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ON COMPUTING THE VULNERABILITY OF SOME GRAPHS AS AVERAGE

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Abstract: We investigate the resistance of a communication network to disruption of operation after the failure of certain stations or communication links, we use several vulnerability measures. If we think of a graph as modeling a network, the average lower independence number of a graph is one measure of graph vulnerability. For a vertex v of a graph $G = (V, E)$, the lower independence number $i_v(G)$ of G relative to v is the minimum cardinality of a maximal independent set of G that contains v. The average lower independence number of G, denoted by $i_{av}(G)$, is the value $\frac{1}{|V(G)|}\sum_{v\in V(G)} i_v(G)$. In this paper, we define and examine this parameter and consider the average lower independence number of binomial trees and middle graphs of some special graphs.

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1. Introduction

In a communication network, the vulnerability measures the resistance of net-

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work to disruption of operation after the failure of certain stations or communication links. The stability of communication networks is of prime importance to network designers.

In analysis of vulnerability of a communication network to disruption, three quantities that come to mind are:

1. The size of the largest remaining group within which the mutual communication can still occur,

2. The number of elements that are not functioning,

3. The number of remaining connected subnetworks.

If we think of a graph as modeling of the communication network, many graph theoretical parameters will be used to describe the stability of communication networks including connectivity, integrity, toughness, tenacity, binding number and scattering number (see [2], [3], [5], [6], [7], [11]).

A graph G is denoted by $G = (V(G), E(G))$, where $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G. The order of G is denoted by n, which is the size of $V(G)$.

The open neighbourhood of v is the set $N(v) = \{u \in V(G) | uv \in E(G)\}$ and the closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$. Also we write $\overline{N}(v) = V(G) - N[v]$. If $S \subseteq V(G)$, then its open neighborhood is $N(S) = \bigcup_{v \in S} N(v)$ and its closed neighborhood is $N[S] = N(S) \cup S$. The *degree* of a vertex v of G, denoted by $\deg(v)$, is the size of its open neighborhood. A vertex v of degree 1 is called an isolated vertex. For a subset A of $V(G)$, $G[A]$ denotes the subgraph induced by the vertices of A.

In a graph $G = (V(G), E(G))$, a subset $S \subseteq V(G)$ of vertices is a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex of S. The *dominating number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. The *independent domination number* (also called *the lower independence number*) $i(G)$ of G is the minimum cardinality of a set that is both independent and dominating. The *independence number* $\beta(G)$ of G is the maximum cardinality of an independent set which is a set of vertices of G whose elements are pairwise nonadjacent.

Henning (see [6]) introduced the concept of average independence. For a vertex v of a graph $G = (V, E)$, the *lower independence number* $i_v(G)$ of G relative to v is the minimum cardinality of a maximal independent set of G that contains v. The *average lower independence number* of G, denoted by $i_{av}(G)$ is the value $\frac{1}{|V(G)|} \sum_{v \in V(G)} i_v(G)$ (see [4], [6]).

The next section includes basic results on the average lower independence number of a graph G. In Section 3, we formalize average lower independence number of binomial trees and middle graphs of some basic graphs.

2. Basic Results

In this section, we will review some of the known results on average lower independence number.

Theorem 1. (see [4], [6]) *For every vertex v in a graph:*

1. $i(G) \leq i_v(G) \leq \beta(G)$.

2. $i(G) \leq i_{av}(G) \leq \beta(G)$.

Theorem 2. (see [6]) *For any graph G of order n with an independent domination number i and an independence number β,*

$$i_{av}(G) \leq \beta - \frac{i(\beta - i)}{n}.$$

Theorem 3. (see [6]) *If T is a tree of order $n \geq 2$, then*

$$i_{av}(G) \leq n - 2 + \frac{2}{n}.$$

3. Average Lower Independence Number of Binomial Trees and Middle Graphs of Some Special Graphs

In this section, the average lower independence number of middle graphs of basic graphs and binomial tree B_n are calculated.

Definition 4. (see [8]) The binomial tree B_n is an ordered tree defined recursively. The binomial tree B_0 consists of a single vertex. The binomial tree B_n consists of two binomial trees B_{n-1} that are linked together: the root of one is the leftmost child of the root of the other.

Definition 5. (see [1], [9], [10]) The middle graph $M(G)$ of a graph is the graph obtained from G by inserting a new vertex into every edge of G and by joining those pairs of these new vertices with edges which lie on adjacent edges of G (see Figure 2).

Theorem 6. *For any binomial tree B_n,*

$$i_{av}(B_n) = 2^{n-1} \qquad (n > 1).$$

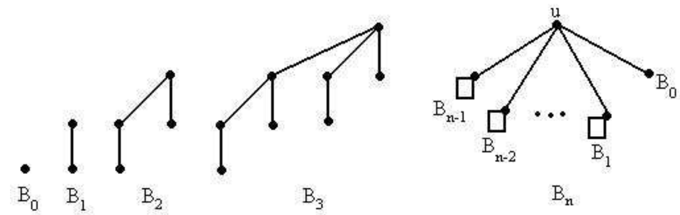

Figure 1: Binomial trees

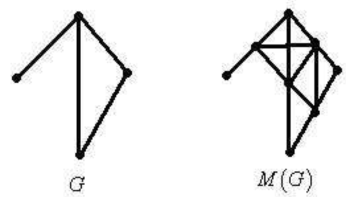

Figure 2: Middle graph

Proof. Any binomial tree B_n consists of 2^n vertices. The degrees of the vertices are $1, 2, \ldots, n$. There are 2^{n-1} leaves, and each of the other 2^{n-1} vertices is adjacent to only one leaf. For a vertex v of a graph B_n, the lower independence number, denoted by $i_v(B_n)$, is the minimum cardinality of a maximal independent set of B_n that contains v.

If we take any leaf as v, then obviously, this leaf is adjacent to only one vertex with a vertex degree different than 1. Thus, a maximal independent set of minimum cardinality containing v has all the other leaves of the graph or has some vertices of the graph that are independent, except the leaves adjacent to those. Thus, $i_v(B_n) = 2^{n-1}$. Since the graph B_n has 2^{n-1} leaves, $\sum_v i_v(B_n) = 2^{n-1}(2^{n-1}) = 2^{2n-2}$.

If we take any vertex as v which is different than a leaf, then since there is only one leaf adjacent to v, an independent set does not include this leaf, but all the other leaves of the graph or some vertices that are independent, except the leaves adjacent to those but not adjacent to v must be in the independent set. Thus, $i_v(B_n) = 2^{n-1}$. Since there is 2^{n-1} vertices except leaves, $\sum_v i_v(B_n) = 2^{n-1}(2^{n-1}) = 2^{2n-2}$. Hence, for a graph B_n, the average lower independence number is,

$$i_{av}(B_n) = \frac{1}{|V(B_n)|}\left(\sum_{v\in V(B_n) i_v}(B_n)\right) = \frac{1}{2^n}(2^{2n-2}+2^{2n-2})$$
$$= 2^{2n-2+1-n} = 2^{n-1}.$$

Thus, the proof is completed. □

Theorem 7. *Let $M(P_n)$ be the middle graph of P_n. Then*

$$i_{av}(M(P_n)) = \frac{n^2+n-1}{2n-1}.$$

Proof. For a vertex v of a graph $M(P_n)$, the lower independence number, denoted by $i_v(M(P_n))$, is the minimum cardinality of a maximal independent set of $M(P_n)$ that contains v. A graph $M(P_n)$ is a connected graph with $|V(M(P_n))| = 2n-1$ vertices. We have two cases for the proof according to the number of vertices of P_n: odd or even.

Case 1. If n is even.

Let the name of the set of new vertices which are added to the graph P_n to obtain $M(P_n)$ be V^*. The independent set which gives the independence number of the induced subgraph $G[V^*] = P_{n-1} = P_m$ is unique and $\beta(G[V^*]) = \frac{m+1}{2} = \frac{n}{2}$. In this independent set, for every vertex v, a maximal independent set with minimum cardinality containing v in graph $M(P_n)$ has $\beta(G[V^*]) = \frac{m+1}{2} = \frac{n}{2}$ vertices. These $\frac{n}{2}$ vertices can be the vertices which give the independence number of graph $G[V^*]$. Thus, $i_v(M(P_n)) = \frac{n}{2}$. Since there exist $\frac{n}{2}$ vertices in the set which gives the number $\beta(G[V^*])$, $\sum_v i_v(M(P_n)) = \frac{n}{2}\left(\frac{n}{2}\right) = \frac{n^2}{4}$. For every vertex v of P_m except those $\frac{n}{2}$ vertices of the independent set, $i_v(M(P_n)) = \frac{n}{2}+1$, including two vertices of P_n and $\frac{n}{2}-1$ vertices of P_m. Since there exist $m - \frac{m+1}{2} = \frac{2m-m-1}{2} = \frac{m-1}{2} = \frac{n-2}{2} = \frac{n}{2}-1$ vertices totally remaining, $\sum_v i_v(M(P_n)) = \left(\frac{n}{2}-1\right)\left(\frac{n}{2}+1\right) = \frac{n^2}{4}-1$.

In graph $M(P_n)$, for every vertex v of P_n, a maximal independent set with minimum cardinality includes two vertices from P_n and $\frac{n}{2}-1$ vertices from P_m. Therefore, for totally n vertices of P_n, $\sum_v i_v(M(P_n)) = (n)\left[2+\left(\frac{n}{2}-1\right)\right] = (n)\left(\frac{n}{2}+1\right) = \frac{n^2}{2}+n$.

Hence, being n even, for a graph $M(P_n)$,

$$i_{av}(M(P_n)) = \frac{1}{|V(M(P_n))|}\sum_{v\in V(M(P_n))} i_v(M(P_n))$$
$$= \frac{1}{2n-1}\left\{\frac{n^2}{4}+\left(\frac{n^2}{4}-1\right)\left(\frac{n^2}{2}+n\right)\right\} = \frac{1}{2n-1}.(4).\left(\frac{n^2+n-1}{4}\right)$$

$$= \frac{n^2+n-1}{2n-1}.$$

Case 2. If n is odd.

The independent set which gives the independence number P_n is unique, and $\beta(P_n) = \frac{n+1}{2}$. For every vertex v of the independent set including $\frac{n+1}{2}$ vertices, the minimum cardinality of a maximal independent set containing v in graph $M(P_n)$ is equal to $\frac{n+1}{2}$, including $\frac{n-1}{2}$ vertices which are added to the graph P_n to obtain $M(P_n)$ and 1 is the vertex v itself. Thus, for these $\frac{n+1}{2}$ vertices, $\sum_v i_v(M(P_n)) = \left(\frac{n+1}{2}\right)\left(\frac{n+1}{2}\right) = \left(\frac{n+1}{2}\right)^2$. For every distinct vertex v remaining in the graph $M(P_n)$ except those $\frac{n+1}{2}$ vertices of the independent set, the minimum cardinality of a maximal independent set containing v is equal to $\frac{n+3}{2}$. Thus, $\sum_v i_v(M(P_n)) = \left(\frac{n-1}{2}\right)\left(\frac{n+3}{2}\right)$.

For every vertex v of graph $M(P_n)$ which is added to the graph P_n to obtain $M(P_n)$, a maximal independent set with minimum cardinality has totally $1 + \frac{n-1}{2} = \frac{n+1}{2}$ vertices; $\frac{n-1}{2}$ vertices are from the new vertices of P_n and 1 is from P_n. Then, for totally $n-1$ vertices, $\sum_v i_v(M(P_n)) = (n-1)\left(\frac{n+1}{2}\right)$.

Hence, being n odd, for a graph $M(P_n)$,

$$\begin{aligned} i_{av}(M(P_n)) &= \frac{1}{|V(M(P_n))|}\sum_{v\in V(M(P_n))} i_v(M(P_n)) \\ &= \frac{1}{2n-1}\left[\left(\frac{n+1}{2}\right)^2\left(\frac{n-1}{2}\right)(\frac{n+3}{2}) + (n-1)\left(\frac{n+1}{2}\right)\right] \\ &= \frac{1}{2n-1}\left(\frac{n^2+2n+1+n^2+3n-n-3+2n^2-2}{4}\right) \\ &= \frac{1}{2n-1}\left(\frac{4n^2+4n-4}{4}\right) = \frac{1}{2n-1}.4.\left(\frac{n^2+n-1}{4}\right) = \frac{n^2+n-1}{2n-1}. \end{aligned}$$

Consequently, from Case 1 and Case 2, obviously, for a graph $M(P_n)$, we have

$$i_{av}(M(P)) = \frac{n^2+n-1}{2n-1}.$$

The proof is completed. □

Theorem 8. *Let $M(C_n))$ be the middle graph of C_n. Then*

$$i_{av}(M(C_n)) = \frac{n+1}{2}.$$

Proof. For a vertex v of graph $M(C_n)$, the lower independence number, denoted by $i_v(M(C_n))$, is the minimum cardinality of a maximal independent set of $M(C_n)$ that contains v. $M(C_n)$ is a connected graph and consists of $|V(M(C_n))| = 2n$ vertices; n vertices with degree 2, n vertices with degree 4. The degrees of the vertices of $M(C_n)$ in the clockwise direction is $2, 4, 2, 4, 2, 4, \ldots$, respectively. Hence, we have two cases according to the number of vertices of C_n: odd or even.

Case 1. If n is even.

If we take any vertex with degree 4 as v, then the other vertices with degree 4 adjacent to v can not be included in the independent set. Then, let us move in the clockwise direction. For our purpose, the other vertices of degree 4 which are not adjacent to each other and v, will be included firstly, in the independent set, as much as possible. Then, a maximal independent set of minimum cardinality has $\frac{n}{2}$ vertices; all of these have degree 4. Thus, $i_v(M(C_n)) = \frac{n}{2}$. Since, there exist n vertices of degree 4, $\sum_v i_v(M(C_n)) = n\left(\frac{n}{2}\right) = \frac{n^2}{2}$.

If we take any vertex with degree 2 as v, then the other two vertices adjacent to v will not be possible to be chosen for the independent set. If we move again in the clockwise direction, for our purpose, the independent set includes all the vertices of degree 4, respectively, in which none of the pair of vertices are adjacent to each other. Eventually, there always be $\frac{n}{2}+1$ vertices in a maximal independent set of minimum cardinality, including v. Thus, $i_v(M(C_n)) = \frac{n}{2} + 1$. Since there exist n vertices of degree 2 in the graph, $\sum_v i_v(M(C_n)) = n\left(\frac{n}{2}+1\right) = \frac{n^2}{2} + n$.

Hence, being n even, for a graph $M(C_n))$,

$$\begin{aligned} i_{av}(M(C_n)) &= \frac{1}{|V(M(C_n))|} \sum_{v \in V(M(C_n))} i_v(M(C_n)) \\ &= \frac{1}{2n}\left(\frac{n^2}{2} + \frac{n^2}{2} + n\right) = \frac{1}{2n}(2n)\left(\frac{n+1}{2}\right) = \frac{n+1}{2}. \end{aligned}$$

Case 2. If n is odd.

If we take any vertex with degree 4 as v, then the other four vertices adjacent to v cannot be included in the independent set. Then, in the clockwise direction, for our purpose, the other vertices with degree 4 which are not adjacent to each other and v, will be included firstly, in the independent set. Eventually, there is always one vertex of degree 2 remaining. A maximal independent set of minimum cardinality has absolutely $1 + \frac{n-1}{2}$ vertices; one is with degree 2 and $\frac{n-1}{2}$ are with degree 4. Thus, $i_v(M(C_n)) = 1 + \frac{n-1}{2}$. Since there exist n vertices of degree 4, $\sum_v i_v(M(C_n)) = n\left(1 + \frac{n-1}{2}\right) = n\left(\frac{n+1}{2}\right)$.

If we take any vertex with degree 2 as v, then the other two vertices adjacent to v cannot be included in the independent set. Then, again in the clockwise direction, for our purpose, firstly the other vertices of degree 4 which are not adjacent to each other and v, will be in the independent set. A maximal independent set of minimum cardinality again has $1+\frac{n-1}{2}$ vertices; one is the vertex v, the others are the vertices of degree 4. Thus, $i_v(M(C_n)) = 1+\frac{n-1}{2}$. Since there exist n vertices of degree 2, $\sum_v i_v(M(C_n)) = n\left(1+\frac{n-1}{2}\right) = n\left(\frac{n+1}{2}\right)$.

Hence, being n odd, for a graph $M(C_n)$,

$$\begin{aligned} i_{av}(M(C_n)) &= \frac{1}{|V(M(C_n))|}\sum_{v\in V(M(C_n))} i_v(M(C_n)) \\ &= \frac{1}{2n}\left(n\left(\frac{n+1}{2}\right)+n\left(\frac{n+1}{2}\right)\right) = \frac{1}{2n}\left(\frac{2n^2+2n}{2}\right) = \frac{n+1}{2}. \end{aligned}$$

Consequently, from Case 1 and Case 2, obviously, for a graph $M(C_n)$, we have

$$i_{av}(M(C_n)) = \frac{n+1}{2}.$$

The proof is completed. □

Theorem 9. *Let $M(K_{1,n})$ be the middle graph of $K_{1,n}$. Then*

$$i_{av}(M(K_{1,n})) = \frac{2n^2+n+1}{2n+1} \qquad (n>1).$$

Proof. The graph $M(K_{1,n})$ is a connected graph with $|V(M(K_{1,n}))| = 2n+1$ vertices; n vertices with degree 1, n vertices with degree $n+1$, and 1 vertex with degree n. For a vertex v of $M(K_{1,n})$, the lower independence number, denoted by $i_v(M(K_{1,n}))$, is the minimum cardinality of a maximal independent set of $M(K_{1,n})$ that contains v.

If we take any vertex with a vertex degree 1 as v, then the vertex with degree $n+1$, that is adjacent to v, cannot be included in the independent set. For our purpose, if we take the other vertex with degree $n+1$ which is not adjacent to the vertex in the independent set, then there are $n-2$ vertices remaining only with the vertex degree 1, which must be included in the independent set. Thus, the minimum cardinality of a maximal independent set that contains v, $i_v(M(K_{1,n}))$, is equal to $1+1+n-2$. Since the graph $M(K_{1,n})$ has n vertices with degree 1, $\sum_v i_v(M(K_{1,n})) = n(1+1+n-2) = n^2$.

If we take any vertex with degree $n+1$ as v, then there are $n-1$ vertices remaining only with degree 1 which must be included in the independent set. Thus, $i_v(M(K_{1,n})) = 1+n-1$. Since the graph $M(K_{1,n})$ has n vertices with degree $n+1, \sum_v i_v(M(K_{1,n})) = n(1+n-1) = n^2$.

If we take the odd vertex with degree n as v, then there are n vertices remaining only with the vertex degree 1 which must be included in the independent set. Thus, $\sum_v i_v(M(K_{1,n})) = 1+n$.

Hence, for a graph $M(K_{1,n})$, the average lower independence number,

$$i_{av}(M(K_{1,n})) = \frac{1}{|V(M(K_{1,n}))|} \sum_{v \in V(M(K_{1,n}))} i_v(M(K_{1,n}))$$
$$= \frac{1}{2n+1}(n^2+n^2+1+n) = \frac{2n^2+n+1}{2n+1}.$$

The proof is completed. □

4. Conclusion

The best known measure of reliability of a graph is its connectivity, defined to be the minimum number of vertices whose deletion results in a disconnected or trivial graph. As the connectivity is a worst case measure, it does not always reflect what happens throughout the graph. For example, a tree and the graph obtained by appending an end-vertex to a complete graph both have connectivity 1. Nevertheless, for a large order the latter graph is far more reliable than the former. The interest in the vulnerability and reliability of networks such as transportation and communication networks, has given rise to a host of other measures of reliability. In this paper we investigate a new measure for the reliability of a graph, the average lower independence number, recently introduced by Henning. This parameter is closely related to the problem of finding large independent sets in graphs. Therefore we investigate the vulnerability of the middle graphs of some special graphs by using the average lower independence number.

References

[1] J. Akiyama, T. Hamada, I. Yoshimura, On characterization of the middle graphs, *TRU Mathematics*, **11** (1975), 35-39.

[2] C.A. Barefoot, R. Entringer, H. Swart, Vulnerability in graphs – A comparative survey, *J. Comb. Math. Comb. Comput.*, **1** (1987), 13-22.

[3] L.W. Beineke, O.R. Oellermann, R.E. Pippert, The average connectivity of a graph, *Discrete Mathematics*, **252** (2002), 31-45.

[4] M. Blidia, M. Chellali, F. Maffray, On average lower independence and domination numbers in graphs, *Discrete Math.*, **295** (2005), 1-11.

[5] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, (1972).

[6] A.M. Henning, Trees with equal average domination and independent domination numbers, *Ars Combinatoria*, **71** (2004), 305-318.

[7] I. Gutman, Distance of thorny graphs, *Publications de l'Institut Mathematique Nouvelle serie torne 63*, **77** (1998), 31-36.

[8] L. Lesniak, G. Chartrand, *Graphs and Digraphs*, California Wadsworth and Brooks (1986).

[9] M. Nihei, On the chromatic number of the middle graph of a graph, *Pi-Mu-Epsilon Jour.*, **10** (1998), 704-708.

[10] M. Nihei, Algebraic connectivity of the line graph, the middle graph and total graph of a regular graph, *Ars Combin.*, **69** (2003), 215-221.

[11] D.B. West, *Introduction to Graph Theory*, Prentice Hall, NJ (2001).