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PHASE FLOWS AND VECTOR LAGRANGIANS IN $J^3(\pi)$

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Abstract: On the basis of Liouville Theorem the generalization of the Nambu mechanics is considered. For three-dimensional phase space the concept of vector Hamiltonian and vector Lagrangian is entered.

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1.

Standard phenomenological approach to the analysis dynamic system is the construction for it the functional of actions $S = \int L \ dt$. We represent this functional as submanifolds in jet bundles $J^n(\pi)$: $E \to M$

$$F(t, x_0, x_1, ..., x_n) = 0,$$

where $t \in M \subset R$, $u = x_0 \in U \subset R$, $x_i \in J^i(\pi) \subset R^n$, $E = M \times U$.

The Euler-Lagrange equation

$$\sum_{k=0}^{n} (-)^k \frac{d^k}{dt^k} \frac{\partial L}{\partial x_k} = 0 \tag{1}$$

describes a lines (jet) in $J^{2n}(\pi)$. Embedding $J^n(\pi) \subset J^1(J^1(J^1(...J^1(\pi))))$ allows us to rewrite differential equation n-th order as a system of n equations of 1-st order

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$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}. \tag{2}$$

According to the Noether Theorem, symmetry of the functional S with respect to generator $X = \partial/\partial t$ gives us the conservation law I, and Hamiltonian form for our dynamics:

$$\dot{\mathbf{x}} = \{H(I), \mathbf{x}\}. \tag{3}$$

2.

Another approach for receiving of the Euler-Lagrange equation (1) for every Hamiltonian set was described by P.A. Griffiths [2]. He find such 1-form

$$\psi = Ldt + \lambda^i \theta_i, \qquad i = 0, ..., n-1,$$

which does not vary at pullback along the vector fields $(\partial/\partial\theta_i, \partial/\partial d\theta_{n-1})$. Here

$$\theta_i = dx_i - x_{i+1}\, dt$$

is the contact distribution, λ^i are the Lagrange multipliers. Bounding of the form $\Psi = d\psi$ on the field $(\partial/\partial\theta_i, \partial/\partial d\theta_{n-1})$ gives the set

$$\begin{cases} \dfrac{\partial L}{\partial x_i} = \dfrac{d\lambda^i}{dt} + \lambda^{i-1}, \qquad i = 0, ..., n-1, \\ \\ \dfrac{\partial L}{\partial x_n} = \lambda^{n-1}, \end{cases}$$

which is equivalent to the equation (1).

Hamiltonian formulation of this theory assumes that the Lagrange's multipliers λ^i are dynamic variables $H = H(x_i, \lambda^i)$:

$$\psi = (L - \lambda^i x_{i+1}) \wedge dt + \lambda^i dx_i = -Hdt + \lambda^i dx_i.$$

Then the bounding of the form $\Psi = d\psi$ on the field $(\partial/\partial x_i, \partial/\partial\lambda^i)$ gives

$$\frac{\partial H}{\partial x_i} = -\frac{\partial \lambda^i}{\partial t}, \qquad \frac{\partial H}{\partial \lambda^i} = \frac{\partial x_i}{\partial t}.$$

3.

Our target is the generalization of the above scheme on a case odd jets. To clear the idea of the method we shall receive the Euler-Lagrange equation for $L \in J^3(\pi)$.

Theorem 1. *For $L \in J^3(\pi)$ the Euler-Lagrange equation has the form*

$$\frac{1}{2}\frac{d}{dt}\left(L^i_{\dot{x}_k} - L^k_{\dot{x}_i}\right) = L^k_{x_i} - L^i_{x_k}. \tag{4}$$

Proof. Let ψ be the Griffiths 2-form:

$$\psi = L^i dx_i \wedge dt + \lambda^i \Theta_i,$$

where $\Theta = \theta \wedge \theta$. The exterior differencial of this form is

$$\begin{aligned} d\psi &= dL^i \wedge \omega_i = (\operatorname{rot} L)^k\, \Theta_k \wedge dt + d\lambda^i \wedge \Theta_i + \lambda^i \wedge d\Theta_i \\ &+ L^2_{\dot{x}_3}\, d\dot{x}_3 \wedge dx_2 \wedge dt + L^3_{\dot{x}_2}\, d\dot{x}_2 \wedge dx_3 \wedge dt \\ &+ L^3_{\dot{x}_1}\, d\dot{x}_1 \wedge dx_3 \wedge dt + L^1_{\dot{x}_3}\, d\dot{x}_3 \wedge dx_1 \wedge dt \\ &+ L^1_{\dot{x}_2}\, d\dot{x}_2 \wedge dx_1 \wedge dt + L^2_{\dot{x}_1}\, d\dot{x}_1 \wedge dx_2 \wedge dt \\ &+ L^1_{\dot{x}_1}\, d\dot{x}_1 \wedge dx_1 \wedge dt + L^2_{\dot{x}_2}\, d\dot{x}_2 \wedge dx_2 \wedge dt + L^3_{\dot{x}_3}\, d\dot{x}_3 \wedge dx_3 \wedge dt. \end{aligned}$$

Limiting it on the vector fields $v = (\partial_{\Theta_k}, \partial_{d\Theta_k})$,

$$(\operatorname{rot} L)^k = -\dot{\lambda}^k, \tag{5}$$

$$L^3_{\dot{x}_2} - L^2_{\dot{x}_3} = 2\lambda^1, \qquad L^1_{\dot{x}_3} - L^3_{\dot{x}_1} = 2\lambda^2, \qquad L^2_{\dot{x}_1} - L^1_{\dot{x}_2} = 2\lambda^3,$$

we get the Euler-Lagrange equation (4). □

4.

Now we consider the construction of the vector Hamiltonian h^i for $L \in J^3(\pi)$. Rewrite the Griffiths 2-form ψ as

$$\begin{aligned} \psi &= L^i \omega_i + \lambda^i \Theta_i \\ &= \left(L^1 - \left(\lambda^3 \dot{x}_2 - \lambda^2 \dot{x}_3\right)\right) dx_1 \wedge dt \\ &+ \left(L^2 - \left(\lambda^1 \dot{x}_3 - \lambda^3 \dot{x}_1\right)\right) dx_2 \wedge dt \\ &+ \left(L^3 - \left(\lambda^2 \dot{x}_1 - \lambda^1 \dot{x}_2\right)\right) dx_3 \wedge dt + \lambda^i dS_i \\ &= -h^i dx_i \wedge dt + \lambda^i dS_i. \end{aligned}$$

Here $dS_i = \varepsilon_{ijk} dx_j \wedge dx_k$ be a Plücker coordinates of the area element dS spanned by vectors dx_i.

Definition 2. The vector field $\mathbf{f}$ is called conservative if

$$\operatorname{div} \mathbf{f} = 0.$$

In other words, the conservative vector field is divergence-free.

Definition 3. The phase trajectory $\mathbf{x}(t)$ is called Lagrange-stable if for all $t > 0$ it remains in some bounded domain of phase space. Geometrically it means, that a phase flow (2) should be divergence-free.

Theorem 4. *The Lagrange-stable phase flow is Hamiltonians.*

Proof. We first calculate the exterior derivatives of closed 2-forms ψ:

$$d\psi = (\text{rot } \mathbf{h})^k \, \Theta_k \wedge dt + \dot{\lambda}^k dt \wedge dS_k + \text{div } \lambda^k \cdot dV = 0.$$

Then from div $\lambda^k = 0$ it follows that

$$\dot{\lambda} = \text{rot } \mathbf{h},$$

i.e. from Hamiltonians point of view the set (2) is described by the dynamics of a generaliszed moments λ, which were defined in (5). □

5.

The base of deformation quantization of dynamical system in $J^2(\pi)$ is the Liouville Theorem about preserved of the volume $\Omega = dx_0 \wedge dx_1$ by phase flows. Geometrically it means, that Lie derivative of the 2-form Ω along vector field X_H^1 is zero: $\mathcal{L}_X \Omega = 0$. In other words if $\{g_t\}$ denotes the one parameter group symplectic diffeomorphisms generated by vector fields X_H^1, then $g_t^* \Omega = \Omega$ and the phase flow $\{g_t\}$ preserve the volume form Ω.

To extend this construction on $J^3(\pi)$ we consider the 3-form of the phase space volume

$$\Omega = dx_0 \wedge dx_1 \wedge dx_2.$$

Theorem 5. *The volume 3-form $\Omega \in \Lambda^3$ supposes existence of two polyvector Hamiltonian fields $X_H^1 \in \Lambda^1$ and $X_H^2 \in \Lambda^2$.*

Proof. By definition, put

$$\mathcal{L}_X \Omega = X \rfloor d\Omega + d\,(X \rfloor \Omega) = 0.$$

Since $\Omega \in \Lambda^3$, we see that $d\Omega = 0$ and

$$d\,(X \rfloor \Omega) = 0.$$

From Poincare's Lemma it follows that the form $X \rfloor \Omega$ is exact, and

$$X \rfloor \Omega = \Theta = d\mathbf{H}.$$

1) If $\mathbf{X}_H^1 \in \Lambda^1$, then $\Theta \in \Lambda^2$, $\mathbf{H} = (\mathbf{h} \cdot d\mathbf{x}) \in \Lambda^1$. The Hamiltonian vector

field has the form

$$\begin{aligned} X^1_H &= (\operatorname{rot}\mathbf{h}\cdot\frac{\partial}{\partial\mathbf{x}}) \\ &= \left(\frac{\partial h_2}{\partial x_1}-\frac{\partial h_1}{\partial x_2}\right)\frac{\partial}{\partial x_0}+\left(\frac{\partial h_0}{\partial x_2}-\frac{\partial h_2}{\partial x_0}\right)\frac{\partial}{\partial x_1}+\left(\frac{\partial h_1}{\partial x_0}-\frac{\partial h_0}{\partial x_1}\right)\frac{\partial}{\partial x_2}. \end{aligned} \tag{6}$$

2) If $X^2_H\in\Lambda^2$, then $\Theta\in\Lambda^1$, $H\in\Lambda^0$ and we see already the Hamiltonian bivector field

$$X^2_H=\frac{1}{2}\left(\frac{\partial H}{\partial x_0}\cdot\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}+\frac{\partial H}{\partial x_1}\cdot\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_0}+\frac{\partial H}{\partial x_2}\cdot\frac{\partial}{\partial x_0}\wedge\frac{\partial}{\partial x_1}\right). \tag{7}$$

□

More generalized (but scalar) this construction was considered in [1].

Poisson's bracket for vectorial Hamiltonian (6) has the form

$$\begin{aligned} \{\mathbf{h},G\} &= X^1_H\rfloor dG \\ &= \left(\frac{\partial h_2}{\partial x_1}-\frac{\partial h_1}{\partial x_2}\right)\frac{\partial G}{\partial x_0}+\left(\frac{\partial h_0}{\partial x_2}-\frac{\partial h_2}{\partial x_0}\right)\frac{\partial G}{\partial x_1}+\left(\frac{\partial h_1}{\partial x_0}-\frac{\partial h_0}{\partial x_1}\right)\frac{\partial G}{\partial x_2}, \end{aligned}$$

and the dynamic equations is

$$\dot{\mathbf{x}}=\{\mathbf{h},\mathbf{x}\}. \tag{8}$$

The Poisson's bracket for bivector fields requires introduction two Hamiltonians

$$\begin{aligned} X^2_H\rfloor(dF\wedge dG)=\{H,F,G\}=\frac{1}{2}\Bigg[&\frac{\partial H}{\partial x_0}\cdot\left(\frac{\partial F}{\partial x_1}\frac{\partial G}{\partial x_2}-\frac{\partial F}{\partial x_2}\frac{\partial G}{\partial x_1}\right) \\ +&\frac{\partial H}{\partial x_1}\cdot\left(\frac{\partial F}{\partial x_2}\frac{\partial G}{\partial x_0}-\frac{\partial F}{\partial x_0}\frac{\partial G}{\partial x_2}\right)+\frac{\partial H}{\partial x_2}\cdot\left(\frac{\partial F}{\partial x_0}\frac{\partial G}{\partial x_1}-\frac{\partial F}{\partial x_1}\frac{\partial G}{\partial x_0}\right)\Bigg], \end{aligned} \tag{9}$$

such that the dynamic equations (2) has the Nambu form [3]

$$\dot{\mathbf{x}}=\{F,G,\mathbf{x}\}.$$

Example. Consider the dynamics of Frenet frame with constant curvature and torsion

$$\begin{cases} \dot{x}=y, \\ \dot{y}=z-x, \\ \dot{z}=-y. \end{cases} \tag{10}$$

The Lax representation for this set has the form

$$\dot{A}=[A,B],\quad A=\begin{pmatrix} x & y & x \\ y & 2z & y \\ x & y & x \end{pmatrix},\quad B=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

and gives us following invariants

$$I_k = \frac{1}{k}\mathrm{Tr}\mathbf{A}^k,$$

$$I_1 = x + z, \quad I_2 = \frac{1}{2}(x^2 + y^2 + z^2), \quad I_3 = \frac{1}{3}\left(x^3 + \frac{3}{2}y^2(x+z) + z^3\right) ...$$

Let $H_1 = x + z$ and $H_2 = \frac{1}{2}(2xz - y^2)$ are the Hamiltonians of the Frenet set, then

$$I_1 = H_1, \quad I_2 = \frac{1}{2}H_1^2 - H_2, \quad I_3 = \frac{1}{3}H_1\left(H_1^2 - 3H_2\right) ...$$

The system (2) is equivalent to the system

$$\dot{\mathbf{x}} = \{H_1, H_2, \mathbf{x}\}$$

with a Poisson bracket (9).

For a finding of vectorial Hamiltonian we write the differential $\Psi = d\psi$ of Lagrange's 1-form for Frenet set (10):

$$\Psi = ydy \wedge dz + (z - x)dz \wedge dx - ydx \wedge dy,$$

and, using homotopy formula, we get an expression for the vectorial Hamiltonians h^i and the vectorial Lagrangian L^i:

$$\mathbf{h} = \frac{1}{3}\begin{pmatrix} y^2 + z^2 - xz \\ -y(x+z) \\ y^2 + x^2 - xz \end{pmatrix}, \qquad \mathbf{L} = \begin{pmatrix} z\dot{y} - y\dot{z} - h_1 \\ x\dot{z} - z\dot{x} - h_2 \\ y\dot{x} - x\dot{y} - h_3 \end{pmatrix}.$$

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