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THE OPERATOR $\circledast^k$ RELATED TO TRIHARMONIC WAVE EQUATION

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Abstract: In this paper, we study the solution of equation $\circledast^k u(x) = f(x)$, where $\circledast^k$ is the operator iterated k times and is defined by

$$\circledast^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^3 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^3\right)^k,$$

where $p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x)$ is an unknown function for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $f(x)$ is the generalized function, k is a positive integer and $u(x)$ is an unknown function.

It is found that the solution $u(x)$ depends on the conditions of p and q and moreover such a solution is related to the solution of the Laplace equation. In particularly if we put $k = 1$ and we obtain solution of the triharmonic wave equation.

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1. Introduction

The operator $\diamondsuit^k$ has been first by A. Kananthai (see [3]) and is named as the diamond operator iterated k times and is defined by

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$$\diamondsuit^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k, \quad p+q=n \tag{1.1}$$

is the dimension of the space $\mathbb{R}^n$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer.

Actually the operator $\diamondsuit^k$ is an extension of the ultra-hyperbolic operator and the Laplacian operator. So the operator $\diamondsuit^k$ can be expressed as a product of the the operator $\Box^k$ and $\triangle^k$, that is $\diamondsuit^k = \Box^k \triangle^k$, where

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \right)^k \tag{1.2}$$

is the Laplacian iterated k times.

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k \tag{1.3}$$

is the ultra-hyperbolic operator iterated k times with $p+q=n$.

A. Kananthai (see [3], Theorem 3.1, p. 33) has shown that the convolution $(-1)^k R_{2k}^e(x) * R_{2k}^H(x)$ is an elementary solution of the operator $\diamondsuit^k$, that is

$$\diamondsuit^k((-1)^k R_{2k}^e(x) * R_{2k}^H(x)) = \delta(x), \tag{1.4}$$

where $\delta(x)$ is Dirac-delta distribution and the functions R_{2k}^e and R_{2k}^H are defined by (2.5) and (2.2) respectively with $\alpha = 2k, k$ is nonnegative integer.

Next, W. Satsanit first introduced $(\circledast)^k$ operator, which is defined by

$$\begin{aligned} \circledast^k &= \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^3 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right]^k \\ &= \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right) \right. \\ &\quad \left. \cdot \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right) + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k = \triangle^k \left(\triangle^2 - \frac{1}{4}(\triangle + \Box)(\triangle - \Box) \right)^k \\ &= \left(\frac{3}{4} \diamondsuit \Box + \frac{1}{4} \triangle^3 \right)^k, \end{aligned} \tag{1.5}$$

where

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2},$$

$$\square = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2},$$

$$\diamondsuit = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2.$$

Now, the purpose of this work is to study the equation

$$(\circledast)^k u(x) = f(x). \tag{1.6}$$

This equation is the generalization of the ultra-hyperbolic equation and it can be applied to the triharmonic wave equation.

Let $K_{\alpha,\beta}$ be a distributional family and be defined by

$$K_{\alpha,\beta} = R^e_\alpha * R^H_\beta, \tag{1.7}$$

where R^e_α is called the elliptic kernel defined by (2.5) and R^H_β is called the ultra-hyperbolic kernel defined by (2.2) and α, β are the complex parameters.

The family $K_{\alpha,\beta}$ is well defined and is tempered distribution, since $R^e_\alpha * R^H_\beta$ is a tempered, (see [1], Lemma 2.2) and R^H_β has a compact support.

In this paper, we can show that

$$u(x) = ((-1)^{3k} K_{4k,4k} * (R^e_{2(k-1)})^m + (-1)^{2k} K_{6k,4k} * f(x)) * (S^{*k})^{*-1}, \tag{1.8}$$

where (S^{*k}) is defined by (2.16) and $(S^{*k})^{*-1}$ is an inverse of (S^{*k}) in the convolution algebra. $u(x)$ is a solution of (1.6), where $m = \frac{n-4}{2} \geq 4$, n is even number and $K_{2k,2k}(x)$ is defined by (1.7). Moreover, we can show that the solution related to the solution of Laplace operator $\triangle^{3k}$ defined by (1.2) and also the triharmonic wave operator is defined by (1.5) with $q = 0$, $k = 1$.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Let us denote by

$$u = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2 \tag{2.1}$$

the nondegenerated quadratic form, $p+q = n$ is the dimension of the space $\mathbb{R}^n$.

Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ and let $\overline{\Gamma}_+$ denote it closure. For any complex number β, define the function

$$R^H_\beta(u) = \begin{cases} \dfrac{u^{\frac{\beta-n}{2}}}{K_n(\beta)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where the constant $K_n(\beta)$ is given by the formula

$$K_n(\beta) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2+\beta-n}{2})\Gamma(\frac{1-\beta}{2})\Gamma(\beta)}{\Gamma(\frac{2+\beta-p}{2})\Gamma(\frac{p-\beta}{2})}. \tag{2.3}$$

The function $R^H_\alpha(u)$ is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Y. Nozaki (see [4]).

It is well known that $R^H_\alpha(u)$ is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of α if $Re(\alpha) < n$. Let supp $R^H_\alpha(u)$ denote the support of $R^H_\alpha(u)$ and suppose supp $R^H_\alpha(u) \subset \bar{\Gamma}_+$, that is supp $R^H_\alpha(u)$ is compact.

From S.E. Trione (see [5], p. 11), R^H_{2k} is an elementary solution of the operator $\Box^k$ that is

$$\Box^k R^H_{2k}(u) = \delta(x). \tag{2.4}$$

Definition 2.2. Let $x = (x_1, x_2, ..., x_n)$ be a point of $\mathbb{R}^n$ and $v = x_1^2 + x_2^2 + ... + x_n^2$, let the function $R^e_\alpha(v)$ denote the elliptic kernel of Marcel Riesz and is defined by

$$R^e_\alpha(v) = \frac{|x|^{\frac{\alpha-n}{2}}}{W_n(\alpha)}, \tag{2.5}$$

where

$$v = |x| = x_1^2 + x_2^2 + ... + x_n^2, \tag{2.6}$$

$$W_n(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}, \tag{2.7}$$

α is a complex parameter and n is the dimension of $\mathbb{R}^n$.

It can be shown that $R^e_{-2k}(x) = (-1)^k \triangle^k \delta(x)$, where $\triangle^k$ is defined by (1.3). It follows that $R^e_0(x) = \delta(x)$ (see [1], p. 118).

Moreover, we obtain that $(-1)^k R^e_{2k}(x)$ is an elementary solution of the operator $\triangle^k$ that is

$$\triangle^k((-1)^k R^e_{2k}(x) = \delta(x), \tag{2.8}$$

see [3], Lemma 2.4, p. 31.

Lemma 2.1. *Given P is a hyper-function, then*

$$P\delta^k(p) + k\delta^{(k-1)}(p) = 0,$$

where $\delta^{(k)}$ is the Dirac-delta distribution with k derivatives.

Proof. See [2], p. 233. ∎

Lemma 2.2. *Given the equation*

$$\triangle^k u(x) = 0, \tag{2.9}$$

where $\triangle^k$ *is defined by (1.2) and* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. *Then* $u(x) = (R^e_{2(k-1)}(v))^{(m)}$ *is a solution of (2.9) with* $m = \frac{n-4}{2}, n \geq 4$ *and* n *is even dimension and* v *is defined by (2.6). The function* $(R^e_{2(k-1)}(v))^{(m)}$ *is defined by (2.5) with* m*-derivatives and* $\alpha = 2(k-1)$

Proof. We first show that the generalized function $u(x) = \delta^{(m)}(r^2)$, where $r^2 = |x|^2 = x_1^2 + x_2^2 + \ldots + x_n^2$ of

$$\triangle u(x) = 0. \tag{2.10}$$

Here $\triangle = (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2})$ is a Laplace operator. Now

$$\frac{\partial}{\partial x_i}\delta^{(m)}(r^2) = 2x_i\delta^{(m+1)}(r^2),$$
$$\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2) = 2\delta^{(m+1)}r^2 + 4x_i^2\delta^{(m+2)}(r^2).$$

Thus

$$\triangle\delta^{(m)}(r^2) = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2) = 2n\delta^{(m+1)}r^2 + 4r^2\delta^{(m+2)}(r^2)$$
$$= 2n\delta^{(m+1)}r^2 - 4(m+2)\delta^{(m+1)}(r^2).$$

By Lemma 2.1 with $P = r^2$ we have

$$\triangle\delta^{(m)}(r^2) = (2n - 4(m+2))\delta^{(m+1)}(r^2) = 0, \quad \text{if} \quad 2n - 4(m+2) = 0,$$

or $m = \frac{n-4}{2}$, $n \geq 4$ and n is even. Thus $\delta^{(m)}(r^2)$ is a solution of (2.10) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even. Now $\triangle^k u(x) = \triangle(\triangle^{k-1}u(x)) = 0$ then from the above proof $\triangle^{k-1}u(x) = \delta^{(m)}(r^2)$ with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even.

Convolving both sides of the above equation by the function $(-1)^{k-1} R^e_{2(k-1)}(x)$ we obtain

$$\begin{aligned}
&(-1)^{k-1}R^e_{2(k-1)} * \triangle^{k-1}u(x) = (-1)^{k-1}R^e_{2(k-1)} * \delta^{(m)}(r^2),\\
&\triangle^{k-1}((-1)^{k-1}R^e_{2(k-1)}) * u(x) = (-1)^{k-1}R^e_{2(k-1)} * \delta^{(m)}(r^2),\\
&\delta * u(x) = u(x) = (-1)^{k-1}R^e_{2(k-1)} * \delta^{(m)}(r^2).
\end{aligned} \tag{2.11}$$

Now from (2.1)

$$R^e_{2(k-1)}(x) = \frac{|x|^{2(k-1)-n}}{W_n(\alpha)} = \frac{(|x|^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} = \frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)},$$

where $r = |x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}$. Hence

$$R^e_{2(k-1)}(x) * \delta^{(m)}(r^2) = \frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} * \delta^{(m)}(r^2) = \left(\frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)}\right)^{(m)}$$
$$= (R^e_{2(k-1)}(x))^{(m)}.$$

It follows that $u(x) = (-1)^{k-1}(R^e_{2(k-1)}(x))^{(m)}$ is a solution of (2.9) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension of R^n. $\square$

Lemma 2.3. *Given the equation*

$$\square^k u(x) = 0, \tag{2.12}$$

where $\square^k$ is defined by (1.3) and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then $u(x) = (R^H_{2(k-1)}(u))^{(m)}$ is a solution of (2.12) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension and u is defined by Definition 2.1. The function $(R^H_{2(k-1)}(u))^{(m)}$ is defined by (2.2) with m-derivatives and $\beta = 2(k-1)$.

Proof. We first to show that the generalized function $\delta^{(m)}(r^2 - s^2)$, where $r^2 = x_1^2 + x_2^2 + \ldots + x_p^2$ and $s^2 = x_{p+1}^2 + x_{p+2}^2 + \ldots + x_{p+q}^2$, $p + q = n$, is a solution of the equation

$$\square u(x) = 0. \tag{2.13}$$

Here $\square$ is defined by (1.3) with $k = 1$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

$$\frac{\partial}{\partial x_i}\delta^{(m)}(r^2 - s^2) = 2x_i\delta^{(m+1)}(r^2 - s^2),$$
$$\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2) = 2\delta^{(m+1)}(r^2 - s^2) + 4x_i^2\delta^{(m+2)}(r^2 - s^2),$$

$$\square\delta^{(m)}(r^2-s^2) = \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2-s^2) = 2p\delta^{(m+1)}(r^2-s^2)+4r^2\delta^{(m+2)}(r^2-s^2)$$
$$= 2p\delta^{(m+1)}(r^2 - s^2) + 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2) + 4s^2\delta^{(m+2)}(r^2 - s^2)$$
$$= 2p\delta^{(m+1)}(r^2 - s^2) - 4(m + 2)\delta^{(m+1)}(r^2 - s^2) + 4s^2\delta^{(m+2)}(r^2 - s^2)$$
$$= (2p - 4(m + 2))\delta^{(m+1)}(r^2 - s^2) + 4s^2\delta^{(m+2)}(r^2 - s^2).$$

By Lemma 2.1 with $P = r^2 - s^2$. Similarly,

$$\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\delta^{(m)}(r^2 - s^2) = (-2q + 4(m + 2))\delta^{(m+1)}(r^2 - s^2) + 4r^2\delta^{(m+2)}(r^2 - s^2).$$

Thus

$$\Box\delta^{(m)}(r^2 - s^2) = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2) - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2)$$

$$= (2(p+q) - 8(m+2))\delta^{(m+1)}(r^2 - s^2) - 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2)$$

$$= (2n - 8(m+2))\delta^{(m+1)}(r^2 - s^2) + 4(m+2)\delta^{(m+1)}(r^2 - s^2)$$

$$= (2n - 4(m+2))\delta^{(m+1)}(r^2 - s^2).$$

If $2n - 4(m+2) = 0$, we have $\Box\delta^{(m)}(r^2 - s^2) = 0$. That is $u(x) = \delta^{(m)}(r^2 - s^2)$ is a solution of (2.12) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension.

Now $\Box^k u(x) = \Box(\Box^{k-1}u(x)) = 0$.

From (2.10) we have $\Box^{k-1}u(x) = \delta^{(m)}(r^2 - s^2)$ with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension. Convolving the above equation by $R^H_{2(k-1)}(u)$, we obtain

$$R^H_{2(k-1)}(u) * \Box^{k-1}u(x) = R^H_{2(k-1)}(u) * \delta^{(m)}(r^2 - s^2),$$

$$\Box^{k-1}(R^H_{2(k-1)}(u)) * u(x) = (R^H_{2(k-1)}(u))^{(m)}, \quad \text{where } v = (r^2 - s^2),$$

$$\delta * u(x) = u(x) = (R^H_{2(k-1)}(u))^{(m)}$$

by (2.3) and $v = r^2 - s^2$ is defined by Definition 2.1.

Thus $u(x) = (R^H_{2(k-1)}(u))^{(m)}$ is a solution of (2.12) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension. $\Box$

Lemma 2.4. *Let L be the operator defined by*

$$L = (\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2)^k, \tag{2.14}$$

where $\triangle$ and $\Box$ are defined by (1.2) and (1.3) respectively. Then we obtain $H(x)$, where

$$H(x) = \left(R^H_{4k}(u) * (-1)^{2k}R^e_{4k}(v)\right) * \left(S^{*k}(x)\right)^{*-1} \tag{2.15}$$

and

$$S(x) = \frac{3}{4}R^e_4(v) + \frac{1}{4}R^H_4(u) \tag{2.16}$$

is an elementary solution of the operator defined by (2.14) iterated k-times, $S^{*k}(x)$ denotes the convolution of S it self k-times, $\left(S^{*k}(x)\right)^{*-1}$ denotes the inverse of $S^{*k}(x)$ in the convolution algebra. Moreover $H(x)$ is a tempered distribution.

Proof. From (3.1), we have

$$\left(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2\right)^k H(x) = \delta(x),$$

or we can write

$$\left(\frac{3}{4}\Box^2+\frac{1}{4}\triangle^2\right)\left(\frac{3}{4}\Box^2+\frac{1}{4}\triangle^2\right)^{k-1}H(x)=\delta(x).$$

Convolving both sides of the above equation by $R_4^H(u)*(-1)^2R_4^e(v)$,

$$\left(\frac{3}{4}\Box^2+\frac{1}{4}\triangle^2\right)(R_4^H(u)*(-1)^2R_4^e(v))\left(\frac{3}{4}\Box^2+\frac{1}{4}\triangle^2\right)^{k-1}H(x)$$
$$=\delta(x)*R_4^H(u)*(-1)^2R_4^e(v),$$

that is

$$\left(\frac{3}{4}\Box^2(R_4^H(u))*(-1)^2R_4^e(v)+\frac{1}{4}\triangle^2((-1)^2R_4^e(v))*R_4^H(u)\right)$$
$$\cdot\left(\frac{3}{4}\Box^2+\frac{1}{4}\triangle^2\right)^{k-1}H(x)=\delta(x)*R_4^H(u)*(-1)^2R_4^e(v)$$

and

$$\left(\frac{3}{4}\Box^2(R_4^H(u))*(-1)^2R_4^e(v))+\frac{1}{4}(\triangle^2(-1)^2R_4^e(v))*R_4^H(u)\right)$$
$$\cdot\left(\frac{3}{4}\Box^2+\frac{1}{4}\triangle^2\right)^{k-1}H(x)=\delta(x)*R_4^H(u)*(-1)^2R_4^e(v).$$

By (2.4) and (2.8)

$$\left(\frac{3}{4}\delta(x)*(-1)^2R_4^e(v)+\frac{1}{4}\delta*R_4^H(u)\right)$$
$$\cdot\left(\frac{3}{4}\Box^2+\frac{1}{4}\triangle^2\right)^{k-1}H(x)=\delta(x)*R_4^H(u)*(-1)^2R_4^e(v).$$

Thus

$$\left(\frac{3}{4}(-1)^2R_4^e(u)+\frac{1}{4}R_4^H(v)\right)\left(\frac{3}{4}\Box^2+\frac{1}{4}\triangle^2\right)^{k-1}H(x)=R_4^H(u)*(-1)^2R_4^e(v)$$

keeping on convolving both sides of the above equation by $R_4^H(u)*(-1)^2R_4^2(v)$ up to $k-1$ times, we obtain

$$S^{*k}(x)*H(x)=\left(R_4^H(u)*(-1)^2R_4^e(v)\right)^{*k}.$$

The symbol $*k$ denotes the convolution of itself $k-$times. By properties of $R_\alpha(u)$, we have

$$\left(R_4^H(u)*(-1)^2R_4^e(v)\right)^{*k}=R_{4k}^H(u)*(-1)^{2k}R_{4k}^e(v).$$
$$S^{*k}(x)*H(x)=\left(R_{4k}^H(u)*(-1)^{2k}R_{4k}^e(v)\right)$$

$$H(x) = \Big(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)\Big) * (S^{*k})^{*-1}$$

is an elementary solution of the operator defined by (2.14). □

Lemma 2.5. *Given the equation*

$$\circledast^k u(x) = 0, \tag{2.17}$$

where $\circledast^k$ *is the operator iterated* k *times defined by (1.5) and* $u(x)$ *is an unknown generalized function. Then*

$$u(x) = (R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)) * (S^{*k})^{*-1} * (R^e_{2(k-1)}(v))^m \tag{2.18}$$

is solution of (2.17) and $(R^e_{2(k-1)}(v))^{(m)}$ *is a function with* m*-derivatives defined by (2.2) and* v *is defined by Definition 2.1 and* $S(x)$ *defined by (2.16).*

Proof. Now

$$\circledast^k u(x) = \triangle^k(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2)^k u(x) = 0. \tag{2.19}$$

By Lemma 2.2,

$$(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2)^k u(x) = (R^e_{2(k-1)}(v))^{(m)}. \tag{2.20}$$

Convolving both sides by $\big(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)\big) * (S^{*k})^{*-1}$, we have

$$\Big(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)\Big) * (S^{*k})^{*-1} * (\frac{3}{4}\Box^2 + \frac{1}{2}\triangle^2)^k u(x)$$
$$= \Big(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)\Big) * (S^{*k})^{*-1} * (R^e_{2(k-1)})(v))^{(m)}.$$

By Lemma 2.4,

$$\delta(x) * u(x) = \Big(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)\Big) * (S^{*k})^{*-1} * (R^e_{2(k-1)}(v))^{(m)}.$$

It follows that

$$u(x) = \Big(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)\Big) * (S^{*k})^{*-1} * (R^e_{2(k-1)}(v))^{(m)}. \tag{2.21}$$

By (1.7) and the equation (2.21) it can be written

$$u(x) = (-1)^{2k} K_{4k,4k} * (S^{*k})^{*-1} * (R^e_{2(k-1)}(v))^{(m)} \tag{2.22}$$

as a solution of (2.17). □

3. Main Results

Theorem 3.1. *Given the equation*

$$\circledast^k G(x) = \delta(x), \tag{3.1}$$

then

$$G(x) = \left(R^H_{4k}(u) * (-1)^{2k} R^e_{6k}(v)\right) * \left(S^{*k}(x)\right)^{*-1} \tag{3.2}$$

is a Green function for the operator $\circledast^k$ *iterated* k*-times, where* $\circledast$ *is defined by (1.5), and*

$$S(x) = \frac{3}{4}(-1)^2 R^e_4(v) + \frac{1}{4} R^H_4(u) \tag{3.3}$$

$S^{*k}(x)$ *denotes the convolution of* S *itself* k*-times,* $\left(S^{*k}(x)\right)^{*-1}$ *denotes the inverse of* $S^{*k}(x)$ *in the convolution algebra. Moreover* $G(x)$ *is a tempered distribution.*

Proof. From (3.1), we have

$$\circledast^k G(x) = \left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^k G(x) = \delta(x),$$

or we can write

$$\left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)\left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^{k-1} G(x) = \delta(x).$$

Convolving both sides of the above equation by $(-1)^3 R^e_2(v) * R^e_2(v) * R^e_2(v)$,

$$\left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)\left((-1)^3 R^e_2(v) * R^e_2(v) * R^e_2(v)\right)\left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^{k-1} G(x)$$
$$= \delta(x) * (-1)^3 R^e_2(v) * R^e_2(v) * R^e_2(v), \tag{3.4}$$

$$\left(\frac{3}{4}\triangle\left((-1)R^e_2(v)\right) * \Box^2\left((-1)^2 R^e_2(v) * R^e_2(v)\right) + \frac{1}{4} * \triangle^3 (-1)^3 R^e_6(v)\right)$$
$$\cdot\left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^{k-1} G(x) = \delta(x) * (-1)^3 R^e_6(v). \tag{3.5}$$

That is

$$\left(\frac{3}{4}\delta * \left(\Box^2 (-1)^2 R^e_4(v)\right) + \frac{1}{4}\delta * \delta * \delta\right)\left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^{k-1} G(x)$$
$$= (-1)^3 R^e_6(v).$$

It follows that

$$\left(\frac{3}{4}\delta * \left(\Box^2 (-1)^2 R^e_4(v)\right) + \frac{1}{4}\delta\right)\left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^{k-1} G(x) = (-1)^3 R^e_6(v).$$

Convolving both sides of the above equation by $R^H_2(u) * R^H_2(u)$, we obtain

$$\left(\frac{3}{4}\delta * \left(\Box^2 (-1)^2 R^e_4(v)\right)\left(R^H_2(u) * R^H_2(u)\right) + \frac{1}{4}\delta * \left((-1)R^e_2(v) * (-1)R^e_2(v)\right)\right)$$

$$\cdot\left(\frac{3}{4}\Diamond\Box+\frac{1}{4}\triangle^3\right)^{k-1} G(x)=(-1)^3R_6^e(v)*\left(R_2^H(u)*R_2^H(u)\right). \quad (3.6)$$

$$\left(\frac{3}{4}(-1)^2R_4^e(v)\left(\Box R_2^H(u)*\Box R_2^H(u)\right)+\frac{1}{4}R_4^H(u)\right)$$
$$\cdot\left(\frac{3}{4}\Diamond\Box+\frac{1}{4}\triangle^3\right)^{k-1} G(x)=(-1)^3R_6^e(v)*R_4^H(u).$$

Thus

$$\left(\frac{3}{4}(-1)^2R_4^e(v)+\frac{1}{4}R_4^H(u)\right)\left(\frac{3}{4}\Diamond\Box+\frac{1}{4}\triangle^3\right)^{k-1} G(x)$$
$$=(-1)^3R_6^e(v)*R_4^H(u), \quad (3.7)$$

keeping on convolving both sides of the above equation by $(-1)^3R_6^e(v)*R_4^H(u)$ up to $k-1$ times, we obtain

$$S^{*k}(x)*G(x)=\left((-1)^3R_6^e(v)*R_4^H(u)\right)^{*k}.$$

The symbol $*k$ denotes the convolution of itself $k-$times. By properties of $R_\alpha(u)$, we have

$$\left((-1)^3R_6^e(v)*R_4^H(u)\right)^{*k}=(-1)^{3k}R_{6k}^e(v)*R_{4k}^H(u).$$

Thus,

$$S^{*k}(x)*G(x)=(-1)^{3k}R_{6k}^e(v)*R_{4k}^H(u). \quad (3.8)$$

Now, consider the function $S^{*k}(x)$, since $(-1)^3R_6^e(v)*R_{4k}^H(u)$ is a tempered distribution. Thus $S(x)$ defined by (3.3) is a tempered distribution, we obtain $S^{*k}(x)$ is a tempered distribution.

Now, $(-1)^{3k}R_{6k}^e(v)*R_{4k}^H(u)\in\mathcal{S}'$, the space of tempered distribution. Choose $\mathcal{S}'\subset\mathcal{D}'_{\mathcal{R}}$ where $\mathcal{D}'_{\mathcal{R}}$ is the right-side distribution which is a subspace of $\mathcal{D}'$ of distribution. Thus $(-1)^{3k}R_{6k}^e(v)*R_{4k}^H(u)\in\mathcal{D}'_{\mathcal{R}}$. It follows that $(-1)^{3k}R_{6k}^e(v)*R_{4k}^H(u)$ is an element of convolution algebra, since $\mathcal{D}'_{\mathcal{R}}$ is a convolution algebra. Hence Zemanian (see [6]), the equation (3.8) has a unique solution

$$G(x)=\left(R_{4k}^H(u)*(-1)^{3k}R_{6k}^e(v)\right)*\left(S^{*k}(x)\right)^{*-1}, \quad (3.9)$$

where $\left(S^{*k}(x)\right)^{*-1}$ is an inverse of S^{*k} in the convolution algebra, $G(x)$ is called the Green function of the operator $\circledast^k$. □

Theorem 3.2. *Given the equation*

$$(\circledast)^k u(x)=f(x), \quad (3.10)$$

where $\circledast^k$ is the operator iterated k times defined by (1.5), $f(x)$ is a general *function, $u(x)$ is an unknown function and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the n-dimensional Euclidean space and n is even. Then (3.10) has the general solution*

$$u(x) = ((-1)^{2k} K_{4k,4k} * (R^H_{2(k-1)}(u))^m + (-1)^{3k} K_{6k,4k} * f(x)) * (S^{*k})^{*-1}, \quad (3.11)$$

where $(R^e_{2(k-1)}(v))^m$ is a function m-derivatives defined by (2.5) and $K_{\alpha,\beta}(x) = R^e_\alpha * R^H_\beta$. If we put $k = 1$, we obtain as solution of triharmonic wave equation.

Proof. Convolving (3.10) both sides by (3.9), we obtain

$$(-1)^{3k} K_{6k,4k}(x) * (S^{*k})^{*-1} * \circledast^k u(x) = (-1)^{3k} K_{6k,4k}(x) * (S^{*k})^{*-1} * f(x),$$

or

$$\circledast^k((-1)^{2k} K_{6k,4k}(x) * (S^{*k})^{*-1}) * u(x) = (-1)^{3k} K_{6k,4k}(x) * (S^{*k})^{*-1} * f(x).$$

By Theorem 3.1

$$\delta(x) * u(x) = u(x) = (-1)^{3k} K_{6k,4k}(x) * (S^{*k})^{*-1} * f(x). \quad (3.12)$$

Since, for a homogeneous equation $\circledast^k u(x) = 0$ we have a solution (see Lemma 2.5)

$$u(x) = (-1)^{2k} K_{4k,4k}(x) * (S^{*k})^{*-1} * (R^e_{2(k-1)}(v))^m). \quad (3.13)$$

Thus the general solution of (3.10) is

$$u(x) = (-1)^{2k} K_{4k,4k}(x) * (S^{*k})^{*-1} * (R^e_{2(k-1)}(u))^m + (-1)^{3k} K_{6k,4k}(x) * (S^{*k})^{*-1} * f(x),$$

or

$$u(x) = ((R^e_{2(k-1)}(v))^m + (-1)^k R^e_{2k}(v) * f(x)) * (-1)^{2k} K_{4k,4k}(x) * (S^{*k})^{*-1}. \quad (3.14)$$

In particular, if $q = 0$ the equation (3.1) becomes the Laplce equation $\triangle^{3k} u(x) = f(x)$ where $x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$ and p is even. Using the formulae $\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z + \frac{1}{2})$ and $\Gamma(\frac{1}{2} + z)\Gamma(\frac{1}{2} - z) = \pi Sec(\pi z)$. Then for $\alpha = 2k$ the function of (2.1) becomes $(-1)^k R^e_{2k}(x)$ where R^e_{2k} is defined by (2.5) and $x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$. Thus by (1.7)

$$(-1)^k K_{2k,2k}(x) = (-1)^k R^e_{2k}(x) * (-1)^k R^e_{2k}(x) = R^e_{4k}(x), \quad \text{see [2], pp. 156-159,}$$

where $x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$ and p is even.

Now, from (2.17) for $q = 0$ we have $\triangle^{3k} u(x) = 0$ or

$$\triangle^k(\triangle^{2k} u(x)) = 0.$$

By Lemma 2.2

$$\triangle^{2k} u(x) = (-1)^{k-1}(R^e_{2(k-1)}(v))^{(m)}, \quad (3.15)$$

$$u(x) = (-1)^{2k} R^e_{4k}(v) * (-1)^{k-1}(R^e_{2(k-1)}(v))^{(m)}$$

$$= (-1)^{3k-1}(R^e_{6k-2}(v))^{(m)} \quad \text{for} \quad x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p.$$

Thus the solution of equation (3.10) becomes

$$u(x) = (-1)^{3k-1}(R^e_{6k-2}(v))^{(m)} + (-1)^{3k}R^e_{6k}(v) * f(x) \tag{3.16}$$

for $x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$ and p is even.

It follows that (3.17) is the general solution of the Laplace equation $\triangle^{3k}u(x) = f(x)$, where $\triangle^{3k}$ is the Laplace operator iterated $3k$-times defined by (1.3) for $x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$ and p is even and if we put $k = 1$, then the equation (3.11) becomes

$$u(x) = (-1)^2(R^e_4(v))^{(m)} + (-1)^3R^e_6(v) * f(x) \tag{3.17}$$

is the general solution of the triharmonic wave equation. □

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