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ON HYPERBOLIC SECOND-ORDER QUASI-LINEAR INITIAL BOUNDARY-VALUE PROBLEMS

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Abstract: The paper deals with the existence and uniqueness of a non-trivial classical solution to a initial boundary-value problem for a quasi-linear second-order system, with homogeneous boundary conditions, in the space $H^{s'}(\bar{\Omega} \times [0,T])$, where Ω is the half-space $\Omega = \mathbf{R}^{d-1} \times (0,\infty)$, $s > d/2+3$ and $s' \in (0,s)$. The proof of the main theorem relies on the existence of the solution in $H^s(\bar{\Omega} \times [0,T])$, to the boundary-value problem, for linear second-order systems with smooth coefficients in the variables x and t. The main result is achieved by defining a suitable iteration scheme that approximates the solution.

AMS Subject Classification: 35L55
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1. Introduction

Let A be the second-order quasi-linear differential operator

$$A[\cdot] = \partial_t^2 + S(\cdot,\nabla_x)\partial_t + \sum_{\alpha=1}^{d} T^{\alpha}(\cdot,\nabla_x)\partial_t\partial_\alpha + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} E^{\alpha,\beta}(\cdot,\nabla_x)\partial_\alpha\partial_\beta + \sum_{\alpha=1}^{d} F^{\alpha}(\cdot,\nabla_x)\partial_\alpha + G(\cdot,\nabla_x), \quad (1)$$

where t is a real variable and the variable x belongs to the half-space $\Omega = \mathbf{R}^{d-1} \times (0,\infty)$, with $d > 1$; moreover, for all α, $\beta = 1,\cdots,d$, the functions $S, T^{\alpha}, E^{\alpha,\beta}, F^{\alpha}, G$; are defined in $\mathbf{R}^d \times \mathbf{R}^{d^2}$ with values in the space of $d \times d$

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real matrices and have smooth coefficients; in addition, we shall assume that $E^{d,d}$ is the null-matrix. Let us define the first-order boundary operator B as follows:

$$B[\cdot] = T^d(\cdot, \nabla_x)\partial_t + 2\sum_{\beta=1}^{d-1} E^{d,\beta}(\cdot, \nabla_x)\partial_\beta + F^d(\cdot, \nabla_x). \tag{2}$$

I shall be concerned with the well-posedness of the following hyperbolic boundary-value problem with homogeneous boundary conditions

$$\begin{cases} A[u] = 0, & \text{in } \Omega \times (0, T), \\ B[u] = 0, & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{3}$$

where $u = u(x, t)$ is the unknown function with values in $\mathbf{R}^d$. I shall study the problem under the assumption that for every α, $\beta = 1, \cdots, d$, the matrix-valued functions $S, T^\alpha, E^{\alpha,\beta}, F^\alpha, G$ belong to $C^\infty(\mathbf{R}^d \times \mathbf{R}^{d^2})$ and are bounded as well as their derivatives. Moreover, I shall assume that $E^{d,d}$ is the null matrix, in order for the boundary-value problem to be well-posed in the half-space $\Omega = \mathbf{R}^{d-1} \times (0, \infty)$.

The main result will be achieved by proving the existence of the solution of the following linear boundary-value problem, in the space $H^s(\bar{\Omega} \times [0, T])$

$$\begin{cases} \partial_t^2 u + \bar{S}(x,t)\partial_t u + \sum_{\alpha=1}^{d} \bar{T}^\alpha(x,t)\partial_t\partial_\alpha u + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} \bar{E}^{\alpha,\beta}(x,t)\partial_\alpha\partial_\beta u \\ + \sum_{\alpha=1}^{d} \bar{F}^\alpha(x,t)\partial_\alpha u + \bar{G}(x,t)u = 0, \qquad \text{in } \Omega \times (0, T), \\ 2\sum_{\beta=1}^{d-1} \bar{E}^{d,\beta}(x,t)\partial_\beta u + \bar{T}^d(x,t)\partial_t u + \bar{F}^d(x,t)u = 0, \qquad \text{on } \partial\Omega \times (0, T); \end{cases} \tag{4}$$

where the functions $\bar{S}, \bar{T}^\alpha, \bar{E}^{\alpha,\beta}, \bar{F}^\alpha, \bar{G}$, for all α, $\beta = 1, \cdots, d$, take values in the space of $d \times d$ real matrices, belong to $C^\infty(\bar{\Omega} \times [0, T])$ and are bounded with their derivatives.

In the paper [3], a linear evolution boundary-value problem in a half-space has been studied in the case where the second-order differential operator has coefficients that depend only on the space-variable. The linear problem has been discussed by studying operators with constant coefficients, through a Fourier-Laplace analysis and subsequently, by applying Hille-Yosida Theorem to prove the well-posedness of the variable-coefficients boundary-value problem in the space $C^1([0, T], H^1(\Omega))$.

The following paper [4] deals with the initial boundary-value problem, in a half-space, for a quasi-linear second-order system, whose coefficients depend on the unknown vector field u and where the functions S, T^α are null-matrices. The main result of [4] states existence and uniqueness in the spaces $C^1([0,T], H^{s'}(\bar{\Omega}))$ as $s' \in (0,s)$ and $s > d/2+2$, of the solution to the initial boundary-value problem, with homogeneous boundary conditions and initial values $u(x,0) = 0$ and $\partial_t u(x,0) = g(x)$, where g belongs to $H^s(\mathbf{R}^d)$. Similarly to the procedure that was introduced in [1] to study Cauchy problems for first-order systems, one of the result of [4] concerns the well-posedness of a boundary-value problem for a linear second-order operator with coefficients which depend on the space-variable x and on the time-variable t. This result was applied subsequently to obtain the existence of a non-null solution of the quasi-linear problem.

In proving existence of the solutions to problems (3) and (4), as well as in the case of the problem treated in [4], I shall assume that the second-order differential operator (3) and the adjoint satisfy suitable conditions, which are analogous to the energy estimates derived in [1] for linear Cauchy problems for first-order symmetrizable systems. Consider, for instance, a linear second-order operator of the form

$$K[u] = \partial_t^2 u + \bar{S}(x,t)\partial_t u + \sum_{\alpha=1}^{d} \bar{T}^\alpha(x,t)\partial_t\partial_\alpha u.$$

Assume that the family of first-order operators $K(t) = \bar{S}(\cdot,t) + \sum_{\alpha=1}^{d} \bar{T}^\alpha(\cdot,t)\partial_\alpha$, admits a functional symmetrizer, in the sense of the definition given in [1] and that the matrices $\bar{S}, \bar{T}^\alpha$ belong to $C^\infty(\Omega \times (0,T))$ and are bounded with their derivatives. Because of the a priori estimates proved in [1] the second-order operator K turns out to satisfy the assumptions of Proposition 2.1, for $K[u]$ can be regarded as a first-order operator with respect to $\partial_t u$.

As a consequence of the existence of a non-trivial solution of problem (4), in the space $H^s(\bar{\Omega} \times [0,T])$, I shall deduce the well-posedness of the initial boundary-value problem in the case where the dependence on the variables x and t of the matrix-valued functions $\bar{S}, \bar{T}^\alpha, \bar{E}^{\alpha,\beta}, \bar{F}^\alpha, \bar{G}$ occurs through a function $v \in H^s(\bar{\Omega} \times [0,T])$. Then I shall define a sequence of initial boundary-value problems, which in accordance with this existence result, admits a solution in the space $H^s(\bar{\Omega} \times [0,T])$. Passing to the limit in the iteration scheme, I shall state the main theorem, concerning the existence and uniqueness of a non-trivial solution to (3) in the spaces $H^{s'}(\bar{\Omega} \times [0,T])$, with $s' \in (0,s)$, $s > d/2+3$.

2. Linear Second-Order Hyperbolic BVP

Let us consider the boundary-value problem (4) and denote by L the linear second-order differential operator associated with (4)

$$L = \partial_t^2 + \bar{S}(x,t)\partial_t + \sum_{\alpha=1}^{d} \bar{T}^{\alpha}(x,t)\partial_t\partial_\alpha + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} \bar{E}^{\alpha,\beta}(x,t)\partial_\alpha\partial_\beta + \sum_{\alpha=1}^{d} \bar{F}^{\alpha}(x,t)\partial_\alpha + \bar{G}(x,t). \quad (5)$$

If we assume that for every $\alpha,\ \beta = 1,\cdots,d$, $\bar{E}^{\alpha,\beta} = \bar{E}^{\beta,\alpha}$, the corresponding adjoint operator turns out to be defined by

$$L^* = \partial_t^2 - \partial_t\bar{S}^T - \bar{S}^T\partial_t + \sum_{\alpha=1}^{d} \left(\partial_t\partial_\alpha\bar{T}^{\alpha\,T} + \partial_t\bar{T}^{\alpha\,T}\partial_\alpha + \partial_\alpha\bar{T}^{\alpha\,T}\partial_t + \bar{T}^{\alpha\,T}\partial_t\partial_\alpha\right) + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\Big(\partial_\beta\partial_\alpha\bar{E}^{\alpha,\beta\,T} + 2\partial_\alpha\bar{E}^{\alpha,\beta\,T}\partial_\beta + \bar{E}^{\alpha,\beta\,T}\partial_\beta\partial_\alpha\Big) - \sum_{\alpha=1}^{d}\left(\bar{F}^{\alpha\,T}\partial_\alpha + \partial_\alpha(\bar{F}^{\alpha\,T})\right) + \bar{G}^T. \quad (6)$$

Let us denote by $H^s_{\bar{\Omega}\times[0,T]}(\mathbf{R}^{d+1})$ the closed subspace of $H^s(\mathbf{R}^{d+1})$ formed by the functions with support in $\bar{\Omega}\times[0,T]$ and by $H^s(\bar{\Omega}\times[0,T])$ the space of distributions admitting an extension in the space $H^s(\mathbf{R}^{d+1})$.

Proposition 2.1. *Let us consider the boundary value problem (4). Let $s \in \mathbf{R}$, $s > d/2 + 2$, and $T > 0$. If the operator L satisfies the following conditions:*

(i) for all $\alpha,\ \beta = 1,\cdots,d$, $\bar{S},\bar{T}^\alpha,\bar{E}^{\alpha,\beta},\bar{F}^\alpha,\bar{G} \in C^\infty(\bar{\Omega}\times[0,T])$ and are bounded as well as their derivatives;

(ii) for all $\alpha,\ \beta = 1,\cdots,d$, $\bar{E}^{\alpha,\beta} = \bar{E}^{\beta,\alpha}$ in $\bar{\Omega}\times[0,T]$ and $\bar{E}^{d,d}$ is the null-matrix;

(iii) there exists a positive real constant c such that for every $\phi \in C_0^\infty(\Omega\times[0,T[)$,

$$\|L^*\phi\|^2_{H^{-s}_{\bar{\Omega}\times[0,T]}(\mathbf{R}^d)} \geq c\|\phi(t)\|^2_{H^{-s}_{\bar{\Omega}}(\mathbf{R}^d)}, \quad (7)$$

with $t \in [0,T]$.

Then there exists a non-null function $u \in H^s(\bar{\Omega}\times[0,T])$ which provides a *solution to problem (4).*

The result can be proved similarly to Proposition 2.1 in [4]. For the sake of completeness, let us outline the proof.

Proof. Let k be the largest integer so that $d/2 < k+1 < s$. Let $g \in H^s(\mathbf{R}^d)$ be a non-null function which satisfies the following compatibility conditions: the function $\bar{T}^d g$ and the derivatives of order less than or equal to k, with respect to the time variable t, of the boundary operator

$$B[u] = 2\sum_{\beta=1}^{d-1} \bar{E}^{d,\beta}\partial_\beta u + \bar{T}^d \partial_t u + \bar{F}^d u,$$

vanish when evaluated at $t = 0$ and $x \in \partial\Omega$.

Let $\phi \in C_0^\infty(\Omega \times [0,T[)$ and define the linear functional f:

$$f(L^*\phi) = < g, \phi(0) >_{H^s, H^{-s}} .$$

Due to the linearity and to condition (iii), the operator L^* is injective and the functional f turns out to be well-defined. Since f is a bounded linear functional, thanks to Hahn-Banach Theorem, there exists a continuous linear form F, defined in the space $H^{-s}_{\bar{\Omega}\times[0,T]}(\mathbf{R}^{d+1})$, which extends the functional f. Thanks to Riesz Representation Theorem, there exists a function $u \in H^s(\bar{\Omega} \times [0,T])$, such that $F(v) =< u, v >_{H^s,H^{-s}}$, for all $v \in H^{-s}_{\bar{\Omega}\times[0,T]}(\mathbf{R}^{d+1})$. As a consequence of this property, the function u turns out to be a solution, in the sense of distributions, of the system $Lu = 0$, in $\Omega \times (0,T)$. □

As far as the initial datum is concerned, let $\psi \in C_0^\infty(\Omega\times[0,T[)$. Integrating by parts, we obtain

$$\begin{aligned} < g, \psi(0) >_{L^2,L^2} =& < \partial_t u(0), \psi(0) >_{L^2,L^2} - < u(0), \partial_t\psi(0) >_{L^2,L^2} \\ & + < \bar{S}u(0), \psi(0) >_{L^2,L^2} + < \sum_{\alpha=1}^{d} \bar{T}^\alpha \partial_\alpha u(0), \psi(0) >_{L^2,L^2} . \end{aligned} \quad (8)$$

By choosing suitable functions $\psi \in C_0^\infty(\Omega \times [0,T[)$ and applying a standard argument, we deduce that u satisfies the following initial conditions: $\partial_t u(x,0) = g(x),\ u(x,0) = 0$.

Furthermore, let $\chi \in C_0^\infty(\mathbf{R}^{d-1}\times[0,+\infty[\times(0,T))$. Then $< Lu, \chi >_{H^s,H^{-s}}= 0$; integrating by parts, we obtain

$$\int_0^T \int_{\mathbf{R}^{d-1}} < 2\sum_{\beta=1}^{d-1} \bar{E}^{d,\beta}\partial_\beta u + \bar{F}^d u + \bar{T}^d\partial_t u,\ \chi > (y,0,\tau)dyd\tau = 0.$$

As a consequence, the function u satisfies the homogeneous boundary conditions.

The result proved above cannot be applied directly to state the main theorem, concerning the existence of a non-null solution to problem (3). We shall extend below the result of Proposition 2.1 by studying boundary-value problems for a class of more general operators.

Let $v \in H^s(\bar{\Omega} \times [0,T])$, with $s > d/2+3$, and denote by L_v the second-order linear differential operator

$$L_v = \partial_t^2 + S(v, \nabla_x v)\partial_t + \sum_{\alpha=1}^{d} T^\alpha(v, \nabla_x v)\partial_t\partial_\alpha + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} E^{\alpha,\beta}(v, \nabla_x v)\partial_\alpha\partial_\beta + \sum_{\alpha=1}^{d} F^\alpha(v, \nabla_x v)\partial_\alpha + G(v, \nabla_x v), \quad (9)$$

and by L_v^* the corresponding adjoint operator.

Proposition 2.2. *Assume that $v \in H^s(\bar{\Omega} \times [0,T])$, with $s > d/2+3$ and consider the boundary-value problem*

$$\begin{cases} L_v u = 0, & \text{in } \Omega \times (0,T), \\ 2\sum_{\beta=1}^{d-1} E^{d,\beta}(v, \nabla_x v)\partial_\beta u + T^d(v, \nabla_x v)\partial_t u + F^d(v, \nabla_x v)u = 0, & \text{on } \partial\Omega \times (0,T); \end{cases} \quad (10)$$

under the following assumptions:

(i) for every α, $\beta = 1, \cdots, d$, the matrix-valued functions $S, T^\alpha, E^{\alpha,\beta}, F^\alpha, G \in C^\infty(\mathbf{R}^d \times \mathbf{R}^{d^2})$ and are bounded with their derivatives; moreover, $E^{\alpha,\beta} = E^{\beta,\alpha}$ and $E^{d,d}$ is the null-matrix;

(ii) there exists a positive constant C, such that for every $\phi \in C_0^\infty(\Omega \times [0,T[)$,

$$\|L_v^* \phi\|^2_{H^{-s}_{\bar{\Omega}\times[0,T]}(\mathbf{R}^d)} \geq C\|\phi(t)\|^2_{H^{-s}_{\bar{\Omega}}(\mathbf{R}^d)},$$

with $t \in [0,T]$.

Then there exists a non-null solution $u \in H^s(\bar{\Omega} \times [0,T])$ to the boundary-value problem (10).

The coefficients of the operators L_v and L_v^*, are continuous and bounded in $\bar{\Omega} \times [0,T]$ due to Sobolev embedding. By means of the same proof carried out to state the result of Proposition 2.1, we can prove there exists a non-null solution to the BVP (10), in the space $H^s(\bar{\Omega} \times [0,T])$.

3. Quasi-Linear Second-Order Hyperbolic Problems

I shall study below the well-posedness of the initial boundary-value problem (3) in the spaces $H^{s'}(\bar{\Omega} \times [0,T])$ with $s' \in (0,s)$, $s > d/2+3$, by means of the definition of a suitable iteration scheme, which yields a sequence of initial boundary-value problems, whose solutions turn out to approach the unique solution of (3).

Theorem 3.1. *Consider the boundary-value problem (3). Let $s > d/2+3$. Let $T > 0$ and $g \in H^s(\mathbf{R}^d)$. Suppose the following conditions are satisfied:*

(i) for every α, $\beta = 1, \cdots, d$, $S, T^\alpha, E^{\alpha,\beta}, F^\alpha, G \in C^\infty(\mathbf{R}^d \times \mathbf{R}^{d^2})$ and are bounded as well as their derivatives; moreover, $E^{\alpha,\beta} = E^{\beta,\alpha}$ and $E^{d,d}$ is the null-matrix;

(ii) for every α, $\beta = 1, \cdots, d$, the matrices $S, T^\alpha, E^{\alpha,\beta}, F^\alpha, G$ vanish when evaluated at $(0,0)$;

(iii) compatibility conditions are satisfied by the function g and the boundary operator

$$B[u] = 2\sum_{\beta=1}^{d-1} E^{d,\beta}(0,0)\partial_\beta u + T^d(0,0)\partial_t u + F^d(0,0)u,$$

as $t = 0$ and $x \in \partial\Omega$.

(iv) if $v \in H^s(\bar{\Omega} \times [0,T])$, there exists a positive constant C, so that for every $\phi \in C_0^\infty(\Omega \times [0,T[)$,

$$\|L_v^* \phi\|^2_{H^{-s}_{\bar{\Omega}\times[0,T]}(\mathbf{R}^d)} \geq C\|\phi(t)\|^2_{H^{-s}_{\bar{\Omega}}(\mathbf{R}^d)},$$

with $t \in [0,T]$;

(v) if r is a positive real number with $2 \leq r \leq s'$ and $s' \in [2,s]$, there exists a positive constant Γ such that for every $v \in H^{s'}(\bar{\Omega} \times [0,T])$ and for every $\phi \in H^{s'}(\bar{\Omega} \times [0,T])$,

$$\|\phi\|^2_{H^r(\bar{\Omega}\times[0,T])} \leq \Gamma\, T\|L_v\, \phi\|^2_{H^{r-2}(\bar{\Omega}\times[0,T])} + \|\phi(0)\|^2_{H^r(\bar{\Omega})} + \|\partial_t\phi(0)\|^2_{H^{r-1}(\bar{\Omega})}. \quad (11)$$

Then there exists $\tilde{T} > 0$ and a function $u \in H^{s'}(\bar{\Omega} \times [0,\tilde{T}])$ with $s' \in (0,s)$, which provides a classical solution, to the initial boundary-value problem (3), with initial data $u(x,0) = 0$ and $\partial_t u(x,0) = g(x)$. Moreover, the solution turns out to be unique.

Proof. Let δ be a positive real number, such that $\|g\|_{H^s(\mathbf{R}^d)} \leq \frac{\delta}{4}$.

According to the notation of the previous section, for every $k \in \mathbf{N}, k > 0$, let us denote by L_{k-1} the linear differential operator $L_{u^{k-1}}$ and by B_{k-1} the corresponding boundary operator. We introduce the following iteration scheme

$$\begin{cases} L_{k-1}u^k = 0, & x \in \Omega,\ t \in [0,T], \\ B_{k-1}u^k = 0, & x \in \partial\Omega,\ t \in [0,T]; \end{cases} \tag{12}$$

as $k \in \mathbf{N}$. Let u^0 be a function belonging to the space $H^s(\bar\Omega \times [0,T])$ so that $\|u^0\|_{H^s(\bar\Omega\times[0,T])} \le \delta$. If the function u^{k-1} belongs to $H^s(\bar\Omega\times[0,T])$, then the operator L_{k-1} satisfies the assumptions of Proposition 2.2. As a consequence of this property, the initial boundary-value problem (12), with initial data $u^k(x,0) = 0$ and $\partial_t u^k(x,0) = g(x)$, admits a unique solution $u^k \in H^s(\bar\Omega \times [0,T])$. We shall prove that the sequence of functions $(u^k)_{k\in\mathbf{N}}$ is convergent to a function u, in the space $H^2(\bar\Omega \times [0,T])$, which provides the solution of the BVP (3). Let us divide the proof in three steps.

Step 1. In accordance with the definition of u^0, we have $\|u^0\|_{H^s(\bar\Omega\times[0,T])} \le \delta$. Let us prove by induction that for all $k \in \mathbf{N}$, $\|u^k\|_{H^s(\bar\Omega\times[0,T])} \le \delta$. We assume that for all $k \le m$, $\|u^k\|_{H^s(\bar\Omega\times[0,T])} \le \delta$, and prove that, as long as T is small enough, the estimate is true in the case where $k \ge m+1$. Since $s > d/2 + 3$ and the functions $u^k \in H^s(\bar\Omega \times [0,T])$, there exists a positive constant μ, which depends on δ, so that for every α, $\beta = 1, \cdots, d$, for all $k \le m$,

$$\begin{array}{l} \|u^k\|_{L^\infty(\Omega\times[0,T])} \le \mu; \quad \|\nabla_x u^k\|_{L^\infty(\Omega\times[0,T])} \le \mu; \\ \|\partial_\alpha\partial_\beta u^k\|_{L^\infty(\Omega\times[0,T])} \le \mu; \\ \|\partial_t u^k\|_{L^\infty(\Omega\times[0,T])} \le \mu; \quad \|\partial_t\partial_\alpha u^k\|_{L^\infty(\Omega\times[0,T])} \le \mu. \end{array} \tag{13}$$

Let us denote by P_j^k the sum

$$P_j^k = -\sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} E^{\alpha,\beta}(u^{k-2}, \nabla_x u^{k-2})\partial_\alpha\partial_\beta u^k - \sum_{\alpha=1}^{d} F^\alpha(u^{k-2}, \nabla_x u^{k-2})\partial_\alpha u^k - G(u^{k-2}, \nabla_x u^{k-2})u^k. \tag{14}$$

Let $k = m+1$. Thanks to Moser estimates (see [1]) and to the conditions satisfied by the matrices $E^{\alpha,\beta}, F^\alpha, G$, for all α, $\beta = 1, \cdots, d$, we obtain

$$\|P_j^k\|_{H^{s-2}(\bar\Omega\times[0,T])} \le \zeta(s,\mu)\,\|u^k\|_{H^s(\bar\Omega\times[0,T])}, \tag{15}$$

where $\zeta(s,\mu)$ is a suitable positive constant that depends on s, μ and on the matrices $E^{\alpha,\beta}, F^\alpha, G$. Similarly,

$$\|S(u^{k-2}, \nabla_x u^{k-2})\partial_t u^k\|_{H^{s-2}(\bar\Omega\times[0,T])} \le \rho_1(s,\mu)\,\|\partial_t u^k\|_{H^{s-1}(\bar\Omega\times[0,T])}, \tag{16}$$

and

$$\|T^\alpha(u^{k-2}, \nabla_x u^{k-2})\partial_t\partial_\alpha u^k\|_{H^{s-2}(\bar\Omega\times[0,T])} \le \rho_2(s,\mu)\,\|\partial_t u^k\|_{H^{s-1}(\bar\Omega\times[0,T])}. \tag{17}$$

As a consequence of the previous estimates, we deduce

$$\|L_{k-2}\, u^k\|_{H^{s-2}(\bar\Omega\times[0,T])} \leq \zeta(s,\mu)\, \|u^k\|_{H^s(\bar\Omega\times[0,T])} + \rho(s,\mu)\|\partial_t u^k\|_{H^{s-1}(\bar\Omega\times[0,T])}, \quad (18)$$

where $\rho(s,\mu)$ is the sum of the positive constants $\rho_1(s,\mu)$, $\rho_2(s,\mu)$.

In view of the condition (v),

$$\|u^k\|^2_{H^s(\bar\Omega\times[0,T])} \leq \Gamma T \left(\zeta(s,\mu)\, \|u^k\|_{H^s(\bar\Omega\times[0,T])} + \rho(s,\mu)\|\partial_t u^k\|_{H^{s-1}(\bar\Omega\times[0,T])}\right)^2 + \|g^k\|^2_{H^s(\bar\Omega)}$$
$$\leq 2\Gamma T \max\left(\zeta^2(s,\mu),\, \rho^2(s,\mu)\right)\, \|u^k\|^2_{H^s(\bar\Omega\times[0,T])} + \frac{\delta^2}{4}. \quad (19)$$

If $T < \frac{1}{4\Gamma\max(\zeta^2(s,\mu),\,\rho^2(s,\mu))}$, then $\|u^k\|^2_{H^s(\bar\Omega\times[0,T])} \leq \delta^2$.

As a consequence of the previous estimate, by induction, we have for all $k \in \mathbf{N}$,

$$\|u^k\|_{H^s(\bar\Omega\times[0,T])} \leq \delta.$$

Step 2. By means of the properties proved above, we shall establish that the sequence $(u^k)_{k\in\mathbf{N}}$ is convergent in the space $H^2(\bar\Omega\times[0,T])$. Let us denote by W^k the difference $u^k - u^{k-1}$, for every $k \in \mathbf{N}$; thus, we have

$$L_{k-1}W^k = (L_{k-2} - L_{k-1})u^{k-1}. \quad (20)$$

Let us set $\sigma = (u, \nabla_x u)$. Thanks to the mean value theorem

$$\|\left(S(u^{k-2},\nabla_x u^{k-2}) - S(u^{k-1},\nabla_x u^{k-1})\right)\partial_t u^{k-1}\|_{H^1(\bar\Omega\times[0,T])}$$
$$\leq 2\|D_\sigma S(a,b)\left(u^{k-2}-u^{k-1}, \nabla_x u^{k-2} - \nabla_x u^{k-1}\right)\|_{H^1(\bar\Omega\times[0,T])}$$
$$\times\, \|\partial_t u^{k-1}\|_{W^{1,\infty}(\Omega\times[0,T])} \leq const(\delta)\, \|W^{k-1}\|_{H^2(\bar\Omega\times[0,T])}. \quad (21)$$

Similar estimates can be computed for the other terms in the r.h.s. of (20). Thus, for all $k \in \mathbf{N}$,

$$\|L_{k-1}W^k\|_{H^1(\bar\Omega\times[0,T])} \leq const(\delta)\, \|W^{k-1}\|_{H^2(\bar\Omega\times[0,T])}. \quad (22)$$

Due to condition (v),

$$\|W^k\|^2_{H^2(\bar\Omega\times[0,T])} \leq \Gamma T\left(const(\delta)^2\|W^{k-1}\|^2_{H^2(\bar\Omega\times[0,T])}\right); \quad (23)$$

whence, as $k \geq 1$,

$$\|W^k\|^2_{H^2(\bar\Omega\times[0,T])} \leq T\left(const\|W^{k-1}\|^2_{H^2(\bar\Omega\times[0,T])}\right). \quad (24)$$

If T is sufficiently small, then the sequence of functions $(u^k)_{k\in\mathbf{N}}$ turns out to be a Cauchy sequence in the space $H^2(\bar\Omega\times[0,T])$, since the constant in (24) is independent of k. Let us denote by u the limit of $(u^k)_{k\in\mathbf{N}}$. Since the sequence $(u^k(t))_{k\in\mathbf{N}}$, as proved in Step 1, is bounded in $H^s(\bar{\Omega}\times[0,T])$, by interpolation, it turns out to be convergent to the function u in the space $H^{s'}(\bar{\Omega}\times[0,T])$ for every $s'\in(0,s)$. As a consequence, the function u belongs to the space $C^2(\bar{\Omega}\times[0,T])$.

Step 3. We discuss now the convergence of the iteration scheme (12).

For the sake of convenience, let us write the operators L_{k-1} and A in the following form

$$L_{k-1}=\partial_t^2+Q_{k-1}+S(u^{k-1},\nabla_x u^{k-1})\partial_t+\sum_{\alpha=1}^{d}T^\alpha(u^{k-1},\nabla_x u^{k-1})\partial_t\partial_\alpha; \quad (25)$$

$$A[u]=\partial_t^2 u+Qu+S(u,\nabla_x u)\partial_t u+\sum_{\alpha=1}^{d}T^\alpha(u,\nabla_x u)\partial_t\partial_\alpha u. \quad (26)$$

Thanks to the convergence properties of the sequence $(u^k)_{k\in\mathbf{N}}$ proved in Step 2, the sequence $(L_{k-1}u^k)_{k\in\mathbf{N}}$ turns out to be convergent to $A[u]$, for every $(x,t)\in\Omega\times(0,T)$, as $k\to\infty$. Thus, the function u provides a classical solution to system (3), $A[u]=0$.

Furthermore, passing to the limit, the solution u turns out to satisfy the boundary condition, i.e. $B[u]=0$, $x\in\partial\Omega$, $t\in[0,T]$.

As far as the initial datum is concerned, we get $u(x,0)=0$ and $\partial_t u(x,0)=g(x)$, where $x\in\Omega$. Since the function g is not null, u is a non-trivial solution to (3).

By means of the same estimates obtained for the sequence of functions $(W^k)_{k\in\mathbf{N}}$ in Step 2, we can prove the uniqueness of the solution u to the problem (3), with initial conditions $u(x,0)=0$ and $\partial_t u(x,0)=g(x)$, $x\in\Omega$. Suppose that u_1 and u_2 are two distinct solutions of the initial boundary-value problem (3) and let $w=u_1-u_2$. Consider the function $L_{u_1}w=(-L_{u_1}+L_{u_2})u_2$. The same estimates proved to find (22), applied to the function w, yield $\|L_{u_1}w\|_{L^2(\Omega\times]0,T[)}\le const(\delta)\|w\|_{H^2(\Omega\times]0,T[)}$. Thus, because of (11),

$$\|w\|^2_{H^2(\Omega\times]0,T[)}\le T\,const\,\|w\|^2_{H^2(\Omega\times]0,T[)}.$$

If T is small enough, then $w=0$, in $\Omega\times]0,T[$. □

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