldsymbol{t}'(s)=\boldsymbol{T}(\cdot,s):\mathbb{R}\to L^2(\mathbb{R})$ are infinitely smooth and vanish at infinity together with all partial and all strong derivatives, respectively. The implementing unitary operator (from $\mathcal{H}$ onto $L^2(\mathbb{R})$) is found by direct construction.

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1. Introduction and the Main Result

The present paper may hopefully prove interesting for researchers who are interested in the development of the theory of non-compact, non-self-adjoint linear integral operators in L^2 spaces (see [6], [10]). In applications of this theory, such

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as occur, for instance, in the theory of singular integral equations of the second kind, it is often desirable to have the kernel function with special properties that make its associated integral operator easier to work with. Of these properties the classically inspired are, for example, those of being bounded, infinitely differentiable, and bi-Carleman (that is, square integrable in each variable separately for almost all values of the other). Here we focus attention on the kernel properties just listed, and try to show that up to a unitary equivalence these properties are simultaneously satisfied.

Precisely, the problem we study in the present paper is to establish the largest class of those closed linear operators S in an abstract separable Hilbert space $\mathcal{H}$ that can be transformed by a suitable unitary operator U_S (from $\mathcal{H}$ onto $L^2(\mathbb{R})$) into an integral operator $T = U_S S U_S^{-1}$ generated in $L^2(\mathbb{R})$ by a bounded, infinitely smooth, bi-Carleman kernel on $\mathbb{R}^2$, or, more concretely, by the K^∞ kernel to be defined in Definition 2 below. It will turn out, and it will be the principal result of this paper (Theorem 4 below), that the operators S so transformable constitute the class which is precisely the same as that which was characterized by Korotkov in [9] to resolve the similar problem when no additional analytic properties other than just being bi-Carleman are required of the measurable kernels of unitary equivalents.

In order to explain in detail the content of our main result, we need some notations, terminology and preliminaries. Throughout this paper, $\mathcal{H}$ is a complex, separable, infinite-dimensional Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$ and inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, and the symbols $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{Z}$, refer to the complex plane, the set of all positive integers, and the set of all integers, respectively.

Let $\mathfrak{C}(\mathcal{H})$ be the set of all closed, linear, densely-defined operators in $\mathcal{H}$, let $\mathfrak{R}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$, and let $\mathfrak{S}_p(\mathcal{H})$ be the Schatten-von Neumann p-ideal of compact linear operators on $\mathcal{H}$ [5, Chapter III, §7].

For an operator S in $\mathfrak{C}(\mathcal{H})$, D_S stands for a linear manifold that is the domain of S, and S^* for the adjoint to S with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. We let $\mathfrak{C}_0(\mathcal{H})$ denote the collection of all those operators S in $\mathfrak{C}(\mathcal{H})$ for which there exists an orthonormal sequence $\{e_k\}_{k=1}^\infty$ in $\mathcal{H}$ such that

$$\{e_1, e_2, e_3, \dots\} \subset D_{S^*}, \quad \lim_{k\to\infty} \|S^* e_k\|_{\mathcal{H}} = 0, \tag{1}$$

and we let $\mathfrak{C}_{00}(\mathcal{H})$ denote the subset of $\mathfrak{C}_0(\mathcal{H})$ consisting of all those operators S in $\mathfrak{C}(\mathcal{H})$ for which there exist a dense linear manifold D in $\mathcal{H}$ and an orthonormal sequence $\{e_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ such that

$$\begin{gathered}\{e_1, e_2, e_3, \dots\} \subset D \subset D_S \cap D_{S^*}, \\ \lim_{k\to\infty} \|Se_k\|_{\mathcal{H}} = 0, \quad \lim_{k\to\infty} \|S^* e_k\|_{\mathcal{H}} = 0.\end{gathered} \tag{2}$$

Let $\mathbb{R}$ be the real line $(-\infty, +\infty)$ equipped with the Lebesgue measure, and let $L^2 = L^2(\mathbb{R})$ be the Hilbert space of (equivalence classes of) measurable complex-valued functions on $\mathbb{R}$ equipped with the inner product $\langle f, g\rangle_{L^2} = \int_{\mathbb{R}} f(s)\overline{g(s)}\, ds$ and the norm $\|f\|_{L^2} = \langle f, f\rangle_{L^2}^{1/2}$. An operator $T \in \mathfrak{C}\left(L^2\right)$ is said to be *integral* if there exists a measurable function $\boldsymbol{T} : \mathbb{R}^2 \to \mathbb{C}$, a *kernel*, such that, for each $f \in D_T$,

$$(Tf)(s) = \int_{\mathbb{R}} \boldsymbol{T}(s,t) f(t)\, dt \quad \text{for almost every } s \text{ in } \mathbb{R}.$$

A kernel $\boldsymbol{T}$ on $\mathbb{R}^2$ is said to be *Carleman* if $\boldsymbol{T}(s,\cdot) \in L^2$ for almost every fixed s in $\mathbb{R}$. To each Carleman kernel $\boldsymbol{T}$ there corresponds a *Carleman function* $\boldsymbol{t} : \mathbb{R} \to L^2$ defined by $\boldsymbol{t}(s) = \overline{\boldsymbol{T}(s,\cdot)}$ for all s in $\mathbb{R}$ for which $\boldsymbol{T}(s,\cdot) \in L^2$. The Carleman kernel $\boldsymbol{T}$ is called *bi-Carleman* in case its conjugate transpose kernel $\boldsymbol{T'}$ ($\boldsymbol{T'}(s,t) = \overline{\boldsymbol{T}(t,s)}$) is also a Carleman kernel. Associated with the conjugate transpose $\boldsymbol{T'}$ of every bi-Carleman kernel $\boldsymbol{T}$ there is therefore a Carleman function $\boldsymbol{t'} : \mathbb{R} \to L^2$ defined by $\boldsymbol{t'}(s) = \overline{\boldsymbol{T'}(s,\cdot)}$ $(= \boldsymbol{T}(\cdot, s))$ for all $s \in \mathbb{R}$ in $\mathbb{R}$ for which $\boldsymbol{T'}(s,\cdot) \in L^2$. With each bi-Carleman kernel $\boldsymbol{T}$, we therefore associate the pair of Carleman functions $\boldsymbol{t}$, $\boldsymbol{t'} : \mathbb{R} \to L^2$, both defined, via $\boldsymbol{T}$, as above. An integral operator whose kernel is Carleman (resp., bi-Carleman) is referred to as the *Carleman* (resp., *bi-Carleman*) operator.

Remark 1. The notions of integral operator, Carleman operator, and bi-Carleman operator, acting in the Hilbert space $L^2(Y,\mu)$ are defined similarly as above in the space L^2 (see [6], [10]); here and throughout (Y,μ) denotes a measure space with a positive, σ-finite, separable, and not purely atomic, measure μ. It follows from the general theory that if T is a *bi-integral* operator on $L^2(Y,\mu)$, that is, both T and its adjoint T^* are integral operators defined on all of $L^2(Y,\mu)$, then T belongs to $\mathfrak{C}_{00}\left(L^2(Y,\mu)\right) \cap \mathfrak{R}\left(L^2(Y,\mu)\right)$ (see, e.g., [6, Theorems 3.10, 15.11]). The bi-integral operators, on the other hand, are generally involved in second-kind integral equations in $L^2(Y,\mu)$, as the adjoint equations to such equations are customarily required to be integral. Note that a bi-integral operator need not be bi-Carleman; and another whole class of not necessarily bounded integral operators that is entirely contained in $\mathfrak{C}_{00}\left(L^2(Y,\mu)\right)$ are just bi-Carleman operators, as will be seen from the discussion below. With regard to these latter operators, it is also relevant to mention the fact (of use in proving the main result, Theorem 4) that if both the operator $T : D_T \to L^2(Y,\mu)$ and its adjoint $T^* : D_{T^*} \to L^2(Y,\mu)$ are Carleman operators, with the kernels $\boldsymbol{T}$ and $\boldsymbol{T^*}$, respectively, then T is a bi-Carleman operator, and $\boldsymbol{T^*}(s,t) = \boldsymbol{T'}(s,t)$ for $(\mu \times \mu)$-almost all $(s,t) \in Y \times Y$ (see [10, Corollary IV.2.17]). The converse is also true. If, that is, $T : D_T \to L^2(Y,\mu)$ is a bi-Carleman operator with the kernel $\boldsymbol{T}$, then its adjoint $T^* : D_{T^*} \to L^2(Y,\mu)$ is a Carleman operator with a kernel equal almost everywhere to $\boldsymbol{T'}$. Therefore, for the same reason as the bi-integral operators, the bi-Carleman operators can intervene in integral equations of the second kind with closed operators. For description of some other important properties which the $\mathfrak{C}_{00}(\mathcal{H})$ operators may possess, we refer to [4, Theorem 5], [3, Theorem 2.3], and [6, Theorems 15.17, 15.18].

Among all possible Carleman and bi-Carleman kernels on $\mathbb{R}^2$, the next definition distinguishes special types of those which, together with their associated Carleman functions, are infinitely smooth and vanish at infinity, and to which we shall restrict our consideration from now on.

Definition 2. A bi-Carleman kernel $\boldsymbol{T} : \mathbb{R}^2 \to \mathbb{C}$ is called a K^∞ *kernel* [11] if it satisfies the three generally independent conditions:

(i) the function $\boldsymbol{T}$ and all its partial derivatives on $\mathbb{R}^2$ of all orders are in $C\left(\mathbb{R}^2, \mathbb{C}\right)$,

(ii) the Carleman function $\boldsymbol{t}$, $\boldsymbol{t}(s) = \overline{\boldsymbol{T}(s,\cdot)}$, and its (strong) derivatives, $\dfrac{d^i \boldsymbol{t}}{ds^i}$, on $\mathbb{R}$ of all orders are in $C\left(\mathbb{R}, L^2\right)$,

(iii) the Carleman function $\boldsymbol{t'}$, $\boldsymbol{t'}(s) = \overline{\boldsymbol{T'}(s,\cdot)} = \boldsymbol{T}(\cdot, s)$, and its (strong) derivatives, $\dfrac{d^i \boldsymbol{t'}}{ds^i}$, on $\mathbb{R}$ of all orders are in $C\left(\mathbb{R}, L^2\right)$.

If a Carleman kernel $\boldsymbol{T} : \mathbb{R}^2 \to \mathbb{C}$ satisfies the above two conditions (i) and (ii), then it is called an SK^∞ *kernel* [11].

Throughout this paper, $C(X,B)$, where B is a Banach space (with norm $\|\cdot\|_B$), denotes the Banach space (with the norm $\|f\|_{C(X,B)} = \sup_{x \in X} \|f(x)\|_B$) of continuous B-valued functions defined on a locally compact space X and *vanishing at infinity* (that is, given any $f \in C(X,B)$ and $\varepsilon > 0$, there exists a compact subset $X(\varepsilon, f) \subset X$ such that $\|f(x)\|_B < \varepsilon$ whenever $x \notin X(\varepsilon, f)$).

Now recall that a bounded linear operator $U : \mathcal{H} \to L^2(Y,\mu)$ is *unitary* if U has range $L^2(Y,\mu)$ and $\langle Uf, Ug \rangle_{L^2(Y,\mu)} = \langle f, g \rangle_{\mathcal{H}}$ for all f, $g \in \mathcal{H}$. In addition, a linear operator $S : D_S \subset \mathcal{H} \to \mathcal{H}$ is *unitarily equivalent* to a linear operator $T : D_T \subset L^2(Y,\mu) \to L^2(Y,\mu)$ if there is a unitary operator $U : \mathcal{H} \to L^2(Y,\mu)$ such that $T = USU^{-1}$, meaning that $D_T = UD_S$.

A characterization theorem for Carleman (resp. bi-Carleman) operators is as follows: *A necessary and sufficient condition that an operator $S \in \mathfrak{C}(\mathcal{H})$ be unitarily equivalent to a Carleman (resp., bi-Carleman) operator in $L^2(Y,\mu)$ is that S belong to the class $\mathfrak{C}_0(\mathcal{H})$ (resp., $\mathfrak{C}_{00}(\mathcal{H})$)* (see Korotkov [8] and Weidmann [14] (resp., Korotkov [9])); in particular case when $S = S^*$ and $\mathcal{H} = L^2(Y,\mu) = L^2(a,b)$, each of these two characterizations turns into a pioneering characterization of self-adjoint Carleman operators in $L^2(a,b)$, given in 1935 by von Neumann [13]. In a concrete setting with the underlying measure space (Y,μ) being $\mathbb{R}$ with the Lebesgue measure, the proof of the characterization theorem for Carleman operators was adjusted so as to yield the following, infinitely smoothing, result.

Proposition 3. *If $S \in \mathfrak{C}_0(\mathcal{H})$, then S is unitarily equivalent to a Carleman operator T in L^2, with an SK^∞ kernel.*

This result was proved in [11], Theorem 1.4. In the present paper, our goal is to state and prove a sharpened version of Proposition 3 when we restrict ourselves from the largest class of Carleman representable operators, $\mathfrak{C}_0(\mathcal{H})$, to the largest class of bi-Carleman representable operators, $\mathfrak{C}_{00}(\mathcal{H})$. The version adds to the hypothesis on the input operator S but sharpens the conclusion regarding quality of the output kernel profitably, and is as follows.

Theorem 4. *If $S \in \mathfrak{C}_{00}(\mathcal{H})$, then there exists a unitary operator $U : \mathcal{H} \to L^2$ such that the operator $T = USU^{-1}$ ($D_T = UD_S$) is a bi-Carleman operator with a K^∞ kernel.*

In comparison with the previous proposition, the result delivers, besides properties (i) and (ii) (see Definition 2), the beneficial property (iii) for the kernel of the unitarily transformed operator T, at the cost of strengthening the spectral assumption on the transformable operator S (cf. (1) and (2)). In no way could Theorem 4 be obtained as an immediate consequence, or a special case, of the statement of Proposition 3. Note also that the truth of the converse of Theorem 4 is immediate from the italicized characterization theorem for bi-Carleman operators above.

The next section of the present paper is entirely devoted to an independent proof of Theorem 4. The proof is, moreover, direct in the sense that it provides a direct method for constructing that unitary operator $U : \mathcal{H} \to L^2$ whose existence the theorem asserts. The method uses no spectral properties of S other than (2) to determine the action of U by specifying two orthonormal bases, of $\mathcal{H}$ and of L^2, one of which is meant to be the image by U of the other, the basis for L^2 may be chosen to be an infinitely smooth wavelet basis.

The result of Theorem 4 has recently been published without proof in [12, Theorem 1].

2. Proof of Theorem 4

The structure of the proof is as follows: first we employ the hypotheses (2) to decompose $\mathcal{H}$ into an orthogonal direct sum $\mathcal{H} = L \oplus L^{\perp}$ of infinite-dimensional subspaces in such a way that if E is the orthogonal projection of $\mathcal{H}$ onto L then the operators SE, S^*E are in the Schatten-von Neumann ideal $\mathfrak{S}_{1/4}(\mathcal{H})$. Also, at this stage, an orthonormal basis $\{f_1, f_2, f_3, \dots\} \subset D_S \cap D_{S^*}$ for $\mathcal{H}$ is formed from orthonormal bases of L and of $L^{\perp}$. In the next step (Step 2) we use the norms of the vectors Sf_k, S^*f_k ($k \in \mathbb{N}$) to give a general description of an orthonormal basis $\{u_1, u_2, u_3, \dots\}$, for L^2, of infinitely smooth functions on $\mathbb{R}$; the description is accompanied by showing that the Lemarié-Meyer wavelet basis (see [1], [7]) can serve as a concrete example of that basis. Then we define a unitary operator $U : \mathcal{H} \to L^2$ by sending in a suitable manner the basis $\{f_1, f_2, f_3, \dots\}$ onto the basis $\{u_1, u_2, u_3, \dots\}$. The last step of the proof (Step 3) consists entirely of proving that the operator $T = USU^{-1} : UD_S \to L^2$ is an integral operator with a K^∞ kernel.

Step 1. If $S \in \mathfrak{C}_{00}(\mathcal{H})$, then, by definition, there is an infinite sequence of orthonormal vectors e_k ($k \in \mathbb{N}$) belonging to a dense linear manifold $D \subset D_S \cap D_{S^*}$ in $\mathcal{H}$ and satisfying the limit relations of (2). By dropping down to a subsequence of $\{e_k\}_{k=1}^{\infty}$, also denoted e_k, assume that

$$\sum_k \left(\|Se_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|S^*e_k\|_{\mathcal{H}}^{\frac{1}{4}} \right) \leq 1 \tag{3}$$

(the sum notation $\sum_k$ will always be used instead of the more detailed symbol $\sum_{k=1}^{\infty}$). Let L be the closed linear span of the e_k's, and let $L^{\perp}$ be the orthogonal complement of L in $\mathcal{H}$. Assume, with no loss of generality, that $\dim L^{\perp} = \dim L = \infty$. Prove that

$$L \subset D_S \cap D_{S^*}. \tag{4}$$

Indeed, if $f = \sum\limits_k \langle f, e_k \rangle_{\mathcal{H}} e_k \in L$, then, by (3), both the series $\sum\limits_k \langle f, e_k \rangle_{\mathcal{H}} Se_k$, and $\sum\limits_k \langle f, e_k \rangle_{\mathcal{H}} S^*e_k$ converge in $\mathcal{H}$; since both S and S^* are closed, the vector f does belong to both D_S and D_{S^*}.

If E is the orthogonal projection of $\mathcal{H}$ onto L and I is the identity operator on $\mathcal{H}$, then it follows from (2) and (4) that

$$(I-E)D \subset (I-E)\left(D_S \cap D_{S^*}\right) \subset D_S \cap D_{S^*},$$

and hence that the linear manifold $(I-E)D$, being dense in $L^\perp$, contains an orthonormal basis $\left\{e_1^\perp, e_2^\perp, e_3^\perp, \dots\right\}$ for the subspace $L^\perp$ such that

$$\left\{e_1^\perp, e_2^\perp, e_3^\perp, \dots\right\} \subset D_S \cap D_{S^*}. \tag{5}$$

Then let $\{f_1, f_2, f_3, \dots\}$ be any orthonormal basis for $\mathcal{H}$ including all terms of the sequences $\{e_k\}_{k=1}^\infty$ and $\left\{e_k^\perp\right\}_{k=1}^\infty$:

$$\{f_1, f_2, f_3, ...\} = \{e_1, e_2, e_3, ...\} \cup \left\{e_1^\perp, e_2^\perp, e_3^\perp, ...\right\}. \tag{6}$$

Note that the operators S, S^* decompose as

$$S = (1-E)S + ES, \quad S^* = (1-E)S^* + ES^*; \tag{7}$$

and observe that the summands $Q = (1-E)S$, $\widetilde{Q} = (1-E)S^*$ here admit the representations

$$\begin{aligned} Qf &= \sum_k \left\langle Qf, e_k^\perp \right\rangle_{\mathcal{H}} e_k^\perp = \sum_k \left\langle f, S^* e_k^\perp \right\rangle_{\mathcal{H}} e_k^\perp, \\ \widetilde{Q}g &= \sum_k \left\langle \widetilde{Q}g, e_k^\perp \right\rangle_{\mathcal{H}} e_k^\perp = \sum_k \left\langle g, S e_k^\perp \right\rangle_{\mathcal{H}} e_k^\perp, \end{aligned} \tag{8}$$

on all f of $D_Q = D_S$ and on all g of $D_{\widetilde{Q}} = D_{S^*}$, respectively, by virtue of (5). Also observe that the adjoints of two other summands in (7) are given by $(ES)^* = S^*E$, $(ES^*)^* = SE$ because $E \in \mathfrak{R}(\mathcal{H})$. Hence, and because of (4), both $J = SE$ and $\widetilde{J} = S^*E$ are in $\mathfrak{R}(\mathcal{H})$. Moreover, J, $\widetilde{J} \in \mathfrak{S}_{1/4}(\mathcal{H})$ as

$$\sum_n \left(\|SEf_n\|_{\mathcal{H}}^{\frac{1}{4}} + \|S^*Ef_n\|_{\mathcal{H}}^{\frac{1}{4}} \right) = \sum_k \left(\|Se_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|S^*e_k\|_{\mathcal{H}}^{\frac{1}{4}} \right) < \infty$$

(see (6), (3)). In particular, it follows that the singular values, s_n, of J and the singular values, $\widetilde{s}_n$, of $\widetilde{J}$ constitute 1/2-summable sequences:

$$\sum_n s_n^{\frac{1}{2}} < \infty, \quad \sum_n \widetilde{s}_n^{\frac{1}{2}} < \infty. \tag{9}$$

If

$$J = \sum_n s_n \left\langle \cdot, p_n \right\rangle_{\mathcal{H}} q_n \quad \text{and} \quad \widetilde{J} = \sum_n \widetilde{s}_n \left\langle \cdot, \widetilde{p}_n \right\rangle_{\mathcal{H}} \widetilde{q}_n \tag{10}$$

are the Schmidt representations of J and $\widetilde{J}$, then the closedness of each of S and S^* yields

$$ES^*f = (ES^*)^{**}f = (SE)^*f = J^*f = \sum_n s_n \left\langle f, q_n \right\rangle_{\mathcal{H}} p_n \quad (f \in D_{S^*}), \tag{11}$$

$$ESg = (ES)^{**}g = (S^*E)^*g = \left(\widetilde{J}\right)^* g = \sum_n \widetilde{s}_n \langle g, \widetilde{q}_n \rangle_{\mathcal{H}} \widetilde{p}_n \quad (g \in D_S),$$

so (7) becomes

$$S = Q + \left(\widetilde{J}\right)^*, \quad S^* = \widetilde{Q} + J^*. \tag{12}$$

Step 2. In this step, we construct a candidate for the desired unitary operator $U : \mathcal{H} \to L^2$ in the theorem. For each $f \in D_S \cap D_{S^*}$ and for each $h \in \mathcal{H}$, let

$$\begin{gathered} z(f) = \|Sf\|_{\mathcal{H}} + \|S^*f\|_{\mathcal{H}}, \\ d(h) = \|Jh\|_{\mathcal{H}}^{\frac{1}{4}} + \|J^*h\|_{\mathcal{H}}^{\frac{1}{4}} + \left\|\widetilde{J}h\right\|_{\mathcal{H}}^{\frac{1}{4}} + \left\|\left(\widetilde{J}\right)^* h\right\|_{\mathcal{H}}^{\frac{1}{4}}, \end{gathered} \tag{13}$$

and use (3) to write

$$\begin{gathered} z(e_k) = \|Se_k\|_{\mathcal{H}} + \|S^*e_k\|_{\mathcal{H}} \le \|Se_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|S^*e_k\|_{\mathcal{H}}^{\frac{1}{4}} \\ \le \|Se_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|J^*e_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|S^*e_k\|_{\mathcal{H}}^{\frac{1}{4}} + \left\|\left(\widetilde{J}\right)^* e_k\right\|_{\mathcal{H}}^{\frac{1}{4}} = d(e_k), \end{gathered} \tag{14}$$

for each $k \in \mathbb{N}$. Moreover, the orthonormality of the e_k and the compactness of each of J and $\widetilde{J}$ guarantee that

$$d(e_k) \to 0 \quad \text{as } k \to \infty. \tag{15}$$

Notation. If an equivalence class $f \in L^2$ contains a function belonging to $C(\mathbb{R}, \mathbb{C})$, then we shall henceforth use $[f]$ to denote that function.

Take any orthonormal basis $\{u_1, u_2, u_3, \dots\}$ for L^2, with the properties:

(a) for each i and for each $n \in \mathbb{N}$, the ith derivative, $[u_n]^{(i)}$, of $[u_n]$ is in $C(\mathbb{R}, \mathbb{C})$ (here and throughout, the letter i is reserved for all non-negative integers),

(b) the sequence $\{u_n\}_{n=1}^{\infty}$ splits into two infinite subsequences $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ satisfying

$$\{h_1, h_2, h_3, \dots\} = \{u_1, u_2, u_3, \dots\} \setminus \{g_1, g_2, g_3, \dots\}, \tag{16}$$

and such that if $H_{k,i} := \left\|[h_k]^{(i)}\right\|_{C(\mathbb{R},\mathbb{C})}$, $G_{k,i} := \left\|[g_k]^{(i)}\right\|_{C(\mathbb{R},\mathbb{C})}$ then, for each i,

$$\sum_k H_{k,i} < \infty, \tag{17}$$

$$\sum_k z(v_k) H_{m(k),i} < \infty, \tag{18}$$

$$\sum_k z\left(e_k^{\perp}\right) H_{n(k),i} < \infty, \tag{19}$$

$$\sum_k d(x_k) G_{k,i} < \infty, \tag{20}$$

where $\{n(k)\}_{k=1}^{\infty}$, $\{m(k)\}_{k=1}^{\infty}$ are two infinite subsequences of the sequence $\{n\}_{n=1}^{\infty}$ associated with each other through

$$\{m(1), m(2), m(3), \dots\} = \mathbb{N} \setminus \{n(1), n(2), n(3), \dots\}, \tag{21}$$

and $\{x_k\}_{k=1}^{\infty}$, $\{v_k\}_{k=1}^{\infty}$ are two infinite subsequences of $\{e_k\}_{k=1}^{\infty}$ related with each other by

$$\{x_1, x_2, x_3, \dots\} = \{e_1, e_2, e_3, \dots\} \setminus \{v_1, v_2, v_3, \dots\}. \tag{22}$$

It is to be noted that since (cf. (14)) $z(v_k) \le 1$ for all $k \in \mathbb{N}$, the requirement (18) becomes superfluous if condition (17) holds; we have recorded this requirement here for convenience of presentation only.

Example. A good example of a basis $\{u_1, u_2, u_3, \dots\}$ with the above properties can be adopted from the wavelet theory, as follows. Let ψ be the Lemarié-Meyer wavelet,

$$[\psi](s) = \frac{1}{2\pi} \int_{\mathbb{R}} \operatorname{sgn} \xi \, e^{\imath \xi (\frac{1}{2} + s)} b(|\xi|) \, d\xi \quad (s \in \mathbb{R}) \tag{23}$$

with the bell function b being infinitely smooth and compactly supported on $[0, +\infty)$ (see, e.g., [1, § 4] or [7, Example D, p. 62] for details). Then $[\psi]$ is of the Schwartz class $\mathcal{S}(\mathbb{R})$, so its every derivative $[\psi]^{(i)}$ is in $C(\mathbb{R}, \mathbb{C})$. In addition, the "mother wavelet" ψ generates an orthonormal basis $\{\psi_{\alpha\beta}\}_{\alpha,\, \beta \in \mathbb{Z}}$ for L^2 by

$$\psi_{\alpha\beta} = 2^{\frac{\alpha}{2}} \psi(2^{\alpha} \cdot - \beta) \quad (\alpha,\, \beta \in \mathbb{Z}).$$

In a completely arbitrary manner, rearrange the two-indexed set $\{\psi_{\alpha\beta}\}_{\alpha,\, \beta \in \mathbb{Z}}$ into a simple sequence so that it becomes $\{u_n\}_{n=1}^{\infty}$. To show that the latter has property (b), suppose $u_n = \psi_{\alpha_n \beta_n}$ whenever $n \in \mathbb{N}$, in accordance with that rearrangement. It is easily verified then that, for each i,

$$\left\| [u_n]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})} = \left\| [\psi_{\alpha_n \beta_n}]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})} \le D_n A_i \quad \text{for all } n \in \mathbb{N},$$

where

$$D_n := \begin{cases} 2^{\alpha_n^2} & \text{if } \alpha_n > 0, \\ 2^{\alpha_n/2} & \text{if } \alpha_n \le 0, \end{cases} \qquad A_i := 2^{(i+1/2)^2} \left\| [\psi]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})}.$$

If a subsequence $\{l(k)\}_{k=1}^{\infty}$ of $\{l\}_{l=1}^{\infty}$ satisfies $\alpha_{l(k)} \to -\infty$ as $k \to \infty$, then split $\{u_n\}_{n=1}^{\infty}$ into $\left\{h_k := u_{l(k)}\right\}_{k=1}^{\infty}$ and $\left\{g_k := u_{r(k)}\right\}_{k=1}^{\infty}$, with

$$\{r(1), r(2), r(3), \dots\} = \mathbb{N} \setminus \{l(1), l(2), l(3), \dots\},$$

and observe that

$$\sum_k D_{l(k)} < \infty. \tag{24}$$

Then, for each i, the sums in (17), (18), (19), and (20), are bounded respectively by

$$A_i \sum_k D_{l(k)}, \quad A_i \sum_k D_{l(m(k))}, \quad A_i \sum_k z\left(e_k^{\perp}\right) D_{l(n(k))}, \quad \text{and}$$
$$A_i \sum_k d(x_k) D_{r(k)},$$

where the last-written two expressions can always be made finite by an appropriate choice of subsequences $\{n(k)\}_{k=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ and $\{x_k\}_{k=1}^{\infty}$ of $\{e_k\}_{k=1}^{\infty}$ (see (24), (15)).

Let us return to the proof of the theorem. Observe, by (6), (22), (16), and (21), that

$$\begin{aligned} \{f_1, f_2, f_3, \dots\} &= \{x_1, x_2, x_3, \dots\} \\ &\quad \cup \{v_1, v_2, v_3, \dots\} \cup \left\{e_1^{\perp}, e_2^{\perp}, e_3^{\perp}, \dots\right\}, \\ \{u_1, u_2, u_3, \dots\} &= \{g_1, g_2, g_3, \dots\} \\ &\quad \cup \left\{h_{m(1)}, h_{m(2)}, h_{m(3)}, \dots\right\} \cup \left\{h_{n(1)}, h_{n(2)}, h_{n(3)}, \dots\right\}, \end{aligned} \tag{25}$$

and define a unitary operator $U : \mathcal{H} \to L^2$ on the basis vectors by setting

$$Ux_k = g_k, \quad Uv_k = h_{m(k)}, \quad Ue_k^{\perp} = h_{n(k)} \quad \text{for all } k \in \mathbb{N}. \tag{26}$$

It is convenient and harmless to assume, in addition, that, for each $n \in \mathbb{N}$,

$$Uf_n = u_n, \quad Uy_n = h_n. \tag{27}$$

Step 3. This step of the proof is to prove that the unitary operator U defined in (26) does indeed possess the property that $T = USU^{-1}$ ($D_T = UD_S$) is a bi-Carleman operator with a K^{∞} kernel. First, for this purpose, verify that the four operators (see the decompositions (12)) $P = UQU^{-1}$ ($D_P = D_T$), $\widetilde{P} = U\widetilde{Q}U^{-1}$ ($D_{\widetilde{P}} = D_{T^*} = UD_{S^*}$), $F = UJ^*U^{-1}$ ($D_F = L^2$), and $\widetilde{F} = U\left(\widetilde{J}\right)^* U^{-1}$ ($D_{\widetilde{F}} = L^2$), are all Carleman operators with SK^{∞} kernels. The checking is straightforward, and goes by representing all pertinent kernels and Carleman functions as infinitely smooth sums of termwise differentiable series of infinitely smooth functions as follows.

Combine (8) with (26) to infer that

$$\begin{aligned} Pf &= \sum_k \left\langle f, T^* h_{n(k)} \right\rangle_{L^2} h_{n(k)} \quad \text{for all } f \in D_T = UD_S, \\ \widetilde{P}g &= \sum_k \left\langle g, T h_{n(k)} \right\rangle_{L^2} h_{n(k)} \quad \text{for all } g \in D_{T^*} = UD_{S^*}, \end{aligned} \tag{28}$$

where

$$\begin{aligned} T^* h_{n(k)} &= \sum_n \left\langle e_k^{\perp}, S f_n \right\rangle_{\mathcal{H}} u_n, \\ T h_{n(k)} &= \sum_n \left\langle e_k^{\perp}, S^* f_n \right\rangle_{\mathcal{H}} u_n \quad (k \in \mathbb{N}), \end{aligned} \tag{29}$$

with the series convergent in L^2. Prove that, for any fixed i, the series

$$\sum_n \left\langle e_k^{\perp}, S f_n \right\rangle_{\mathcal{H}} [u_n]^{(i)}(s), \quad \sum_n \left\langle e_k^{\perp}, S^* f_n \right\rangle_{\mathcal{H}} [u_n]^{(i)}(s) \quad (k \in \mathbb{N})$$

converge in the norm of $C(\mathbb{R}, \mathbb{C})$. Indeed, all these series are pointwise dominated on $\mathbb{R}$ by one series

$$\sum_n \left(\|S f_n\|_{\mathcal{H}} + \|S^* f_n\|_{\mathcal{H}} \right) \left| [u_n]^{(i)}(s) \right|,$$

which converges uniformly on $\mathbb{R}$ because its component subseries (see (13), (26), (25))

$$\sum_k z(x_k) \left| [g_k]^{(i)}(s) \right|, \quad \sum_k z(v_k) \left| \left[h_{m(k)} \right]^{(i)}(s) \right|,$$
$$\sum_k z\left(e_k^{\perp}\right) \left| \left[h_{n(k)} \right]^{(i)}(s) \right|$$

are in turn dominated by the series of (20), of (18), and of (19), respectively (see also (14), (22)). Whence it follows that, for each $k \in \mathbb{N}$,

$$\left\| \left[T^* h_{n(k)} \right]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})} \le C_i^*, \quad \left\| \left[T h_{n(k)} \right]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})} \le C_i, \tag{30}$$

with constants C_i^* and C_i independent of k. From (13) it also follows via the unitarity of U that

$$\left\| T^* h_{n(k)} \right\|_{L^2} \le z\left(e_k^{\perp}\right), \quad \left\| T h_{n(k)} \right\|_{L^2} \le z\left(e_k^{\perp}\right) \quad (k \in \mathbb{N}). \tag{31}$$

If functions $\boldsymbol{P}$, $\widetilde{\boldsymbol{P}} : \mathbb{R}^2 \to \mathbb{C}$ and Carleman functions $\boldsymbol{p}$, $\widetilde{\boldsymbol{p}} : \mathbb{R} \to L^2$ are defined as

$$\begin{aligned}\boldsymbol{P}(s,t) &= \sum_k \left[h_{n(k)}\right](s)\overline{\left[T^*h_{n(k)}\right](t)},\\ \widetilde{\boldsymbol{P}}(s,t) &= \sum_k \left[h_{n(k)}\right](s)\overline{\left[Th_{n(k)}\right](t)},\\ \boldsymbol{p}(s) = \overline{\boldsymbol{P}(s,\cdot)} &= \sum_k \overline{\left[h_{n(k)}\right](s)}T^*h_{n(k)},\\ \widetilde{\boldsymbol{p}}(s) = \overline{\widetilde{\boldsymbol{P}}(s,\cdot)} &= \sum_k \overline{\left[h_{n(k)}\right](s)}Th_{n(k)},\end{aligned} \tag{32}$$

whenever s, $t \in \mathbb{R}$, then, for all non-negative integer i and j,

$$\begin{aligned}\frac{\partial^{i+j}\boldsymbol{P}}{\partial s^i\partial t^j}(s,t) &= \sum_k \left[h_{n(k)}\right]^{(i)}(s)\overline{\left[T^*h_{n(k)}\right]^{(j)}(t)},\\ \frac{\partial^{i+j}\widetilde{\boldsymbol{P}}}{\partial s^i\partial t^j}(s,t) &= \sum_k \left[h_{n(k)}\right]^{(i)}(s)\overline{\left[Th_{n(k)}\right]^{(j)}(t)},\\ \frac{d^i\boldsymbol{p}}{ds^i}(s) &= \sum_k \overline{\left[h_{n(k)}\right]^{(i)}(s)}T^*h_{n(k)},\\ \frac{d^j\widetilde{\boldsymbol{p}}}{ds^j}(s) &= \sum_k \overline{\left[h_{n(k)}\right]^{(j)}(s)}Th_{n(k)},\end{aligned}$$

because, in view of (30), (17), (31), and (19), the series just displayed converge (and even absolutely) in $C(\mathbb{R}^2,\mathbb{C})$ and $C(\mathbb{R},L^2)$, respectively. Thus,

$$\frac{\partial^{i+j}\boldsymbol{P}}{\partial s^i\partial t^j},\ \frac{\partial^{i+j}\widetilde{\boldsymbol{P}}}{\partial s^i\partial t^j} \in C(\mathbb{R}^2,\mathbb{C}), \quad \frac{d^i\boldsymbol{p}}{ds^i},\ \frac{d^j\widetilde{\boldsymbol{p}}}{ds^j} \in C(\mathbb{R},L^2), \tag{33}$$

whenever i and j are non-negative integers. Also, from (31) and (19), it follows that the series of (28) (viewed, of course, as series with terms belonging to $C(\mathbb{R},\mathbb{C})$) converge and even absolutely in the $C(\mathbb{R},\mathbb{C})$ norm, and therefore that their pointwise sums are none other than $[Pf]$ and $\left[\widetilde{P}g\right]$, respectively. On the other hand, the established properties of the series of (32) make it possible to write, for each temporarily fixed $s \in \mathbb{R}$, the following chains of relations

$$\begin{aligned}\sum_k \left\langle f, T^*h_{n(k)}\right\rangle_{L^2}\left[h_{n(k)}\right](s) &= \left\langle f, \sum_k \overline{\left[h_{n(k)}\right](s)}T^*h_{n(k)}\right\rangle_{L^2}\\ &= \int_{\mathbb{R}}\left(\sum_k \left[h_{n(k)}\right](s)\overline{\left[T^*h_{n(k)}\right](t)}\right)f(t)\,dt = \int_{\mathbb{R}}\boldsymbol{P}(s,t)f(t)\,dt,\end{aligned}$$

$$\sum_k \left\langle g, Th_{n(k)} \right\rangle_{L^2} \left[h_{n(k)}\right](s) = \left\langle g, \sum_k \overline{\left[h_{n(k)}\right](s)} Th_{n(k)} \right\rangle_{L^2}$$
$$= \int_{\mathbb{R}} \left(\sum_k \left[h_{n(k)}\right](s) \overline{\left[Th_{n(k)}\right](t)} \right) g(t)\,dt = \int_{\mathbb{R}} \widetilde{\boldsymbol{P}}(s,t) g(t)\,dt,$$

whenever f is in D_P and g is in $D_{\widetilde{P}}$. These and (33) imply that $P : D_P \to L^2$, $\widetilde{P} : D_{\widetilde{P}} \to L^2$ are integral operators with the SK^∞ kernels $\boldsymbol{P}$ and $\widetilde{\boldsymbol{P}}$, respectively.

Now define two auxiliary operators B, $\widetilde{B} \in \mathfrak{S}_1(\mathcal{H})$ by (cf. (10))

$$B = \sum_n s_n^{\frac{1}{4}} \left\langle \cdot, p_n \right\rangle_{\mathcal{H}} q_n, \quad \widetilde{B} = \sum_n \widetilde{s}_n^{\frac{1}{4}} \left\langle \cdot, \widetilde{p}_n \right\rangle_{\mathcal{H}} \widetilde{q}_n, \tag{34}$$

and apply the Schwarz inequality to infer that if $\|f\|_{\mathcal{H}} = 1$ then

$$\begin{aligned} b(f) &:= \|Bf\|_{\mathcal{H}} + \|B^*f\|_{\mathcal{H}} + \left\|\widetilde{B}f\right\|_{\mathcal{H}} + \left\|\left(\widetilde{B}\right)^* f\right\|_{\mathcal{H}} \\ &= \sqrt{\sum_n s_n^{\frac{1}{2}} \left|\left\langle f, p_n \right\rangle_{\mathcal{H}}\right|^2} + \sqrt{\sum_n s_n^{\frac{1}{2}} \left|\left\langle f, q_n \right\rangle_{\mathcal{H}}\right|^2} \\ &+ \sqrt{\sum_n \widetilde{s}_n^{\frac{1}{2}} \left|\left\langle f, \widetilde{p}_n \right\rangle_{\mathcal{H}}\right|^2} + \sqrt{\sum_n \widetilde{s}_n^{\frac{1}{2}} \left|\left\langle f, \widetilde{q}_n \right\rangle_{\mathcal{H}}\right|^2} \\ &= \left\|(J^*J)^{\frac{1}{8}} f\right\|_{\mathcal{H}} + \left\|(JJ^*)^{\frac{1}{8}} f\right\|_{\mathcal{H}} \\ &+ \left\|\left(\left(\widetilde{J}\right)^* \widetilde{J}\right)^{\frac{1}{8}} f\right\|_{\mathcal{H}} + \left\|\left(\widetilde{J}\left(\widetilde{J}\right)^*\right)^{\frac{1}{8}} f\right\|_{\mathcal{H}} \\ &\le \|Jf\|_{\mathcal{H}}^{\frac{1}{4}} + \|J^*f\|_{\mathcal{H}}^{\frac{1}{4}} + \left\|\widetilde{J}f\right\|_{\mathcal{H}}^{\frac{1}{4}} + \left\|\left(\widetilde{J}\right)^* f\right\|_{\mathcal{H}}^{\frac{1}{4}} = d(f). \end{aligned} \tag{35}$$

Then observe that the inducing kernels of the integral operators $F = UJ^*U^{-1}$, $\widetilde{F} = U\left(\widetilde{J}\right)^* U^{-1}$ of $\mathfrak{S}_{1/4}(L^2)$ are the sums of the bilinear series

$$\begin{aligned} &\sum_n s_n^{\frac{1}{2}} UB^*q_n(s) \overline{UBp_n(t)} \quad \left(= \sum_n s_n Up_n(s) \overline{Uq_n(t)} \right), \\ &\sum_n \widetilde{s}_n^{\frac{1}{2}} U\left(\widetilde{B}\right)^* \widetilde{q}_n(s) \overline{U\widetilde{B}\widetilde{p}_n(t)} \quad \left(= \sum_n \widetilde{s}_n U\widetilde{p}_n(s) \overline{U\widetilde{q}_n(t)} \right), \end{aligned} \tag{36}$$

in the sense of almost everywhere convergence on $\mathbb{R}^2$ (see (11), (34)). The functions used in these expansions can be written as the series

$$UBp_k = \sum_n \left\langle p_k, B^*f_n \right\rangle_{\mathcal{H}} u_n, \quad UB^*q_k = \sum_n \left\langle q_k, Bf_n \right\rangle_{\mathcal{H}} u_n,$$

$$U\widetilde{B}\widetilde{p}_k = \sum_n \left\langle \widetilde{p}_k, \left(\widetilde{B}\right)^* f_n \right\rangle_{\mathcal{H}} u_n, \quad U\left(\widetilde{B}\right)^* \widetilde{q}_k = \sum_n \left\langle \widetilde{q}_k, \widetilde{B} f_n \right\rangle_{\mathcal{H}} u_n$$

all converging in L^2. Show that, for any fixed i, the functions $[UBp_k]^{(i)}$, $[UB^*q_k]^{(i)}$, $\left[U\widetilde{B}\widetilde{p}_k\right]^{(i)}$, $\left[U\left(\widetilde{B}\right)^*\widetilde{q}_k\right]^{(i)}$ ($k \in \mathbb{N}$) make sense, are all in $C(\mathbb{R}, \mathbb{C})$, and their $C(\mathbb{R}, \mathbb{C})$ norms are bounded independent of k. Indeed, all the series

$$\sum_n \langle p_k, B^* f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s), \quad \sum_n \langle q_k, B f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s),$$

$$\sum_n \left\langle \widetilde{p}_k, \left(\widetilde{B}\right)^* f_n \right\rangle_{\mathcal{H}} [u_n]^{(i)}(s), \quad \sum_n \left\langle \widetilde{q}_k, \widetilde{B} f_n \right\rangle_{\mathcal{H}} [u_n]^{(i)}(s) \quad (k \in \mathbb{N})$$

are dominated by one series

$$\sum_n \left(\|B^* f_n\|_{\mathcal{H}} + \|B f_n\|_{\mathcal{H}} + \left\| \left(\widetilde{B}\right)^* f_n \right\|_{\mathcal{H}} + \left\| \widetilde{B} f_n \right\|_{\mathcal{H}} \right) \left| [u_n]^{(i)}(s) \right|$$

that converges uniformly on $\mathbb{R}$, because it is composed of the two uniformly convergent subseries on $\mathbb{R}$ (see (26), (27)):

$$\sum_k b(y_k) \left| [h_k]^{(i)}(s) \right|, \quad \sum_k b(x_k) \left| [g_k]^{(i)}(s) \right|,$$

where the first series is dominated by the series of (17) multiplied by $2(\|B\| + \|\widetilde{B}\|)$, and the second by the series of (20), because of (35).

Now define functions $\boldsymbol{F}$, $\widetilde{\boldsymbol{F}} : \mathbb{R}^2 \to \mathbb{C}$ and Carleman functions $\boldsymbol{f}$, $\widetilde{\boldsymbol{f}} : \mathbb{R} \to L^2$ by

$$\boldsymbol{F}(s,t) = \sum_n s_n^{\frac{1}{2}} [UB^*q_n](s) \overline{[UBp_n](t)},$$

$$\widetilde{\boldsymbol{F}}(s,t) = \sum_n \widetilde{s}_n^{\frac{1}{2}} \left[U\left(\widetilde{B}\right)^* \widetilde{q}_n\right](s) \overline{\left[U\widetilde{B}\widetilde{p}_n\right](t)},$$

$$\boldsymbol{f}(s) = \overline{\boldsymbol{F}(s,\cdot)} = \sum_n s_n^{\frac{1}{2}} \overline{[UB^*q_n](s)} UBp_n,$$

$$\widetilde{\boldsymbol{f}}(s) = \overline{\widetilde{\boldsymbol{F}}(s,\cdot)} = \sum_n \widetilde{s}_n^{\frac{1}{2}} \overline{\left[U\left(\widetilde{B}\right)^* \widetilde{q}_n\right](s)} U\widetilde{B}\widetilde{p}_n,$$

whenever s, $t \in \mathbb{R}$ (cf. (36)). Then, for all non-negative integers i, j and all s, $t \in \mathbb{R}$,

$$\frac{\partial^{i+j}\boldsymbol{F}}{\partial s^i \partial t^j}(s,t) = \sum_n s_n^{\frac{1}{2}} [UB^*q_n]^{(i)}(s) \overline{[UBp_n]^{(j)}(t)},$$

$$\frac{\partial^{i+j}\widetilde{\boldsymbol{F}}}{\partial s^i \partial t^j}(s,t) = \sum_n \widetilde{s}_n^{\frac{1}{2}} \left[U\left(\widetilde{B}\right)^* \widetilde{q}_n\right]^{(i)}(s) \overline{\left[U\widetilde{B}\widetilde{p}_n\right]^{(j)}(t)},$$

$$\frac{d^i \boldsymbol{f}}{ds^i}(s) = \sum_n s_n^{\frac{1}{2}} \overline{[UB^* q_n]^{(i)}(s)} UBp_n,$$
$$\frac{d^j \widetilde{\boldsymbol{f}}}{ds^j}(s) = \sum_n \widetilde{s}_n^{\frac{1}{2}} \overline{\left[U\left(\widetilde{B}\right)^* \widetilde{q}_n\right]^{(j)}(s)} U\widetilde{B}\widetilde{p}_n,$$

as the series just written converge (and even absolutely) in $C(\mathbb{R}^2, \mathbb{C})$ and $C(\mathbb{R}, L^2)$, respectively, due to (9). Therefore, it follows that $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ are the SK^∞ kernels of F and of $\widetilde{F}$, respectively.

Now, since (12) implies $T = P + \widetilde{F}$, $T^* = \widetilde{P} + F$, it follows from the above that both the operators $T : D_T \to L^2$ and $T^* : D_{T^*} \to L^2$ admit the integral representations,

$$(Tf)(s) = \int_{\mathbb{R}} \boldsymbol{T}(s,t) f(t)\, dt \ (f \in D_T), \quad (T^* g)(s) = \int_{\mathbb{R}} \boldsymbol{T^*}(s,t) g(t)\, dt \ (g \in D_{T^*}),$$

where $\boldsymbol{T} = \boldsymbol{P} + \widetilde{\boldsymbol{F}}$, $\boldsymbol{T^*} = \widetilde{\boldsymbol{P}} + \boldsymbol{F}$ are SK^∞ kernels. Then (cf. Remark 1) $\boldsymbol{T}(s,t) = \overline{\boldsymbol{T^*}(t,s)}$ for all $(s,t) \in \mathbb{R}^2$; hence $\boldsymbol{T}(\cdot,t) = \overline{\boldsymbol{T^*}(t,\cdot)}$ in the L^2 sense for each fixed $t \in \mathbb{R}$, which shows conclusively that $\boldsymbol{T}$ is a K^∞ kernel of T. The proof of the theorem is complete.

3. Concluding Remark

In virtue of Theorem 4 and Remark 1, one can confine one's attention (with no essential loss of generality) to second-kind integral equations with K^∞ kernels. One of the main technical advantages of dealing with such kernels is that their restrictions to compact rectangles in $\mathbb{R}^2$ are quite amenable to the methods of the classical theory of ordinary integral equations, and approximate their original kernels with respect to norms $\|\cdot\|_{C(\mathbb{R}^2,\mathbb{C})}$ and $\|\cdot\|_{C(\mathbb{R},L^2)}$. This, for instance, can be used directly to establish an explicit theory of spectral functions for any Hermitian K^∞ kernel ($\boldsymbol{T}(s,t) = \boldsymbol{T'}(s,t)$) by a development conceptually the same as the one given by Carleman [2, pp. 25-51] for a symmetric Carleman kernel on $[a,b] \times [a,b]$ that induces an unbounded integral operator in $L^2(a,b)$, but is, by construction, the pointwise limit of its symmetric Hilbert-Schmidt subkernels satisfying the mean square continuity condition. We believe that, with respect to K^∞ kernels, this Carleman's line of development can be extended far beyond the restrictive assumption of a Hermitian kernel.

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