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ONE-DIMENSIONAL CONES

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let $Y \subset \mathbb{P}^n$ be a union of d lines passing through O (a one-dimensional cones). Here we study the arithmetic genera of the unions of lines through O containing Y.

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1. Introduction

Here "one-dimensional abstract cone" (from now on just "abstract cone") is a projective and reduced scheme Y with a unique singular point, O, such that each irreducible component of Y is smooth, rational and contains O. For any reduced scheme A let $\mathcal{B}(A)$ denote the set of the irreducible components of A. The degree $\deg(Y)$ of an abstract cone Y is the integer $\sharp(\mathcal{B}(Y))$. The singular point of an abstract cone is called its vertex. We also say that $\mathbb{P}^1$ is an abstract cone of degree 1 and that any point of it is a vertex. Let $S(d,g)$ denote the set of all abstract cones with degree d and arithmetic genus g. In Section 2 we collect a few results about them. In Section 3 we consider gemometric cones $Y \subset \mathbb{P}^n$, i.e. we fix $O \in \mathbb{P}^n$ and take unions Y of d distinct lines, each of them containing O. For any integer $n \geq 1$ let $S(d,g,n)$ denote the set of all reduced one-dimensional reduced subcones of $\mathbb{P}^n$ (i.e. their irreducible components are lines) with degree d and arithmetic genus g. Any inclusion $\mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$ as a

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hyperplane induces an inclusion $S(d,g,n) \subseteq S(d,g,n{+}1)$ whose image are, up to projective equivalence, the set of all degenerate cones. Let $S(d,g,n)'$ denote the set of all $X \in S(d,g,n)$ which are "in linearly general position", i.e. such that, taking any hyperplane $H \subset \mathbb{P}^n$ not through the vertex of X the d points $X \cap H$ are in linearly general position in H. For all $d > d'$, $g \geq g'$ and $Y \in S(d',g')$ (resp. $Y \in S(d',g',n)$) let $S_Y(d,g)$ (resp. $S_Y(d,g,n)$, resp. $S_Y(d,g,n)'$) denote the set of all $X \in S(d,g)$ (resp. $X \in S(d,g,n)$, resp. $X \in S(d,g,n)'$) containing an abstract subcone isomorphic to Y (resp. a subcone projectively equivalent to Y). We stress that the non-emptiness part for the sets $S(d,g,n)$ is trivial (use Lemma 1 and the description of all Hilbert functions of finite subsets of $\mathbb{P}^{n-1}$). Only stronger statements, related to $S(d,g,n)'$, $S_Y(d,g,n)$ and $S_Y(d,g,n)'$ are perhaps new.

2. Abstract Cones

Proposition 1. *Fix integers $g \geq 0$ and $d \geq 2$. Every degree d abstract cone with arithmetic genus g is isomorphic to a projective cone if and only if $g = 0$.*

Proof. First assume $g = 0$. Take homogeneous coordinates $x_0, \dots, x_d$ of $\mathbb{P}^d$. Every genus 0 abstract cone with degree d has embedding dimension d and it is isomorphic to the union the d coordinate axis $L_j := \{x_j = 0 \text{ for all } j \neq i\}$, $i \in \{1, \dots, d\}$, containing $(1; 0; \dots; 0)$. Now assume $g > 0$ and $d = 2$. Let Y_g be any abstract cone of arithmetic genus g obtained gluing two copies of the pair $(\mathbb{P}^1, O)$ at O along a the scheme $(g+1)O$. Any degree 2 reduced cone of a projective space is a reducible conic. Hence Y_g cannot be embedded as a cone. If $d > 2$ take as non-embeddable abstract cone the union $Y_{g,d}$ of Y_g and $d-2$ $\mathbb{P}^1$'s through O so that $\dim T_O(Y_{g,d}) = \dim(T_O(Y_g))$ where $T_O(X)$ denote the Zariski tangent space at $O \in X$ of the scheme X. The abstract cone $Y_{g,d}$ is uniquely determined, up to isomorphisms by Y_g. $Y_{g,d}$ cannot be embedded in a projective space as a cone, because its subcurve Y_g cannot be embedded in a projective space as a cone. □

Remark 1. Fix a reduced projective curve Y and a proper closed subcurve A. Set $B := \overline{Y \backslash A}$. For every $L \in \mathrm{Pic}(Y)$ we have a Mayer-Vietoris exact sequence

$$0 \to L \to L|A \oplus L|B \to L|A \cap B \to 0. \tag{1}$$

The restriction map $\rho : \mathrm{Pic}(Y) \to \mathrm{Pic}(A) \times \mathrm{Pic}(B)$ is surjective. If A and B are connected, then ρ is injective if and only if $\mathrm{length}(A \cap B) \leq 1$, i.e. if and only if either $A \cap B = \emptyset$ or the scheme-theoretic intersection $A \cap B$ is a unique point. In the latter case we have $p_a(Y) = p_a(A) + p_B(B)$ (take $L := \mathcal{O}_Y$ in (1)).

Proposition 2. *Fix integers $d \geq 2$ and $g \geq q$. Let Y be an abstract cone which may be embedded as a cone by a very ample line bundle R. Assume $\deg(Y) = d-1$ and $p_a(Y) = q$. Every abstract cone X with degree d and genus g containing Y is embeddable as a cone if and only if $g = q$. If $g = q$, then X may be embedded by a very ample line bundle L such that $L|Y \cong R$.*

Proof. The "only if" part may be proved as the case $d = 2$ of the "only if" part of Proposition 1. For the "if"part fix X and apply Remark 1 with $X := Y$ and $A := Y$. Using (1) we see the very ampleness of every $L \in \mathrm{Pic}(X)$ such that $L|Y \cong R$ and $\deg(L|\overline{X \backslash Y}) > 0$. □

Proposition 3. *Fix integers $d \geq 2$ and $g \geq 0$.*

(a) For every $X \in S(d,g)$ there is a very ample $L \in Pic(X)$ such that $\deg(L|T) \leq 2g+3$ for every $T \in \mathcal{B}(X)$.

(b) The set $S(d,g)$ is bounded.

Proof. Part (a) is proved by induction on d using (1). Part (b) follows from part (a) and the existence of the Hilbert scheme of projective spaces. □

3. One-Dimensional Geometric Cones

We recall the following result (see [3], Theorem 5)

Lemma 1. *Let $X \subset \mathbb{P}^n$ be an embedded one-dimensional and reduced cone. Fix a hyperplane $H \subset \mathbb{P}^n$ not containing the vertex of X. Then*

$$p_a(X) = \sum_{t \geq 1} h^1(H, \mathcal{I}_{X \cap H}(t)).$$

Proof. Let $u : Y \to X$ be the normalization map. Since u is finite and Y is a disjoint union of $\deg(X)$ $\mathbb{P}^1$'s, we have $\chi(\mathcal{O}_X) + \chi(u_*(\mathcal{O}_Y)/\mathcal{O}_X)) = \deg(X)$. Since the sheaf $u_*(\mathcal{O}_Y)/\mathcal{O}_X$ is supported by the vertex of X, $\chi(u_*(\mathcal{O}_Y)/\mathcal{O}_X) = \mathrm{length}(u_*(\mathcal{O}_Y)/\mathcal{O}_X))$. The latter integer may be computed working in the affine cone $X \backslash X \cap H$ of $\mathbb{P}^n \backslash H \cong \mathbb{A}^n$. In this case the lemma is just the statement of [3], Theorem 5. □

For all integers $d \geq n \geq 2$ set $g_{d,n} := \sum_{t \geq 1} \max\{0, d - \binom{n+t-1}{n-1}\}$. Lemma 1 (i.e. [3], Theorem 5), [3], Theorem 4, and semicontinuity give the following result.

Corollary 1. *Fix integers $d \ge n \ge 2$. The set $S(d, g_{d,n}, n)$ is parametrized by a non-empty quasi-projective variety and a dense open subset of this variety parametrizes $S(d, g_{d,n}, n)'$. We have $S(d, g, n) = \emptyset$ for all $g < g_{d,n}$.*

Since $S(d, g_{d,n}, n)$ is parametrized by an integral variety, we may speak about its general element.

Theorem 1. *Fix integers d, n, x, w such that $d \ge n \ge 4$, $1 \le x \le 2n-2$, $d + x \le \binom{n+2}{3}$, and $0 \le w \le x$. If $x \le n-1$, then assume $w = 0$. There is $Y \in S(d, g_{d,n}, n)'$ such that $S_Y(d + x, g_{d+x,n} + x + w, n)' \neq \emptyset$.*

Proof. Fix $O \in \mathbb{P}^n$ and a hyperplane $H \subset \mathbb{P}^n$ such that $O \notin H$. We will only consider cones with vertex O. Thus it is equivalent to work with finite subsets of H. If $w = 0$, then set $q := 0$. If $w > 0$, then set $q := w+n-1-x$. In all cases we have $0 \le q \le n-1$. Fix a linearly normal smooth curve $T \subset H$ such that $\deg(T) = n - 1 + q$ and $p_a(T) = q$. Thus T is projectively normal. Fix a general $A \subset T$ such that $\sharp(A) = 2(n-1+q)+1-q = 2n+q-1$. After fixing A take a general $B \subset H$ such that $\sharp(B) = d - \sharp(A)$. Since T is projectively normal, $h^0(T, \mathcal{O}_T(2)) = \sharp(A)$ and A is general in T, we obtain $h^1(H, \mathcal{I}_A(2)) = 0$ and $H^0(H, \mathcal{I}_A(2)) = H^0(H, \mathcal{I}_T(2))$. Since $h^1(H, \mathcal{I}_A(2)) = 0$, we have $h^1(H, \mathcal{I}_A(2)) = 0$ for all $t \ge 3$. The generality of B gives $h^0(H, \mathcal{I}_{A\cup B}(t)) = \max\{0, \binom{n+t-1}{n-1}\}$ for all $t \ge 1$. Let Y be the cone with base $A \cup B$ and vertex O. Fix a general $(P_1, \dots, P_x) \in T^x$ and write $S := \{P_1, \dots, P_x\}$. Let X be the cone with vertex O and base $A \cup B \cup S$. For general A, S we have $h^1(H, \mathcal{I}_{A\cup S}(t)) = \max\{0, \sharp(A) + x - t(n - 1 + q) + q - 1\}$ for all $t \ge 1$. For general B we have $h^1(H, \mathcal{I}_{A\cup S\cup B}(t)) = \max\{0, \sharp(A) + x - t(n-1+q) + q - 1, d - \binom{n+t-1}{n-1}\}$ for all $t \ge 1$. Since $x + 2n + q - 1 = 3(n + q - 1) + 1 - q + w$ and T is projectively normal, we have $h^1(H, \mathcal{I}_{A\cup S}(3)) = w$. Notice that we may take B general after fixing $A\cup S$. Since $4(n-1+q)+1-q = 4n-3+3q \ge (2n+q-1)+x = \sharp(A\cup S)$ and T is projectively normal, $h^1(H, \mathcal{I}_{A\cup S}(t)) = 0$ for all $t \ge 4$. Let $S' \subset H$ be a general subset of H such that $\sharp(S') = d + x$. Since $d + x \le \binom{n+2}{3}$ and we may take B general after fixing A and B, we get

$$\sum_{t\ge1} h^1(H, \mathcal{I}_{A\cup S\cup B}(t)) = \sum_{t\ge1} h^1(H, \mathcal{I}_{S'}(t)) + h^1(H, \mathcal{I}_{A\cup S}(2)) + h^1(H, \mathcal{I}_{A\cup S}(3)),$$

i.e. $p_a(X) = g_{d+x,n} + x + w$ (Lemma 1). □

Proposition 4. *Fix $Y \in S(d, g, n)$. Then $S_Y(d + 1, g, n) \neq \emptyset$ if and only if Y is degenerate. If Y is degenerate, then all the elements of $S_Y(d + 1, g, n)$ are projectively equivalent.*

Proof. Let O be the vertex of Y. The Zariski tangent space T_OY is the linear span of Y. Let $X \subset \mathbb{P}^n$ be any union of Y and a line D such that $(Y \cap D)_{red} = \{O\}$. We have $p_a(X) = p_a(Y)$ if and only if $\{O\}$ is the scheme-theoretic intersection of Y and D. This is the case if and only if D is not contained in $T_O Y$. Thus there is such a line D if and only if Y is degenerate. Assume $T_O Y \neq \mathbb{P}^n$. Let $D, D' \subset \mathbb{P}^n$ be any two lines such that $D \cap T_O Y = D' \cap T_O Y = \{O\}$. There is $h \in \mathrm{Aut}(\mathbb{P}^n)$ such that $h|T_O Y = \mathrm{Id}_{T_O Y}$ and $h(D) = D'$, proving the last assertion. □

Remark 2. Fix integers $n \geq d \geq 2$. Then $S(n, g, n)' \neq \emptyset$ if and only if $g = 0$ and all elements of $S(d, 0, n)'$ are projectively equivalent. If $n + 1 \leq d \leq 2n - 1$, then $S(d, g, n)' \neq \emptyset$ if and only if $0 \leq g \leq d - n$. Obviously $S(d, g, 2) \neq \emptyset$ if and only if $g = (d - 1)(d - 2)/2$.

For all integers $d \geq 2n \geq 3$ set $\pi(d, n) := \binom{m}{2}(n - 1) + m\epsilon$, where $m := \lfloor (d - 1)/(n - 1) \rfloor$ and (if $d \geq 2n + 1 \geq 9$) $\pi_1(d, n) := \binom{m_1}{2} n + m_1(\epsilon_1 + 1) + \mu_1$, where $m_1 := \lfloor (d - 1)/n \rfloor$, $\epsilon_1 := d - m_1 n - 1$, $\mu_1 := 1$ if $\epsilon_1 = n - 1$ and $\mu_1 := 0$ if $\epsilon_1 \neq n - 1$. Notice that for $d \gg n$ we have $\pi(d, n) \sim d^2/(2n - 2)$, $\pi(d, n) - \pi(d - 1, n) \sim d/(n - 1)$ and $\pi_1(d, n) \sim d^2/2n$.

Proposition 5. *Fix integer $d \geq 2n + 1 \geq 6$. Then $\pi(d, n)$ is the maximal integer g such that $S(d, g, n)' \neq \emptyset$. Any $X \in S(d, \pi(d, n), n)'$ is contained in the cone over a rational normal curve of $\mathbb{P}^{n-1}$. Every degree d cone contained in the cone over a rational normal curve of $\mathbb{P}^{n-1}$ is an element of $S(d, \pi(d, n), n)'$.*

Proof. Castelnuovo's upper bound for the genus of integral and non-degenerate curves in $\mathbb{P}^n$ works, because it only use the linearly general position of a general hyperplane section (the uniform position property is only used for some of its refinements). Thus we get the first statement. For the second statement use [1], Theorem4, and that the rational normal curve $T_H \subset X \cap H$, H a hyperplane not containing the vertex of X, is the intersection of all quadric hypersurfaces of H containing $X \cap H$: the 2-dimensional cone is obtained varying H, because it is just the intersection of all quadrics containing X. The last assertion follows from Lemma 1, because for any rational normal curve $T \subset H$, any finite subset of T is in linearly general position in H. □

Proposition 5 implies that $S(d, \pi(d, n), n)'$ is parametrized by an integral quasi-projective variety.

Proposition 6. *Fix integers d, n, s, g such that $d \geq 2n + 1 \geq 6$, $1 \leq s \leq 2n - 1$ and $g < \pi(d, n) + s$. Fix any $Y \in S(d, \pi(d, n), n)'$. Then $S_Y(d, g, n) = \emptyset$ for all $g < \pi(d, n) + s$ and $S_Y(d + s, \pi(d, n) + s, n)' \neq \emptyset$.*

Proof. The first assertion follows from Proposition 4. Let X be the union of Y and s general lines. Fix a hyperplane $H \subset \mathbb{P}^n$ not containing the vertex of X. Let T be the rational normal curve of H containing $X \cap H$. Since $h^0(H, \mathcal{O}_H(t)) - h^0(H, \mathcal{I}_T(t)) = (n-1)t+1$ for all integers $t \geq 1$ and $s \leq 2n-1$, we get $h^1(H, \mathcal{I}_{X\cap H}(t)) = h^1(H, \mathcal{I}_{Y\cap H}(t))$ for all integers $t \geq 2$. Apply Lemma 1. $\square$

The proof of Proposition 6 implies that $S_Y(d, \pi(d,n)+s, n)'$ is parametrized by an integral quasi-projective variety.

Lemma 2. *Fix a rational normal curve $T \subset H := \mathbb{P}^{n-1}$ and a zero-dimensional scheme $Z \subset T$. Set $d :=$ length(Z) and assume $d \geq 2n-1$. Let s be the minimal integer such that $d \leq (n-1)s$. Then the homogeneus ideal of Z in H is generated by the quadric hypersurfaces containing T and by $(n-1)s+1-d$ forms of degree s. Thus Z is the scheme-theoretic base locus of the form of degree s containing it.*

Proof. We computed the cohomology of Z. Moreover $h^0(T, \mathcal{O}_T(s)(-Z)) = (n-1)s - d$. T is the base locus of $\mathcal{I}_Z(t)$ for all integers t such that $2 \leq t \leq s-1$. The dual of the Euler's sequence of the tangent bundle of H shows that the lemma is true if $h^1(H, \mathcal{I}_H \otimes \Omega^1_H(s+1)) = 0$. This vanishing is true, because $T \cong \mathbb{P}^1$, $\Omega^1_H(s+1)$ is the direct sum of $n-1$ line bundles of degree $-n+(n-1)(s+1) = (n-1)s-1$ and $d \leq (n-1)s$. $\square$

Proposition 7. *Fix integers d, n, g such that $d \geq 2n+1 \geq 6$, $g \geq \pi(d,n)+2$ and $g \neq \pi(d+1,n)$. Fix any $Y \in S(d, \pi(d,n), n)'$. Then $S_Y(d+1, g, n)' = \emptyset$.*

Proof. Take a line $D \subset \mathbb{P}^n$ passing through the vertex O of X, but not contained in Y. Set $X := Y \cup D$ and $P := D \cap H$. Fix a hyperplane $H \subset \mathbb{P}^n$ such that $O \notin H$. Call Z the 2-dimensional minimal degree cone containing Y. Thus $T := Z \cap H$ is a rational normal cone containing $Y \cap H$. If $P \in T$, then $p_a(X) = \pi(d+1,n)$ (use Lemma 1). Now assume $P \notin Y$. Since T is the base locus of $\mathcal{I}_{Y\cap H,H}(2)$, we get $h^1(H, \mathcal{I}_{X\cap H}(2)) = h^1(H, \mathcal{I}_{Y\cap H}(2))$. We get in the same way the relation $h^1(H, \mathcal{I}_{X\cap H}(t)) = h^1(H, \mathcal{I}_{Y\cap H}(t))$ for all integers t such that $(n-1)t+1 \leq d$. Since for every finite set $W \subset H$ the function $\mathbb{N} \to \mathbb{N}$ defined by $x \mapsto h^1(H, \mathcal{I}_W(x))$ is strictly decreasing until it arrives to 0, we get that (apart from $h^1(H, \mathcal{I}_{X\cap H}(1))$) the addition of P to $H \cap Y$ contributes at most 1 and at most at one t. If this contribution arises, then it arises at the integer s such that $(n-1)(s-1)+1 \leq d \leq (n-1)s-1$. Lemma 2 shows that the absence of this contribution. $\square$

Proposition 8. *Fix integers n, d such that $d \geq n \geq 3$. Let k be the minimal positive integer such that $d \leq \binom{k+n-1}{n-1}$. Assume $\binom{k+n-1}{n-1} - (n-1) \leq$*

$d < \binom{k+n-1}{n-1}$. *Fix a general* $Y \in S(d, g_{d,n}, n)$. *Then* $S(d, g_{d+1,n} + 1, n) \neq \emptyset$.

Proof. Fix $O \in \mathbb{P}^n$ and a hyperplane $H \subset \mathbb{P}^n$ such that $O \notin H$. We will only consider cones with vertex O. Thus it is equivalent to work with finite subsets of H. Y corresponds to a general $S \subset H$. The generality of S and the definition of the integer k gives $h^1(H, \mathcal{I}_S(t)) = 0$ for all $t \geq k$ and $h^0(H, \mathcal{I}_S(t)) = 0$ for all $t \leq k+1$. Since $\binom{n+k-1}{n-1} - d \leq n-1$, the base locus B of $|\mathcal{I}_S(k)|$ strictly contains S (and indeed contains at least $k^{n-1} - d$ other points). We claim that we may take one of these points, $P \in B \backslash S$, and take as X the cone with $S \cup \{P\}$ as its basis. Indeed, for every $P \in B \backslash S$ we get $p_a(X) \geq g_{d+1,n} + 1$. To get the reverse inequality it is sufficient to find $P \in B \backslash S$ such that $h^1(H, \mathcal{I}_{S \cup \{P\}}(k+1)) = 0$. This is possible because the base locus of $|\mathcal{I}_S(k+1)|$ is strictly contained in B. □

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