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ON THE CLASSIFICATION OF COMPACT SIMPLE LIE BIALGEBRAS

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Abstract: We begin by reviewing the definition of a Lie bialgebra and establish a basic equivalence type theorem for the Lie bialgebra structure. We then present a new proof of the Soibelman classification of such structures for a compact simple Lie bialgebra. The proof uses only well known properties of compact simple Lie algebras and does rely on the Drinfeld classification for complex simple Lie bialgebras.

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1. Introduction

Lie bialgebras were introduced in the early 80's due mostly to V.G. Drinfeld and M.A. Semenov-Tian-Shansky as semiclassical limits of the quantum Yang, Baxter equation. In [1] a classification was obtained for Lie bialgebras structures over a complex simple Lie algebra, and using this theory, Soibelman [13] was able to classify Lie bialgebra structures over compact simple Lie algebras. Since the classification in [1] uses ideas from the study of the quantum Yang, Baxter equation, it would be of interest to find a proof of the classification theorem using only the classical theory of Lie algebras. In the following we present such a proof of the classification of the Lie bialgebra structures over a compact simple Lie algebra using only the traditional theory as developed in [6] and [2]. The outline of the paper is as follows: In Section 2 we recall the definition of a

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Lie bialgebra and the basic theory. In Section 3 we present the new proof of the classification. In Section 4 we close with a brief presentation of the standard example of a Manin triple.

2. Definitions and Basic Concepts

We begin this section by recalling the definition of a Lie bialgebra as given in [4] and [10] and setting up the notation we will need later on. Let $\mathfrak{g}$ be a finite dimensional vectorspace with $\mathfrak{g}^\star$ its dual and suppose both spaces are equipped with Lie algebra structures. We will require a certain compatibility between these structures that can be expressed in a number of equivalent ways. Before stating this equivalence, we introduce the following "dot" convention for our notation. This is the convention that all natural Lie algebra actions of $\mathfrak{g}$ and $\mathfrak{g}^\star$ are denoted by a "$\cdot$", with the specific action being determined by the spaces of the vectors involved in the expression. For example, for $x, y \in \mathfrak{g}$, and $\alpha \in \mathfrak{g}^\star$, we see that $x \cdot y$ is the adjoint action of x on y, $x \cdot \alpha$ is the coadjoint action of x on α, and $\alpha \cdot x$ is the coadjoint action of α on x. The traditional bracket notation will be reserved for a Lie algebra called the double which will be constructed below. Now let

$$\wedge^{nk} = \wedge^n \mathfrak{g}^\star \bigotimes \wedge^k \mathfrak{g}$$

$p \in \wedge^{21}$ the Lie algebra tensor of $\mathfrak{g}$ and $\pi \in \wedge^{12}$ the Lie algebra tensor of $\mathfrak{g}^\star$. If we identify

$$\wedge^{nk} = Hom(\wedge^n \mathfrak{g}, \wedge^k \mathfrak{g})$$

as the space of n cochains on $\mathfrak{g}$ with values in the $\mathfrak{g}$-space $\wedge^k \mathfrak{g}$ [5], we can define a coboundary operator

$$d : \wedge^{nk} \to \wedge^{n+1,k}.$$

Similarly if we identify

$$\wedge^{nk} = Hom(\wedge^k \mathfrak{g}^\star, \wedge^n \mathfrak{g}^\star)$$

as the space of k cochains on $\mathfrak{g}^\star$ with values in the $\mathfrak{g}^\star$-space $\wedge^n \mathfrak{g}^\star$, we can define a coboundary operator

$$\delta : \wedge^{nk} \to \wedge^{n,k+1}.$$

Finally, let $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^\star$. and define a bracket on $\mathfrak{d}$ by

$$[x + \alpha, y + \beta] = x \cdot y + \alpha \cdot \beta + \alpha \cdot y + x \cdot \beta - y \cdot \alpha - \beta \cdot x$$

$\forall x, y \in \mathfrak{g}$ and $\forall \alpha, \beta \in \mathfrak{g}^\star$. We have (see [8], [7])

Theorem 1. *The following are equivalent:*

— $x \cdot (\alpha \cdot \beta) = (x \cdot \alpha) \cdot \beta - (\alpha \cdot x) \cdot \beta - (x \cdot \beta) \cdot \alpha + (\beta \cdot x) \cdot \alpha$
$\forall x \in \mathfrak{g}$ *and* $\forall \alpha, \beta \in \mathfrak{g}^\star$.

— *The bracket on* $\mathfrak{d}$ *defined above satisfies the Jacobi identity.*
($\mathfrak{d}$ *is a Lie algebra).*

— $d\pi = 0$.

— $\delta p = 0$.

— $d\delta + \delta d = 0$ *i.e.* d *and* δ *form a bicomplex.*

Proof. We begin the proof by recalling the definition of the coboundary operator as given in [5]. Let V be a $\mathfrak{g}$-module then for

$$f \in Hom(\wedge^n \mathfrak{g}, V)$$

we have

$$(df)(x_0, ..., x_n) = \sum_{i=0}^{n} (-1)^i x_i \cdot f(x_0, ..., \hat{x}_i, ..., x_n) + \sum_{i<j} (-1)^{i+j} f(x_i \cdot x_j, x_0, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_n).$$

The definition immediately gives

$$d\mathbf{1} = p \qquad \delta\mathbf{1} = \pi$$

for $\mathbf{1} \in \wedge^{1,1}$ the identity function. We also have

$$d\pi = \delta p$$

A quick computation will now prove the equivalence of the first and second items. We compute the third item

$$< \alpha \wedge \beta, d\pi(x,y) >=< \alpha \wedge \beta, x \cdot \pi(y) - x \cdot \pi(x) - \pi(x \cdot y) >=$$
$$- < (x \cdot \alpha) \wedge \beta + \alpha \wedge (x \cdot \beta), \pi(y) > + < (y \cdot \alpha) \wedge \beta + \alpha \wedge (y \cdot \beta), \pi(x) > - < \alpha \cdot \beta, x \cdot y >$$
$$= - < (x \cdot \alpha) \cdot \beta + \alpha \cdot (x \cdot \beta), y > + < (y \cdot \alpha) \cdot \beta + \alpha \cdot (y \cdot \beta), x > - < \alpha \cdot \beta, x \cdot y >$$
$$=< \beta, (x \cdot \alpha) \cdot y - x \cdot (\alpha \cdot y) - (y \cdot \alpha) \cdot x + y \cdot (\alpha \cdot x) + \alpha \cdot (x \cdot y) > .$$

Thus the first and third items are equivalent and similarly, the first and fourth items are equivalent. Clearly, the fifth item implies the first four items. To prove that the first four items imply the fifth, we show that $d + \delta$ is in fact D the coboundary operator for cohomology of the Lie algebra $\mathfrak{d}$ with values in the trivial representation. Starting with

$$\wedge^n \mathfrak{d}^\star = \sum_{i+j=n} \wedge^{ij}$$

it is clear that

$$D : \wedge^{ij} \to \wedge^{i+1,j} \oplus \wedge^{i,j+1}.$$

Let D_1 be D restricted to $\wedge^{ij}$ followed by projection onto $\wedge^{i+1,j}$ and

$$f = (\beta_1 \wedge ... \wedge \beta_i) \otimes (y_1 \wedge ... \wedge y_j) \in \wedge^{ij}.$$

We will show that $D_1 f = df$.

$$D_1 f(x_0, ..., x_i, \alpha_1, ..., \alpha_j)$$

$$= -\sum_{0 \le k < l \le i} (-1)^{k+l} f(x_k \cdot x_l, x_0, ..., \hat{x_k} ..., \hat{x_l}, ..., x_i, \alpha_1, ..., \alpha_j)$$

$$- \sum_{0 \le k \le i, 1 \le l \le j} (-1)^{k+(l+i)+(i-1)} f(x_0, ..., \hat{x_k}, ..., x_i, x_k \cdot \alpha_l, \alpha_1, ..., \hat{\alpha_l}, ..., \alpha_j)$$

$$= -\sum_{0 \le k \le i, 1 \le l \le j} (-1)^{k+l} < \beta_1 \wedge ... \wedge \beta_i, x_k \cdot x_l \wedge x_0 \wedge ... \hat{x_k} ... \hat{x_l} ... \wedge x_l >$$

$$\times < \alpha_1 \wedge ... \wedge \alpha_j, y_1 \wedge ... y_j >$$

$$+ \sum_{0 \le k \le i} (-1)^{k-1} < \beta_1 \wedge ... \wedge \beta_i, x_0 \wedge ... \hat{x_k} ... \wedge x_i > \times < x_k \cdot (\alpha_1 \wedge ... \wedge \alpha_j), y_1 \wedge ... \wedge y_j >$$

$$= < df(x_0, ..., x_k), \alpha_1 \wedge ... \wedge \alpha_j > .$$

Similarly, D followed by the other projection is δ. Thus $D = d + \delta$ and since $\mathfrak{d}$, $\mathfrak{g}$, and $\mathfrak{g}^\star$ are Lie algebras $D^2 = d^2 = \delta^2 = 0$. This shows item five and proves the theorem. We call the first item the Drinfeld identity. □

We call $(\mathfrak{g}, \mathfrak{g}^\star)$ a Lie bialgebra pair or simply a Lie bialgebra and $\mathfrak{d}$ is called the double if one of the above conditions is satisfied. It is well known [10] that the double becomes a Lie bialgebra when we equip its dual $\mathfrak{d}^* = \mathfrak{g}^\star \oplus \mathfrak{g}$ with the direct sum product structure and we will always assume this to be the case. We also say that a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^\star)$ has property X if $\mathfrak{g}$ has property X. Thus $(\mathfrak{g}, \mathfrak{g}^\star)$ compact means $\mathfrak{g}$ is the Lie algebra of a compact Lie group. The collection of finite dimensional Lie bialgebras forms a category with ordered pairs $(\mathfrak{g}, \mathfrak{g}^\star)$ as objects and with an arrow $(\mathfrak{g}_1, \mathfrak{g}_1^\star) \to (\mathfrak{g}_2, \mathfrak{g}_2^\star)$ given by a Lie algebra map from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ such that the dual map is also a Lie algebra map. This category admits an arrow reversing flip functor, $(\mathfrak{g}, \mathfrak{g}^\star) \mapsto (\mathfrak{g}^\star, \mathfrak{g})$ which sends sub-Lie bialgebras to quotient Lie bialgebras and visa-versa. It is clear that the automorphism group of $(\mathfrak{g}, \mathfrak{g}^\star)$ is

$$Aut(\mathfrak{g}, \mathfrak{g}^\star) = Aut(\mathfrak{g}) \cap Aut(\mathfrak{g}^\star).$$

Use of the Drinfeld identity shows that the space $(\mathfrak{g}^\star \cdot \mathfrak{g}^\star)^\perp$ is a subalgebra of $\mathfrak{g}$ and that any $x \in (\mathfrak{g}^\star \cdot \mathfrak{g}^\star)^\perp$ generates a 1-parameter group of automorphisms of $\mathfrak{g}^\star$. A similar condition holds for $\alpha \in (\mathfrak{g} \cdot \mathfrak{g})^\perp$. Thus, we can define the inner automorphisms as the group generated by all such x and α.

Suppose we are given a fixed Lie algebra tensor p for $\mathfrak{g}$. It is a natural question to ask for all the Lie algebra tensors π for $\mathfrak{g}^\star$ so that $d\pi = 0$. We will now investigate this question for $\mathfrak{g}$ compact simple and recover the classification given in [13] and also [3]. To begin, note that if $\pi = dr$ for some $r \in \wedge^{02}$ then π will automatically satisfy $d\pi = 0$. We call a Lie bialgebra a coboundary Lie bialgebra if π is a d-coboundary. If $\mathfrak{g}$ is semisimple and $d\pi = 0$ then $\pi = dr$ for a unique $r \in \wedge^{02}$. Thus, $d\delta r = -\delta dr = -\delta\pi = 0$. Since δ depends linearly on π we can think of $d\delta r = 0$ as a quadratic equation in r and to find all the π we just need all such r. The equation $\delta r = 0$ is known in [1] as the classical Yang Baxter equation, $d\delta r = 0$ the modified classical Yang Baxter equation, and r is called an r-matrix if it satisfies the modified classical Yang Baxter equation. A coboundary Lie bialgebra which satisfies the classical Yang Baxter equation is called a triangular Lie bialgebra. r can be considered as a map from $\mathfrak{g}^\star$ to $\mathfrak{g}$ via

$$< \alpha \wedge \beta, r > = < \beta, r(\alpha) > = - < \alpha, r(\beta) >$$

and note that the action of δ is consistent up to scalar with this identification. The $\mathfrak{g}^\star$ actions can now be given by

$$< \alpha \cdot \beta, x > = < \alpha \wedge \beta, x \cdot r > = < \beta, x \cdot r(\alpha) - r(x \cdot \alpha) > = < r(\alpha) \cdot \beta - r(\beta) \cdot \alpha, x >,$$

thus

$$\alpha \cdot \beta = r(\alpha) \cdot \beta - r(\beta) \cdot \alpha$$

and

$$\alpha \cdot x = r(x \cdot \alpha) + r(\alpha) \cdot x.$$

We see that $ker(r)$ is an Abelian subalgebra of $\mathfrak{g}^\star$ and that

$$\delta r(\alpha, \beta) = 2r(\beta \cdot \alpha) - 2r(\beta) \cdot r(\alpha).$$

Indeed,

$$\begin{aligned} \delta r(\alpha, \beta) &= \alpha \cdot r(\beta) - \beta \cdot r(\alpha) - r(\alpha \cdot \beta) \\ &= r(r(\beta) \cdot \alpha) + r(\alpha) \cdot r(\beta) - r(r(\alpha) \cdot \beta) - r(\beta) \cdot r(\alpha) - r(\alpha \cdot \beta) \\ &= 2r(\beta \cdot \alpha) - 2r(\beta) \cdot r(\alpha). \end{aligned}$$

So if $\mathfrak{g}$ is triangular then r is a Lie algebra homomorphism and $ker(r)$ will be an ideal. It is a direct consequence of the action of the r-matrix that

Lemma 2. *If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a coboundary Lie bialgebra with $\mathfrak{g}_1$ and $\mathfrak{g}_2$ ideals then $\mathfrak{g}_1^\star$ and $\mathfrak{g}_2^\star$ are subalgebras of $\mathfrak{g}^\star = \mathfrak{g}_1^\star \oplus \mathfrak{g}_2^\star$. Hence, $(\mathfrak{g}_1, \mathfrak{g}_1^\star)$ and $(\mathfrak{g}_2, \mathfrak{g}_2^\star)$ are coboundary Lie bialgebras.*

The condition $d\delta r = 0$ can be reinterpreted as

Lemma 3. *For any coboundary Lie bialgebra* $(\mathfrak{g}, \mathfrak{g}^\star)$, *the map*

$$p \circ (r \otimes r) - r \circ \pi : \mathfrak{g}^\star \wedge \mathfrak{g}^\star \to \mathfrak{g}$$

is a $\mathfrak{g}$*-map*

Proof. We can prove this directly using the Drinfeld identity as follows. Let $x \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^\star$. Then

$$\begin{aligned}(x \cdot (r \circ \pi))(\alpha \wedge \beta) &= x \cdot r(\alpha \cdot \beta) - r((x \cdot \alpha) \cdot \beta + \alpha \cdot (x \cdot \beta)) \\ &= (x \cdot r)(\alpha \cdot \beta) + r(x \cdot (\alpha \cdot \beta)) - r(x \cdot (\alpha \cdot \beta) + (\alpha \cdot x) \cdot \beta - (\beta \cdot x) \cdot \alpha) \\ &= -(\alpha \cdot \beta) \cdot x - \beta \cdot (\alpha \cdot x) + r(\beta) \cdot (\alpha \cdot x) + \alpha \cdot (\beta \cdot x) - r(\alpha) \cdot (\beta \cdot x) \\ &= r(\beta) \cdot r(x \cdot \alpha) - r(\alpha) \cdot r(x \cdot \beta) + r(\beta) \cdot (r(\alpha) \cdot x) - r(\alpha) \cdot (r(\beta) \cdot x) \\ &= (x \cdot (p \circ (r \otimes r)))(\alpha \wedge \beta).\end{aligned}$$ □

Corollary 4. *The map*

$$p \circ (r \otimes r) \circ p + p \circ \pi \circ r : \mathfrak{g}^\star \to \mathfrak{g}$$

is a $\mathfrak{g}$ *map.*

3. The Classification

We now assume that $\mathfrak{g}$ is compact semisimple and pick a basis $\{e_i\}$ orthonormal with respect to the negative of the Killing form with dual basis $\{\epsilon^i\}$. Since $r \in \wedge^{02}$, we can hit it with p and define the m-vector [9] to be

$$m = p(r).$$

The m-vector will be the key to unlocking the structure of $\mathfrak{g}^\star$. We begin with

Lemma 5. $m = -\epsilon^i \cdot e_i$ *is* $\mathfrak{g}^\star$ *invariant.*

Proof. In our given basis π is given as (employing the Einstein summation convention)

$$\pi = \pi_k^{ij} \epsilon^k \otimes e_i \wedge e_j,$$

$$\begin{aligned}\pi_k^{ij} &=< \epsilon^i \cdot \epsilon^j, e_k >=< \epsilon^i, r(\epsilon^j) \cdot e_k > - < \epsilon^j, r(\epsilon^i) \cdot e_k > \\ &=< \epsilon^i, r^{jl} e_l \cdot e_k > - < \epsilon^j, r^{il} e_l \cdot e_k >= r^{jl} p_{lk}^i - r^{il} p_{lk}^j,\end{aligned}$$

so then m will be

$$m = r^{ij} e_i \cdot e_j = r^{ij} p_{ij}^k e_k = (\pi_i^{ik} - r^{kj} p_{ji}^i) e_k = \pi_i^{ik} e_k = -\epsilon^i \cdot e_i.$$

Since $p^i_{jk} = p^j_{ki}$ in an orthonormal basis. The invariance of this vector is equivalent to the well known fact that the Killing form is symmetric. Indeed,

$$< \alpha \cdot \beta, -\epsilon^i \cdot e_i >= - < (\alpha \cdot \beta) \cdot \epsilon^i, e_i >= \kappa^*(\beta, \alpha) - \kappa^*(\alpha, \beta) = 0,$$

where κ^* is the Killing form on $\mathfrak{g}^\star$. □

The above proof really only uses the existence of a nondegenerate invariant form on $\mathfrak{g}$ so it holds for $\mathfrak{g}$ reductive. We also see that the m-vector is proportional to the modular vector for $\mathfrak{g}^\star$ which is the obstruction for $\mathfrak{g}^\star$ to be the Lie algebra of a unimodular Lie group. Invariance of m shows $m \cdot r = 0$ or equivalently, r is an m-map, i.e.

$$r(m \cdot \alpha) = m \cdot r(\alpha) \quad \forall \alpha \in \mathfrak{g}^\star,$$

so we have

$$m \in Z(r) \subset Z(m) \subset \mathfrak{g},$$

where $Z(r)$ denotes the $\mathfrak{g}$-stabilizer of r and similarly for m. We denote the Killing map from $\mathfrak{g}$ to $\mathfrak{g}^\star$ by $x \mapsto \overline{x}$ and note $\overline{x \cdot y} = x \cdot \overline{y}$. The next lemma is given in [3].

Lemma 6. *If $\mathfrak{g}$ is compact simple then*

$$r(\overline{x}) \cdot r(\overline{y}) - r(\overline{x} \cdot \overline{y}) = cx \cdot y$$

for some constant c. $c = 0$ if and only if $m = 0$, in which case $Im(r)$ is Abelian.

Proof. For $\mathfrak{g}$ semisimple, the dimension of the space of $\mathfrak{g}$-maps from $\mathfrak{g} \wedge \mathfrak{g}$ to $\mathfrak{g}$ is easily seen to be equal to the dimension of $H^3(\mathfrak{g})$ which is 1 when the complexification of $\mathfrak{g}$ is simple. This shows that the identity holds and we call this the main identity. Suppose now that $c = 0$. Then, r is a homomorphism, $\mathfrak{a} = Im(r)$ is a subalgebra and $r \in \mathfrak{a} \wedge \mathfrak{a}$ is a nondegenerate r-matrix on $\mathfrak{a}^\star$. Following [1], we let $b : \mathfrak{a} \to \mathfrak{a}^\star$ be the inverse of r. The identity

$$b(x \cdot y) = b(x) \cdot b(y) = x \cdot b(y) - y \cdot b(x)$$

shows that $db = 0$ on $\mathfrak{a}$. Since $\mathfrak{a}$ is a subalgebra of a compact Lie algebra we can decompose $\mathfrak{a}$ as

$$\mathfrak{a} = \mathfrak{s} \oplus \mathfrak{l}$$

with $\mathfrak{s}$ semisimple and $\mathfrak{l}$ the Abelian center of $\mathfrak{a}$. Now b restricted to $\mathfrak{s}$ is still closed so there exists an $\alpha \in \mathfrak{a}^\star$ so that $b(x) = x \cdot \alpha$ for all $x \in \mathfrak{s}$. This shows that $b|_{\mathfrak{s}}$ is not injective. Thus $\mathfrak{s} = 0$ and $\mathfrak{a}$ is Abelian. Finally, $m = 0$ since $m \in \mathfrak{a} \cdot \mathfrak{a} = 0$. This proves one direction. To prove the other direction, we begin by noting

$$< \pi p(\alpha), x >=< p(\alpha), \pi(x) >=< \alpha, p(x \cdot r) >=< \alpha, x \cdot m >=< m \cdot \alpha, x >,$$

so that

$$\pi p(\alpha) = m \cdot \alpha.$$

Using Corollary 4 and the main identity, we see that

$$(p \circ (r \otimes r) \circ p + p \circ \pi \circ r)(\overline{x}) = cx.$$

Let $t(x) = p \circ (r \otimes r) \circ p(\overline{x})$ and $s(x) = (p \circ \pi \circ r)(\overline{x}) = -r(\overline{m \cdot x})$. We will show that $tr(s) = 2tr(t)$. This will prove that $m = 0$ implies $c = 0$. Now

$$tr(t) =< \epsilon^i, p \circ (r \otimes r) \circ p(\overline{e_i}) >= \sum_i p^i_{jk} r^{jl} r^{kn} p^i_{ln}$$

while

$$tr(s) =< \epsilon^i, p \circ \pi \circ r(\overline{e_i}) >= \sum_i r^{ij} p^i_{kl} \pi^{kl}_j$$

$$= \sum_i r^{ij} p^i_{kl} (r^{ln} p^k_{nj} - r^{kn} p^l_{nj}) = 2tr(t).$$

This proves the lemma. □

We note that $< \overline{m}, m >= tr(s)$ so $c \geq 0$. We also have

$$t(x) = -\sum_i r(\epsilon^i) \cdot r(\overline{e_i \cdot x}).$$

Our next lemma is

Lemma 7. *Assume $c = 1$ and let $W = \{x | r(\overline{r(\overline{x})}) = -x\}$. Then $\overline{W}$ is an ideal of $\mathfrak{g}^\star$ and the action of $\mathfrak{g}^\star$ on $\overline{W}$ preserves the complex structure define by r.*

Proof. Let $x \in W$ and $y \in \mathfrak{g}$. Then

$$\overline{x} \cdot \overline{y} = \overline{r(\overline{x}) \cdot y} - \overline{r(\overline{y}) \cdot x} = -\overline{x \cdot r(\overline{y})} - \overline{r(\overline{r(\overline{x})} \cdot \overline{y})} - \overline{r(\overline{y}) \cdot x}$$
$$= \overline{r(\overline{x \cdot y} + r(\overline{y}) \cdot r(\overline{x}))} = -\overline{r(\overline{r(\overline{x} \cdot \overline{y})})}.$$

So that $\overline{x} \cdot \overline{y} \in \overline{W}$. Moreover,

$$\overline{y} \cdot \overline{r(\overline{x})} = \overline{r(\overline{y}) \cdot r(\overline{x})} + \overline{x \cdot y} = \overline{r(\overline{y} \cdot \overline{x})}.$$

This proves the lemma. □

The following theorem will allow us to classify the r-matrices on a compact simple Lie algebra, repeating the results of [1], [13] and [3].

Theorem 8. *For a compact simple Lie bialgebra, if $m \neq 0$ then $Z(m)$ is Abelian and $Z(r) = Z(m)$.*

Proof. Assume $m \neq 0$ so that $c \neq 0$. Note that if the theorem holds for a scaled r then it holds for r so we can assume without loss of generality that $c = 1$. Then if $x \in \mathfrak{g}$ with $r(\overline{x}) = 0$

$$x \cdot (m \cdot x) = r(\overline{x}) \cdot r(\overline{m \cdot x}) - r(\overline{x} \cdot (\overline{m \cdot x})) = 0$$

which implies $< \overline{m \cdot x}, m \cdot x >=< \overline{m}, x \cdot (m \cdot x) >= 0$. Thus, $ker(r) \subset \overline{Z(m)}$. Now let $V = m \cdot \mathfrak{g}$, so that $\mathfrak{g} = Z(m) \oplus V$. Since $m \in Z(r)$ we have

$$r \in Z(m) \wedge Z(m) \oplus V \wedge V$$

with corresponding decomposition $r = r_0 + r_1$. We see that $r(\overline{Z(m)}) \subset Z(m)$. Hence, $\overline{Z(m)}$ is a subalgebra of $\mathfrak{g}^\star$ and $(Z(m), \overline{Z(m)})$ is a coboundary Lie bialgebra. We write

$$V = \bigoplus V_\lambda$$

as the eigenspace decomposition with respect to ad_m^2 and claim that the map $x \mapsto r(\overline{x})$ acts as a scalar times ad_m on V_λ. This will show $Z(m) \subset Z(r_1)$. Clearly $r(\overline{V_\lambda}) \subset V_\lambda$ for each λ since $m \in Z(r)$. Let t and s be defined as in Lemma 6. Then s is a symmetric invertible map on V and hence on all V_λ. We also define $f(x) = r(\overline{r(\overline{x})})$ a symmetric map commuting with s and ad_m. Let $x \in V_\lambda$ with $s(x) = \mu x$ and $f(x) = \sigma x$, and let $y = r(\overline{x})$. The main identity gives

$$x \cdot y = \sigma y \cdot x,$$

since

$$\overline{x} \cdot \overline{y} = y \cdot \overline{y} - \sigma x \cdot \overline{x} = 0.$$

We also have $x \cdot y \neq 0$ since the kernel of s is $Z(m)$, so that

$$< \overline{m}, x \cdot y >= -\mu < \overline{x}, x >\neq 0.$$

This shows $\sigma = -1$ and $f(x) = -x$ on V. Finally, we compute

$$\lambda y = m \cdot (m \cdot y) = \mu m \cdot x$$

to see that

$$r(\overline{x}) = \frac{\mu}{\lambda} m \cdot x.$$

To compute μ we note that $ad_m^2 \circ f = s^2$ which gives $\mu^2 = -\lambda$. To see $\mu > 0$ we show that the eigenvalues of t restricted to V are ≤ 1. Since $s + t = 1$, this will show the eigenvalues of s cannot be negative. Let W be as in Lemma 7. Clearly $V \subset W$. Let K be the perpendicular space to W consisting of the eigenspaces of f corresponding to the eigenvalues different from -1. Since $\overline{W}$ is an ideal, $\overline{W} \cdot K = 0$ showing $K \subset Z(r_1)$. Now

$$p(\overline{V}) \subset \overline{V} \wedge \overline{W} \oplus \overline{V} \wedge \overline{K} \subset \overline{W} \wedge \overline{W} \oplus \overline{V} \wedge \overline{K}$$

so we can split $t = p \circ (r \otimes r) \circ p$ restricted to V into two pieces $t = t_1 + t_2$.

To show the eigenvalues of t restricted to V are ≤ 1 we first show that $t_2 = 0$. Let $\{e_1, ..., e_N\}$ be a basis of K diagonalizing f with the property that if $r(\overline{e_i}) \neq 0$ then $r(\overline{e_i}) = q_i e_j$ for some j with $f(e_i) = -q_i^2 e_i$. Since $q_i^2 \neq 1$ we have $e_i \cdot r(\overline{e_i}) = 0$. Finally, $e_i, r(\overline{e_i}) \in Z(r_1)$ gives (no i sum)

$$e_i \cdot r(r(\overline{e_i}) \cdot \overline{x}) = r(\overline{e_i}) \cdot r(\overline{e_i \cdot x})$$

for all $x \in V$. Thus for each i we have it's corresponding j with

$$q_i e_j \cdot r(\overline{e_i \cdot x}) = q_j e_i \cdot r(\overline{e_j \cdot x}).$$

Since $q_j = -q_i$ these terms will in fact cancel in the sum over all i so that

$$t_2(x) = -\sum_{i=1}^{N} r(\overline{e_i}) \cdot r(\overline{e_i \cdot x}) = 0.$$

Thus $t = t_1$ on V. Now for any $x, y \in \mathfrak{g}$ we have

$$< \overline{x} \wedge \overline{y}, x \wedge y > \geq < \overline{x \cdot y}, x \cdot y >$$

thus p has norm ≤ 1. Since the action of r on $\overline{W}$ defines a complex structure we see that the eigenvalues of t_1 are indeed ≤ 1. This proves the claim that the map $x \mapsto r(\overline{x})$ is a scalar times ad_m on V_λ and shows $Z(m) \subset Z(r_1)$.

To see that $Z(m) \subset Z(r_0)$ we argue as follows. Since s restricted to $Z(m)$ is the zero map we see that t restricted to $Z(m)$ is a $Z(m)$-map. This tells us that

$$p \circ (r_0 \otimes r_0) \circ p : \overline{Z(m)} \to Z(m)$$

is a $Z(m)$-map. Let $\mathfrak{s}_i$ be a simple ideal of $\mathfrak{s}$, the semisimple part of $Z(m)$. By Lemma 2 we know that $(\mathfrak{s}_i, \overline{\mathfrak{s}_i})$ is a coboundary Lie bialgebra, so let r_i be its r-matrix. For $x, y \in \mathfrak{s}_i$ the main identity together with the fact that $< (r - r_i)(\overline{\mathfrak{s}_i}), \mathfrak{s}_i >= 0$ shows that

$$r_i(\overline{x}) \cdot r_i(\overline{y}) - r_i(\overline{x} \cdot \overline{y}) = r(\overline{x}) \cdot r(\overline{y}) - r(\overline{x} \cdot \overline{y}) = cx \cdot y.$$

Let

$$p_i = p|_{\mathfrak{s}_i \wedge \mathfrak{s}_i}.$$

Then

$$t_i = p_i \circ (r_i \otimes r_i) \circ p_i = p \circ (r_0 \otimes r_0) \circ p|_{\overline{\mathfrak{s}_i}}$$

is an $\mathfrak{s}_i$-map. Thus, $s_i(x) = -m_i \cdot r_i(\overline{x})$ is an $\mathfrak{s}_i$-map on $\mathfrak{s}_i$. But $s_i(m_i) = 0$. This implies that $m_i = 0$ and thus $c = 0$, a contradiction. Thus we see that the semisimple part of $Z(m)$ is trivial and $Z(m)$ is Abelian. Clearly then, $Z(m) \subset Z(r_0)$ so that $Z(m) \subset Z(r)$. This proves the theorem. □

It can be easily checked that for any Abelian subalgebra $\mathfrak{a}$, any element of $\mathfrak{a} \wedge \mathfrak{a}$ gives an r-matrix with zero m-vector and conversely, by Lemma 4, if the

m-vector is zero then $Im(r)$ is an Abelian subalgebra. This classifies r-matrices with zero m-vector. Now suppose that r is an r-matrix with nonzero m-vector and let

$$\mathfrak{t} = Z(m) = Z(r)$$

a Cartan subalgebra of $\mathfrak{g}$. We decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

where Δ is a set of positive roots and each $\mathfrak{g}_\alpha$ is a 2 dimensional subspace with generators x_α and y_α of Killing length 1 so that for all $x \in \mathfrak{t}$

$$x \cdot x_\alpha = 2 < \alpha, x > y_\alpha,$$

$$x \cdot y_\alpha = -2 < \alpha, x > x_\alpha.$$

Since $\mathfrak{t} = Z(m)$ we can assume that we chose Δ so that $< \alpha, m >> 0$ for all $\alpha \in \Delta$. We can decompose r as before as $r = r_0 + r_1$. Since $\mathfrak{t} = Z(r) \subset Z(r_1)$ we must have

$$r_1 = \sum_{\alpha\in\Delta} C_\alpha x_\alpha \wedge y_\alpha$$

for some C_α.

Proposition 9. *For any $s \in \mathfrak{t} \wedge \mathfrak{t}$, $r_1 + s$ is an r-matrix. In particular, r_1 is an r-matrix.*

Proof. We begin by showing r_1 is an r-matrix. Let π' be the Lie product on $\mathfrak{g}^\star$ generated by r_1. We need to show that π' satisfies the Jacobi identity. Decompose $\mathfrak{g}^\star$ as

$$\mathfrak{g}^\star = \bar{\mathfrak{t}} \oplus \mathfrak{t}^\perp$$

and note that $\mathfrak{g}^\star \cdot \mathfrak{g}^\star \subset \mathfrak{t}^\perp$ implies $r(\alpha \cdot \beta) = r_1(\alpha \cdot \beta)$. Thus the Jacobi identity for π' holds for all $\alpha, \beta, \gamma \in \mathfrak{t}^\perp$. Since $r_1(\bar{\mathfrak{t}}) = 0$ it trivially holds for $\alpha, \beta, \gamma \in \bar{\mathfrak{t}}$. For $\alpha, \beta \in \bar{\mathfrak{t}}$ and $\gamma \in \mathfrak{t}^\perp$ we compute

$$\pi'(\alpha, \pi'(\beta, \gamma)) = \alpha \cdot (\beta \cdot \gamma) - r_0(\alpha) \cdot (\beta \cdot \gamma)$$
$$-\alpha \cdot (r_0(\beta) \cdot \gamma) + r_0(\alpha) \cdot (r_0(\beta) \cdot \gamma))$$

while

$$\pi'(\pi'(\alpha, \beta), \gamma) + \pi'(\beta, \pi'(\alpha, \gamma)) = 0 + \beta \cdot (\alpha \cdot \gamma) - r_0(\beta) \cdot (\alpha \cdot \gamma)$$
$$-\beta \cdot (r_0(\alpha) \cdot \gamma) + r_0(\beta) \cdot (r_0(\alpha) \cdot \gamma).$$

Comparing shows they are equal. The Jacobi identity for $\alpha \in \bar{\mathfrak{t}}$ and $\beta, \gamma \in \mathfrak{t}^\perp$ is proved similarly. This shows r_1 is an r-matrix. Now if $s \in \mathfrak{t} \wedge \mathfrak{t}$, working the above proof backwards with r_0 replaced by s will verify that $s + r_1$ is also an

r-matrix. This completes the proof. □

To complete the classification we need only compute the allowed values of the C_α. This is achieved by noting $r(\overline{x_\alpha}) = C_\alpha y_\alpha$ and $r(\overline{y_\alpha}) = -C_\alpha x_\alpha$. thus, $\overline{x_\alpha} \cdot \overline{y_\alpha} = 0$. Our main identity now gives

$$cx_\alpha \cdot y_\alpha = r(\overline{x_\alpha}) \cdot r(\overline{y_\alpha}) - r(\overline{x_\alpha} \cdot \overline{y_\alpha}) = C_\alpha^2 x_\alpha \cdot y_\alpha.$$

Thus

$$C_\alpha = \pm\sqrt{c}$$

for every $\alpha \in \Delta$. The next lemma will allow us to determine the signs.

Lemma 10. $r(\overline{m}) = 0$ *if and only if*

$$\overline{m} \cdot \overline{x} = -\overline{\overline{m} \cdot x} \qquad \forall x \in \mathfrak{g}.$$

In this case, the coadjoint action of $\overline{m}$ *is symmetric wrt to the Killing form. Thus,* $\overline{m}$ *act diagonally on* $\mathfrak{g}$ *and hence on* $\mathfrak{g}^\star$.

Proof. We check

$$\overline{m} \cdot \overline{x} = r(\overline{m}) \cdot \overline{x} - r(\overline{x}) \cdot \overline{m} = m \cdot \overline{r(\overline{x})} + r(\overline{m}) \cdot \overline{x},$$

$$\overline{m} \cdot x = r(x \cdot \overline{m}) + r(\overline{m}) \cdot x = -m \cdot r(\overline{x}) + r(\overline{m}) \cdot x.$$

Now applying the Killing map to the lower equalities shows the equivalence since the center of $\mathfrak{g}$ is trivial. Symmetry of the coadjoint action is shown by

$$\kappa(x, \overline{m} \cdot y) =< \overline{x}, \overline{m} \cdot y >= - < \overline{m} \cdot \overline{x}, y >=< \overline{\overline{m} \cdot x}, y >= \kappa(\overline{m} \cdot x, y)$$

This proves the lemma. □

To compute the signs we assume that $r_0 = 0$ so that $\overline{m}$ acts diagonally with eigenvectors

$$\overline{m} \cdot \overline{x_\alpha} = -r(x_\alpha) \cdot \overline{m} = -C_\alpha \overline{y_\alpha \cdot m} = -2C_\alpha < \alpha, m > \overline{x_\alpha}$$

and similarly for y_α. Now let $\alpha, \beta \in \Delta$ be simple with $\alpha + \beta \in \Delta$. Then

$$\mathfrak{g}_\alpha \cdot \mathfrak{g}_\beta \subset \mathfrak{g}_{\alpha+\beta}$$

which implies

$$\overline{\mathfrak{g}_\alpha} \cdot \overline{\mathfrak{g}_\beta} \subset \overline{\mathfrak{g}_{\alpha+\beta}}.$$

Applying $\overline{m}$ to the product $x_\alpha \cdot x_\beta$ shows

$$C_{\alpha+\beta} < \alpha + \beta, m >= C_\alpha < \alpha, m > + C_\beta < \beta, m > .$$

This implies that $C_{\alpha+\beta}, C_\alpha$, and C_β all have the same sign. We can now split the simple roots into 2 sets depending on the sign of C_α. Since no root is a sum of two simple roots, one from each set, we must have one of these sets empty since otherwise the Dynkin diagram would not be connected. But $\mathfrak{g}$ is simple and so has a connected Dynkin diagram [6]. We restate the results of [13].

Theorem 11. *If $\mathfrak{g}$ is a compact simple Lie bialgebra with r-matrix r, we can find a Cartan subalgebra $\mathfrak{t}$ for which r has the decomposition*

$$r = r_0 + C \sum_{\alpha \in \Delta} x_\alpha \wedge y_\alpha,$$

where r_0 is any element of $\mathfrak{t} \wedge \mathfrak{t}$ and C is any constant. For $C \neq 0$ the automorphism Lie algebra of the Lie bialgebra is $\mathfrak{t}$.

We note that for nonzero m, the r-matrix chooses a Cartan and a positive Weyl chamber for us and $\overline{m}$ is proportional to ρ, half the sum of the positive roots. Conversely, if we choose a Cartan subalgebra and a Weyl chamber (Δ) we get a unique r-matrix by picking $C = 1$ and the orientation $x_\alpha \wedge y_\alpha$ via $< \alpha, x_\alpha \cdot y_\alpha >> 0$. If $C = 0$ then the derived algebra of $\mathfrak{g}^\star$ is Abelian. If $C \neq 0$, then since $\overline{m} \cdot \overline{x_\alpha} = -2C < \alpha, m > \overline{x_\alpha}$ for all α, we see that

$$\mathfrak{g}^\star_\alpha \cdot \mathfrak{g}^\star_\beta \subset \mathfrak{g}^\star_{\alpha+\beta}$$

for all α and β. Define the height of a root α to be the sum of the integer coefficients of the expansion of α as a sum of simple roots and let N be the height of the highest root.

Corollary 12. *For $C \neq 0$, the derived algebra of $\mathfrak{g}^\star$ is N-step nilpotent. Hence, $\mathfrak{g}^\star$ is solvable and the simply connect Lie group corresponding to $\mathfrak{g}^\star$ is contractable.*

4. The Standard Example

As an example we recall the construction of [10]. Consider the invariant nondegenerate bilinear form on $\mathfrak{sl}(n, \mathbb{C})$ defined by

$$< X, Y >= IM(tr(XY)).$$

It is straightforward to check that this form vanishes on $\mathfrak{g} = \mathfrak{su}(n)$ and on the subalgebra $\mathfrak{sb}(n)$ defined as the subalgebra consisting of uppertriangular matrices which are real along the diagonal. This condition, defining what is known as a Manin triple, sets up a nondegenerate pairing which allows us to identify $\mathfrak{g}^\star = \mathfrak{sb}(n)$ and results in a Lie bialgebra with $\mathfrak{d} = \mathfrak{sl}(n, \mathbb{C})$. The r-matrix is given by

Proposition 13. *For the Lie bialgebra $(\mathfrak{su}(n), \mathfrak{sb}(n))$ the r-matrix is given by*

$$2r(\alpha) = \alpha - \alpha^*,$$

where α^* denotes the conjugate transpose of the matrix α. $Z(r)$ is equal to the *standard Cartan subalgebra consisting of the diagonal elements of* $\mathfrak{su}(n)$.

Proof. Let $s : \mathfrak{g}^\star \to \mathfrak{g}$ denote the map $\alpha \mapsto \alpha - \alpha^*$. Since $< X, Y >= - < X^*, Y^* >$, s is antisymmetric. We also know that $[x, Y^*] = [x, Y]^*$ for all $x \in \mathfrak{g}$ and $Y \in \mathfrak{d}$. We check

$$\begin{aligned}(x \cdot s)(\alpha) &= x \cdot s(\alpha) - s(x \cdot \alpha)\\ &= [x, \alpha - \alpha^*] - x \cdot \alpha + (x \cdot \alpha)^*\\ &= x \cdot \alpha - \alpha \cdot x - (x \cdot \alpha - \alpha \cdot x)^* - x \cdot \alpha + (x \cdot \alpha)^*\\ &= -2\alpha \cdot x = 2(x \cdot r)(\alpha).\end{aligned}$$

Thus, $s - 2r$ is a $\mathfrak{g}$ invariant antisymmetric map from $\mathfrak{g}^\star$ to $\mathfrak{g}$. Since the only such map is the zero map, the equality is proven. The rest follows easily. □

The map $L : \mathfrak{g} \to \mathfrak{g}$ defined by

$$L(x) = \frac{\imath}{2}(\overline{x} + \overline{x}^*)$$

can be checked to be $\mathfrak{g}$ invariant. This means that L is a scalar times the identity and we scale the Killing form so that this map is in fact the identity. Thus we have

$$x = \frac{\imath}{2}(\overline{x} + \overline{x}^*),$$

$$r(\overline{x}) = \overline{x} + \imath x,$$

and

$$r(\overline{x}) \cdot r(\overline{y}) - r(\overline{x} \cdot \overline{y}) = x \cdot y.$$

Finally, it can be easily seen that

$$m = 2\imath \operatorname{diag}(n-1, n-3, n-5, ..., 3-n, 1-n).$$

References

[1] A.A. Belavin, V.G. Drinfeld, Triangle equations and simple Lie algebras, *Soviet Scientific Reviews*, Section C, **4** (1984), 93-165.

[2] T. Bröcker, T. Dieck, Representations of compact Lie groups, *Graduate Text in Mathematics*, **98**, Springer-Verlag (1985).

[3] M. Cahen, S. Gutt, J. Rawnsley, Some remarks on the classification of poisson Lie groups, *Contemp. Math.*, **179** (1994), 1-15.

[4] V.G. Drinfeld, Quantum groups, In: *Proceedings of the International Congress of Mathematicians*, Berkeley (1987), 798-820.

[5] G. Hochschild, J.-P. Serre, Cohomology of Lie algebras, *Ann. of Math.* (1953), 591-603.

[6] J. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag (1972).

[7] Y. Kosmann-Schwarzbach, Jacobian quasi-bialgebras and quasi-Poisson Lie groups, In: *Mathematical Aspects of Classical Field Theory, Contemp. Math.*, **132**, Amer. Math Soc. (1992), 459-489.

[8] P. Lecomte, C. Roger, Modules et cohomologies des bialgébres de Lie, *C.R. Acad. Sci. Paris*, **310**, Série I (1990), 405-410.

[9] J.H. Lu, Classical dynamical r-matrices and homogeneous Poisson structures on G/H and K/T, *Univ. of Arizona Preprint*, math.SG/9909004.

[10] J.H. Lu, A. Weinstein, Poisson Lie groups, dressing transformation, and Bruhat decompositions, *J. Diff. Geom.*, **31** (1990), 501-526.

[11] M.A. Semenov-Tian-Shansky, What is a classical r-matrix?, *Functional Anal. Appl.*, **17** (1983), 259-272.

[12] M.A. Semenov-Tian-Shansky, Dressing transformations and Poisson Lie group actions, *Publ. Res. Inst. Math. Sci.*, **21** (1985), 1237-1260.

[13] Y. Soibelman, Algebra of functions on quantum group $su(n)$ and Schubert cells, *Russian Doklady*, **307** (1989).