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# OPTIMAL CONTROL PROBLEMS WITH RANDOM FINAL TIME

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**Abstract:** Problems involving controlled one-dimensional diffusion processes $X(t)$ over a random time interval are considered. The infinitesimal variance of the processes depends on the control variable. The processes are controlled until they leave the interval $(d_1, d_2)$. The cost criterion whose expected value we want to minimize is such that, in addition to the quadratic control costs, a final cost is incurred if $X[\tau(x)] = d_2$, where $\tau(x)$ is the random final time. Exact and explicit solutions are obtained in special cases both for the optimal control and for the value function. A related game theory problem is also presented.

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## 1. Introduction

Suppose that $X(t)$ is a one-dimensional controlled diffusion process defined by the stochastic differential equation (s.d.e.)

$$dX(t) = f[X(t)]\,dt + b[X(t)]u(t)\,dt + \{v[X(t)]\}^{1/2}\,dW(t)\,,$$

where the function $v$ is strictly positive and $W(t)$ is a standard Brownian motion. Let

$$T_0(x) = \inf\{t > 0 : X(t) \in D \mid X(0) = x\}\,,$$

where $x \notin D$ ($D$ being a subset of the real line). Whittle [9] has termed *LQG*

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*homing* the problem of minimizing the expected value of the cost criterion

$$J_0(x) = \int_0^{T_0(x)} \tfrac{1}{2} q[X(t)]u^2(t)\, dt + K_0\{X[T_0(x)]\}\,,$$

where $q$ is a strictly positive function and $K_0$ is a general termination cost function. Actually, Whittle considered the $n$-dimensional case, and the functions $f$, $b$, $v$, $q$ and $K$ could theoretically all depend on the variable $t$. Furthermore, the problem can be generalized by using a risk-sensitive cost criterion instead of $J_0(x)$ (see Kuhn [6] and Whittle [10]).

Next, in Lefebvre [7] the author considered the one-dimensional process defined by

$$dX(t) = b[X(t)]u(t)\, dt + \{v[X(t)]|u(t)|\}^{1/2}\, dW(t)\,.$$

Notice that not only the function $f$ was chosen identical to zero, but, more importantly, the control variable $u(t)$ appeared both in the infinitesimal mean and infinitesimal variance of $X(t)$. The cost criterion was taken to be

$$J_1(x) = \int_0^{T_1(x)} \left\{\tfrac{1}{2} q[X(t)]u^2(t) + \lambda\right\}\, dt,$$

where, in that paper, $\lambda$ was a positive parameter. The random final time $T_1(x)$ was defined by

$$T_1(x) = \inf\{t \geq 0 : |X(t)| = d \mid X(0) = x \in [-d, d]\}.$$

Under some symmetry assumptions, it was found that the control

$$u^* = \left(\frac{2\lambda}{q(x)}\right)^{1/2} \quad \text{when } 0 \leq x \leq d$$

minimizes the expected value of $J_1(x)$. The value function $F(x)$ (see (4)) was computed explicitly in important cases, in particular in the case when $X(t)$ is a Wiener process, or a geometric Brownian motion, if $u(t) \equiv 1$.

In another paper, Lefebvre [8] modified the previous problem by considering the controlled (one-dimensional) process $X(t)$ that obeys the s.d.e.

$$dX(t) = f[X(t)]\, dt + \{v[X(t)]|u(t)|\}^{1/2}\, dW(t),$$

where $f(x) > 0$ and $v(x) > 0$ for $x > d_1 \geq 0$. That is, this time the function $b$, rather than $f$, in the original formulation was assumed to be identical to zero.

The cost criterion was

$$J(x) = \int_0^{\tau(x)} \left\{\tfrac{1}{2} q[X(t)]u^2(t) + \lambda\right\}\, dt + K\{X[\tau(x)]\}, \tag{1}$$

where

$$\tau(x) = \inf\{t \geq 0 : X(t) \in \{d_1, d_2\} \mid X(0) = x \in [d_1, d_2]\} \tag{2}$$

and the parameter $\lambda$ can now take any real value. The termination cost function $K(\cdot)$ was the following:

$$K\{X[\tau(x)]\} = \begin{cases} 0 & \text{if } X[\tau(x)] = d_1, \\ k_0 & \text{if } X[\tau(x)] = d_2, \end{cases}$$

in which $k_0$ is a (large) positive constant.

When $\lambda > 0$ (respectively $< 0$), we want the controlled process $X(t)$ to leave $(d_1, d_2)$ as soon (resp. late) as possible. In both cases, we would of course like $X[\tau(x)]$ to be equal to $d_1$ and we must take the quadratic control costs into account.

The optimal value of the control variable was computed explicitly and the case when $f$, $v$ and $q$ are proportional to $X^n(t)$, where $n \in \{0, 1, 2\}$, was treated thoroughly.

## 2. Problem Formulation

Let $X(t)$ be the one-dimensional controlled stochastic process defined by the s.d.e.

$$dX(t) = f[X(t)]\,dt + \{v[X(t)] + |u(t)|\}^{1/2}\,dW(t), \tag{3}$$

where $v(\cdot)$ is a positive function. Notice that, again, the infinitesimal variance of the controlled process $X(t)$ is control-dependent. Usually, it is assumed that (only) the infinitesimal mean is control-dependent. Moreover, the control is now additive, rather than multiplicative in the variance, which changes the problem entirely. Actually, in many applications, it is probably more realistic to assume that the control is additive.

**Remarks.** i) To be precise, in the theory, the case when the variance depends on the control variable, as here, has been considered (see Fleming and Soner [2], for instance). However, in practice, there are not many examples of problems of this type for which an explicit solution has been obtained. One such example is the so-called Merton's portfolio problem (see Fleming and Soner [2, p. 174] and also Karatzas et al [4], Fleming and Zariphopoulou [3] and Fitzpatrick and Fleming [1]).

ii) Suppose that $u(t)$ is the percentage of his portfolio that a man invests in a risky asset. If we admit that this percentage does not change the future expected value of the portfolio, but that its volatility increases with it, then an equation such as (3) could be appropriate. Note that if $u(t)$ is chosen equal to 0, then there is still some volatility present. In the case when the control is multiplicative, choosing $u(t) \equiv 0$ leads to no volatility at all, which, in general, is not realistic.

We want to minimize the expected value of the cost criterion defined in (1), with $k_0$ now any positive constant (not necessarily large). Let $F(x)$ be the value function defined by

$$F(x) = \inf_{\substack{u(t) \\ 0 \le t \le \tau(x)}} E[J(x)]. \tag{4}$$

In the next section, we will find the ordinary differential equation (o.d.e.) satisfied by $F(x)$. This o.d.e. is, in general, non-linear and very hard to solve. However, in some cases, for example when $f[X(t)] \equiv 0$, the equation we have to solve becomes tractable. In Section 3, we will also consider a variant of the main problem. Sections 4 and 5 will deal with related optimal control problems and a game theory problem, respectively. Finally, concluding remarks will end the paper.

## 3. Optimal Control

First, we will show the following proposition.

**Proposition 3.1.** *Assuming that the value function $F(x)$ defined in (4) exists and is twice differentiable, if $F''(x) \le 0$, then $F(x)$ is a solution of the second-order non-linear o.d.e.*

$$0 = \lambda + f(x)F'(x) + \tfrac{1}{2}v(x)F''(x) - \frac{1}{8q(x)}[F''(x)]^2 \tag{5}$$

*subject to the boundary conditions*

$$F(d_1) = 0 \quad \textit{and} \quad F(d_2) = k_0. \tag{6}$$

*Proof.* Under the assumptions in the proposition, we find that $F(x)$ satisfies the dynamic programming equation (d.p.e.) (see Whittle [9])

$$0 = \inf_u \left\{ \tfrac{1}{2}q(x)u^2 + \lambda + f(x)F'(x) + \tfrac{1}{2}\left[v(x) + |u|\right] F''(x) \right\} \tag{7}$$

for $x$ in the interval $(d_1, d_2)$, where $u$ is the value of $u(t)$ at time $t = 0$.

Next, notice that $u(t)$ appears either in absolute value (in the s.d.e. (3)) or squared (in the cost criterion (1)). Therefore, we may state that the sign of $u(t)$ does not matter. We assume, without loss of generality, that $u$ is actually non-negative. It is then a simple matter to find that the optimal control $u^*$ is given by

$$u^* = -\frac{1}{2q(x)} F''(x) \quad (\geq 0).$$

Finally, substituting $u^*$ into the d.p.e. (7), we deduce that $F(x)$ is indeed a solution of (5). Moreover, the boundary conditions are trivially the ones given in (6). □

**Remarks.** i) If the function $v$ is such that the d.p.e. (7) is *uniformly elliptic*, then, because the interval $(d_1, d_2)$ is bounded, this equation, subject to (6), has generally a smooth solution which is *unique* (see Fleming and Soner [2, p. 172]).

ii) We must verify, when we solve particular problems, that the value of the optimal control is non-negative or, equivalently, that $F''(x)$ is negative or zero. Otherwise, the solution is not valid.

As we mentioned in the previous section, obtaining the general solution of equation (5) is quite arduous. However, we were able to obtain an exact and explicit solution in a special, but not trivial case.

**Particular Case.** Assume that $f(x) \equiv 0$. The o.d.e. (5) becomes

$$[F''(x)]^2 - 4\,q(x)v(x)F''(x) - 8\,\lambda q(x) = 0 \tag{8}$$

from which we deduce that

$$F''(x) = 2q(x)v(x) \pm \left[4q^2(x)v^2(x) + 8\lambda q(x)\right]^{1/2}.$$

Since we want $F''(x) \leq 0$, we must choose the "$-$" sign above and assume that the parameter $\lambda$ is non-negative, so that survival in the interval $(d_1, d_2)$ is penalized (when $\lambda > 0$). Finally, we obtain that

$$u^* = -\frac{1}{2q(x)} F''(x) = -v(x) + \left[v^2(x) + (2\lambda/q(x))\right]^{1/2}. \tag{9}$$

**Remarks.** i) Equation (8) has two solutions:

$$F_1(x) = \int \left( \int \left\{ 2q(x)v(x) + \left[4q^2(x)v^2(x) + 8\lambda q(x)\right]^{1/2} \right\} dx \right) dx + c_1 x + c_0$$

and

$$F_2(x) = \int \left( \int \left\{ 2q(x)v(x) - \left[4q^2(x)v^2(x) + 8\lambda q(x)\right]^{1/2} \right\} dx \right) dx + c_1 x + c_0.$$

Because $F''(x)$ must be negative or equal to zero, we can discard $F_1(x)$. Moreover, we have two boundary conditions and two unknown constants. Therefore, under the assumption that $F''(x)$ exists (as mentioned above), we can state that we have found the value function and that $u^*$ is indeed given by equation (9).

ii) If we assume that $X(t)$ is a controlled Wiener process with infinitesimal parameters $\mu = 0$ and $\sigma^2$ (see Karlin and Taylor [5], for instance), and if we take $q(x) \equiv q_0 > 0$, we find that

$$F''(x) = 2q_0\sigma^2 - \left[4q_0^2\sigma^4 + 8\lambda q_0\right]^{1/2} := F_0$$

(a constant), so that

$$F(x) = \tfrac{1}{2}F_0x^2 + c_1x + c_0,$$

where $c_0$ and $c_1$ are constants that, again, are uniquely determined from the boundary conditions (6).

iii) We deduce from (9) that $u^* = 0$ if $\lambda = 0$.

Consider now the problem for which the controlled process is defined by the s.d.e.

$$dX(t) = f[X(t)]\,dt + \{v[X(t)] + u^2(t)\}^{1/2}\,dW(t)$$

and the cost function is given by

$$J(x) = \int_0^{\tau(x)} \left\{\tfrac{1}{2}q[X(t)]|u(t)| + \lambda\right\}\,dt + K\{X[\tau(x)]\}.$$

The d.p.e. becomes

$$0 = \inf_u \left\{\tfrac{1}{2}q(x)|u| + \lambda + f(x)F'(x) + \tfrac{1}{2}\left[v(x) + u^2\right]F''(x)\right\}. \tag{10}$$

Again, we may assume that $u^*$ is non-negative. We then find that

$$u^* = -\frac{q(x)}{2F''(x)} \quad (\geq 0),$$

where $F''(x)$ must be strictly less than 0. Substituting into (10), we obtain the following proposition.

**Proposition 3.2.** *Under the assumptions in Proposition 3.1, the value function $F(x)$ is a solution of the second-order non-linear o.d.e.*

$$0 = \lambda F''(x) + f(x)F'(x)F''(x) + \tfrac{1}{2}v(x)[F''(x)]^2 - \tfrac{1}{8}q^2(x) \tag{11}$$

*subject to the same boundary conditions: $F(d_1) = 0$ and $F(d_2) = k_0$. Furthermore, $F(x)$ must be such that $F''(x) < 0$ for the solution to be valid.*

**Particular Case.** As in the previous problem, we consider the special case when $f(x) \equiv 0$. Then, we may write that

$$F''(x) = \frac{1}{v(x)}\left\{-\lambda \pm \left[\lambda^2 + \tfrac{1}{4}v(x)q^2(x)\right]^{1/2}\right\}.$$

To satisfy the condition $F''(x) < 0$, we must choose the "$-$" sign. Moreover, the parameter $\lambda$ must be such that

$$-\lambda - \left[\lambda^2 + \tfrac{1}{4}v(x)q^2(x)\right]^{1/2} < 0,$$

which is actually true for any real $\lambda$ because $v$ and $q$ are strictly positive.

The optimal control is given by

$$u^* = \frac{v(x)q(x)}{2\left\{\lambda + \left[\lambda^2 + \frac{1}{4}v(x)q^2(x)\right]^{1/2}\right\}}.$$

Notice that, this time, if $\lambda = 0$ then we have $u^* = v^{1/2}(x)$.

**Remark.** As in the previous case, we can state that there exists a unique solution of equation (11), with $f(x) \equiv 0$, that satisfies the appropriate boundary conditions.

## 4. Related Problems

In this section, we present two problems of the same type as those considered in the previous section and for which explicit solutions can be found.

*Case I.* We replace $f[X(t)]$ by $b[X(t)]u^2(t)$ in the s.d.e. (3) and we take $q(x) \equiv 0$ in the cost criterion (1), so that

$$dX(t) = b[X(t)]u^2(t)\,dt + \{v[X(t)] + |u(t)|\}^{1/2}\,dW(t) \tag{12}$$

and

$$J(x) = \lambda\tau(x) + K\{X[\tau(x)]\}. \tag{13}$$

We may still assume that $u(t) \geq 0$. If $b(x)$ [and $F'(x)$] $\neq 0$, we obtain that

$$u^* = -\frac{F''(x)}{4b(x)F'(x)}.$$

**Remarks.** i) Actually, we cannot have $F'(x) = 0$, since $F(x)$ should be strictly increasing in $x$ in the interval $(d_1, d_2)$.

ii) To simplify, we assume that $b(x) \neq 0$ for all $x$ in the interval $[d_1, d_2]$.

We must solve the second-order non-linear o.d.e.

$$0 = \lambda + \tfrac{1}{2}v(x)F''(x) - \frac{[F''(x)]^2}{16\,b(x)F'(x)}.$$

We can solve this equation in the case when $\lambda = 0$. The o.d.e. becomes

$$0 = v(x)b(x)F'(x)F''(x) - \tfrac{1}{8}[F''(x)]^2. \tag{14}$$

That is, assuming that $F''(x) \neq 0$,

$$0 = v(x)b(x)F'(x) - \tfrac{1}{8}F''(x).$$

We find that

$$F(x) = c_1 \int \exp\left\{8 \int b(x)v(x)\,dx\right\} dx + c_0,$$

where the constants $c_1$ and $c_0$ are uniquely determined from the boundary conditions (6). However, we then obtain that

$$u^* = -2v(x) < 0,$$

which contradicts the assumption that $u^* \geq 0$. Therefore, we must conclude that $F''(x) = 0$, so that $u^* = 0$.

**Remark.** We find at once that $F''(x) = 0$ is a solution of equation (14). Therefore, there exists a smooth solution, say $W$ (with $W'' = 0$), to this equation. Making use of a result known as a *verification theorem* (see Fleming and Soner [2, p. 172]), or simply by uniqueness of the solution under the appropriate assumptions, we can state that $F(x) = W$. Actually, from the boundary conditions (6), we deduce that this solution is given by

$$F(x) = k_0 \frac{(x - d_1)}{(d_2 - d_1)} \quad \text{for } x \in [d_1, d_2].$$

*Case II.* Next, we replace equation (12) by

$$dX(t) = b[X(t)]u(t)\,dt + \{v[X(t)] + u^2(t)\}^{1/2}\,dW(t)$$

and we choose $\lambda = 0$ in the cost criterion (13). Notice that here we do not have to assume that $u(t) \geq 0$.

If $F''(x) \neq 0$, we find that

$$u^* = -\frac{b(x)F'(x)}{F''(x)} \tag{15}$$

and the o.d.e. that we must solve is

$$0 = v(x)[F''(x)]^2 - b^2(x)[F'(x)]^2.$$

It is easy to show that

$$F(x) = c_1 \int \exp\left\{\int |b(x)|v^{-1/2}(x)\,dx\right\} dx + c_0,$$

where again the constants $c_1$ and $c_0$ are uniquely determined from (6).

We deduce from (15) that (with $b(x) \neq 0$)

$$u^* = -\frac{b(x)}{|b(x)|}v^{1/2}(x) = -\text{sign}[b(x)]\,v^{1/2}(x).$$

**Particular Case.** If $b(x) \equiv b_0 > 0$ and $v(x) \equiv v_0 > 0$, we obtain that

$$F(x) = c_1 \frac{v_0^{1/2}}{b_0} \exp\left\{\frac{b_0}{v_0^{1/2}} x\right\} + c_0$$

and

$$u^* = -v_0^{1/2}.$$

If, for simplicity, we choose $d_1 = 0$, then we find that

$$F(x) = k_0 \frac{\exp\left\{\frac{b_0}{v_0^{1/2}} x\right\} - 1}{\exp\left\{\frac{b_0}{v_0^{1/2}} d_2\right\} - 1}$$

for $0 \leq x \leq d_2$.

## 5. A Game Theory Problem

Let $(X_1(t), X_2(t))$ be the two-dimensional controlled stochastic process defined by the system of s.d.e.'s

$$dX_i(t) = f_i[X_i(t)]\,dt + \{v_i[X_i(t)] + |u_i(t)|\}^{1/2} dW_i(t)$$

for $i = 1, 2$, where $W_1(t)$ and $W_2(t)$ are independent standard Brownian motions and $f_i(\cdot)$ is a real function.

The objective is to optimize the mathematical expectation of the cost criterion

$$J(x_1, x_2) = \int_0^T [\lambda + u_1^2(t) - u_2^2(t)]\,dt + K[X_1(T), X_2(T)],$$

where

$$K[X_1(T), X_2(T)] = \begin{cases} 0 & \text{if } X_1(T) - X_2(T) = 0, \\ k_0 & \text{if } X_1(T) - X_2(T) = d, \end{cases}$$

with $k_0 > 0$ and

$$T(x_1, x_2) = \inf\{t > 0 \colon X_1(t) - X_2(t) = 0 \ \text{ or } \ d \mid X_i(0) = x_i\}$$

in which we assume that $0 < x_1 - x_2 < d$.

In our game theory problem, there are two optimizers. The first one, using $u_1(t)$, tries to minimize the expected value of the cost criterion $J$, while the second one, using $u_2(t)$, wants to maximize this expected value. We consider the value function

$$F(x_1, x_2) = \inf_{\substack{u_1(t) \\ 0 \leq t \leq T}} \sup_{\substack{u_2(t) \\ 0 \leq t \leq T}} E[J(x_1, x_2)].$$

Assuming that $F$ exists and is twice differentiable with respect to $x_1$ and $x_2$, we can show that

$$0 = \inf_{u_1} \sup_{u_2} \left\{ \lambda + u_1^2 - u_2^2 + \sum_{i=1}^{2} \left( f_i F_{x_i} + \tfrac{1}{2} \left[ v_i + |u_i| \right] F_{x_i x_i} \right) \right\}. \tag{16}$$

From this d.p.e., we deduce the following proposition.

**Proposition 5.1.** *If $F_{x_1x_1} \geq 0$, then $u_1^* = 0$. Similarly, when $F_{x_2x_2} \leq 0$, we have $u_2^* = 0$. When $F_{x_1x_1} < 0$ and $F_{x_2x_2} > 0$, we find that*

$$u_1^* = -\tfrac{1}{4} F_{x_1x_1} \quad \text{and} \quad u_2^* = \tfrac{1}{4} F_{x_2x_2}. \tag{17}$$

*Proof.* As in the optimal control problems, we may assume, without loss of generality, that $u_i \geq 0$. The first two results can be stated simply by looking at the d.p.e. (16), whereas the formulas in (17) are obtained from (16), by differentiation. □

**Remark.** Again, for the sake of simplicity, we assume that the signs of the second-order derivatives remain the same in the whole continuation region.

To solve the game theory problem, we will use a particular case of the technique known as "the method of similarity solutions." We assume that

$$F(x_1, x_2) = G(z),$$

where $z := x_1 - x_2$. It follows that

$$F_{x_1x_1}(x_1, x_2) = F_{x_2x_2}(x_1, x_2) = G''(z). \tag{18}$$

Therefore, we cannot have $F_{x_1x_1} < 0$ and $F_{x_2x_2} > 0$ at the same time. Thus, one of the two optimal controls will be equal to zero, while the other one will be non-negative.

Suppose that $F_{x_1x_1} \geq 0$ (and $F_{x_2x_2} \geq 0$). We must solve the second-order non-linear partial differential equation (p.d.e.)

$$\lambda + \tfrac{1}{16} F_{x_2x_2}^2 + \tfrac{1}{2} \sum_{i=1}^{2} v_i F_{x_i x_i} + \sum_{i=1}^{2} f_i F_{x_i} = 0 \tag{19}$$

under the boundary conditions

$$F(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 - x_2 = 0, \\ k_0 & \text{if } x_1 - x_2 = d, \end{cases} \tag{20}$$

where $d > 0$. With $F(x_1, x_2) = G(z)$, the p.d.e. is transformed into the second-order non-linear o.d.e.

$$\lambda + \tfrac{1}{16} \left[ G''(z) \right]^2 + \tfrac{1}{2}(v_1 + v_2) G''(z) + (f_1 - f_2) G'(z) = 0 \tag{21}$$

and the boundary conditions become $G(0) = 0$ and $G(d) = k_0$. For this equation to make sense, $v_1 + v_2$ and $f_1 - f_2$ must of course be functions of $z$ only.

**Remarks.** i) We assume that the functions $f_i$ and $v_i$ are such that the solution of (16), subject to (20), is *unique*. Hence, we can make use of any method, such as the method of similarity solutions above, to find a particular solution.

ii) When $F_{x_i x_i} \leq 0$, for $i = 1, 2$, we find that

$$\lambda - \frac{1}{16}\left[G''(z)\right]^2 + \frac{1}{2}(v_1 + v_2)G''(z) + (f_1 - f_2)G'(z) = 0$$

subject to the same boundary conditions.

**Particular Case.** Suppose that $f_1[X_1(t)] \equiv f_2[X_2(t)]$ and that $v_i[X_i(t)] \equiv \sigma_i^2$, a positive constant, for $i = 1, 2$. Then, equation (21) reduces to

$$\lambda + \frac{1}{16}\left[G''(z)\right]^2 + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)G''(z) = 0.$$

We deduce that

$$G''(z) = -4(\sigma_1^2 + \sigma_2^2) \pm 4\left[(\sigma_1^2 + \sigma_2^2)^2 - \lambda\right]^{1/2}.$$

Since we assumed that $G''(z) = F_{x_i x_i} \geq 0$, we must choose the "+" sign and impose the condition $\lambda \leq 0$. Therefore, we must give a reward (if $\lambda > 0$) for survival in the continuation region.

Finally, the optimal control $u_2^*$ is given (under the above assumptions) by

$$u_2^* = \frac{1}{4}G''(z) = -(\sigma_1^2 + \sigma_2^2) + \left[(\sigma_1^2 + \sigma_2^2)^2 - \lambda\right]^{1/2}.$$

## 6. Conclusion

We have obtained explicit solutions to optimal control problems, as well as to a game theory problem. We could try to solve other particular problems, or (hopefully) more general cases. Moreover, we could consider two-dimensional versions of the optimal control problems and/or use a risk-sensitive cost criterion rather than $J(x)$. This type of problems could also be considered in discrete time.

In the case of the game theory problem, we could try other techniques to solve the dynamic programming equation. Finally, the problem formulation could be modified, by replacing the termination cost by

$$K[X_1(T), X_2(T)] = \begin{cases} 0 & \text{if } X_1(T) = X_2(T), \\ k_0 & \text{if } X_1(T) = dX_2(T), \end{cases}$$

with

$$T(x_1, x_2) = \inf\{t > 0 \colon X_1(t) = X_2(t) \ \text{ or } \ X_1(t) = dX_2(t)\},$$

where $x_1 = X_1(0)$ and $x_2 = X_2(0)$ are such that $1 < x_1/x_2 < d$. Then, instead of looking for solutions of the form $F(x_1, x_2) = G(x_1 - x_2)$, we would try to solve the d.p.e. by assuming that $F(x_1, x_2) = H(x_1/x_2)$.

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