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A STRONG MAXIMUM PRINCIPLE FOR LINEAR ELLIPTIC OPERATORS

Paola Cavaliere[1 §], Maria Transirico[2]

[1]Department of Engineering of Information and Applied Mathematics
University of Salerno
Via Ponte Don Melillo, Fisciano (Salerno), I-84084, ITALY
e-mail: pcavaliere@unisa.it

[2]Department of Mathematics and Informatics
University of Salerno
Via Ponte Don Melillo, Fisciano (Salerno), I-84084, ITALY
e-mail: mtransirico@unisa.it

Abstract: This paper is concerned with a maximum principle for subsolutions, in the class $W^{2,p}_{\text{loc}}$, of second order linear elliptic equations in non-divergence form in arbitrary open (bounded or not) subsets of $\mathbb{R}^n$, $n \geq 2$, when $p > \frac{n}{2}$.

The main coefficients are required to be just locally in $L^\infty \cap VMO$. A uniqueness result for the related homogenous Dirichlet problem is also obtained.

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1. Introduction

Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 2$, and consider a uniformly elliptic operator in non-divergence form

$$L_0 = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j},$$

whose coefficients a_{ij} are measurable in Ω. Moreover, let u be a solution of the

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§Correspondence author

Dirichlet problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap C^0(\bar{\Omega})\,, \\ L_0 u = f \in L^p(\Omega)\,, \\ u_{|\partial\Omega} = 0\,, \end{cases} \tag{1.1}$$

with $p > \frac{n}{2}$.

If $p = n$, then the classical Aleksandrov-Bakelman-Pucci estimate asserts that

$$\sup_{\Omega} |u| \leq c\, ||f||_{L^p(\Omega)}\,, \tag{1.2}$$

where $c \in \mathbb{R}_+$ depends only on n, Ω and the ellipticity constant (see, for instance, [11], [1], and also [2] for an interesting improving of the above result). However, an example in [10] points out that, when the ellipticity constant is small enough, the a priori estimate (1.2) fails for L^p-norms, $p < n$, in place of the L^n-norm, and c depending only on n, Ω and the ellipticity constant.

On the other side, it is well known that, under additional regularity assumptions on the coefficients a_{ij}, (1.2) still holds for $p < n$ even if with c depending also on p and on the regularity of the a_{ij}'s. This has been proved in the case of Hölder continuous coefficients $a_{ij}(x)$ in [7], when the a_{ij} are continuous in [8] and if the a_{ij} belong to $W^{1,n}(\Omega)$ in [9] (we refer to [6] for a more recent treatment of such theory).

Therefore, it appears natural to investigate the problem (1.1) when $p < n$ assuming that the coefficients a_{ij} belong to a space wider than those already considered in the literature. Let us remember that both hypotheses a_{ij} uniformly continuous and $a_{ij} \in W^{1,n}$ imply $a_{ij} \in VMO$ (see [5]).

Note also that it follows from (1.2) that the problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap C^0(\bar{\Omega})\,, \\ L_0 u = 0\,, \\ u_{|\partial\Omega} = 0\,, \end{cases} \tag{1.3}$$

with $p > \frac{n}{2}$, has only the zero solution.

In this paper we fix an arbitrary open (bounded or not) subset Ω of $\mathbb{R}^n$, $n \geq 2$, a real number $p > \frac{n}{2}$, and we consider the second-order locally uniformly elliptic differential operator

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a\,,$$

where the main coefficients a_{ij} are locally bounded and locally VMO, the lower orders terms a_i and a satisfy suitable local summability conditions, and a is locally negative. In this situation, we will establish that if u is a solution of the problem

$$u \in W^{2,p}_{\text{loc}}(\Omega)\,, \quad Lu \geq\, 0\,, \tag{1.4}$$

then u does not have any positive relative maximum in Ω. Then, as an application of this result, we will show that the problem

$$\begin{cases} u \in W^{2,p}_{\text{loc}}(\Omega)\,, \quad Lu = 0\,, \\ \lim_{x \longrightarrow x_0} u(x) = 0 \quad \forall x_0 \in \partial\Omega\,, \\ \lim_{|x| \longrightarrow +\infty} u(x) = 0 \quad \text{if } \Omega \text{ is unbounded,} \end{cases} \tag{1.5}$$

admits only the zero solution.

2. Some Notation

Let Ω be an open subset of $\mathbb{R}^n$. If $p > 1$ and $g \in L^p(\Omega)$, a function $\omega_p[g,\Omega] : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a *modulus of continuity* of g in $L^p(\Omega)$ if

$$\sup_{\substack{E \in \Sigma(\Omega) \\ |E| \leq t}} ||g||_{L^p(E)} \leq \omega_p[g,\Omega](t) \quad \forall t \in R_+\,, \quad \lim_{t \to 0^+} \omega_p[g,\Omega](t) = 0\,.$$

If Ω has the property

$$|\Omega(x,r)| \,\geq\, A\, r^n \quad \forall x \in \Omega\,, \forall r \in]0,1]\,, \tag{2.1}$$

where $\Omega(x,r) = \Omega \cap B(x,r)$ and A is some positive constant independent of x and r, one can consider the space $BMO(\Omega,t)$, $t \in \mathbb{R}_+$, consisting of all functions g in $L^1_{\text{loc}}(\bar{\Omega})$ such that

$$[g]_{BMO(\Omega,t)} = \sup_{\substack{x \in \Omega \\ r \in]0,t]}} ⨍_{\Omega(x,r)} |g - ⨍_{\Omega(x,r)} g| \, < +\infty\,.$$

If $g \in BMO(\Omega) = BMO(\Omega, t_A)$, with

$$t_A = \sup_{t \in \mathcal{R}_+} \Big(\sup_{\substack{x \in \Omega \\ r \in]0,t]}} \frac{r^n}{|\Omega(x,r)|} \leq \frac{1}{A} \Big)\,,$$

we shall say that g is in $VMO(\Omega)$ when $[g]_{BMO(\Omega,t)} \to 0$ as $t \to 0^+$. Moreover, a function $\eta[g,\Omega] : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a *modulus of continuity* of g in $VMO(\Omega)$ if

$$[g]_{BMO(\Omega,t)} \leq \eta[g,\Omega](t) \quad \forall t \in \mathbb{R}_+\,, \quad \lim_{t \to 0^+} \eta[g,\Omega](t) = 0\,.$$

We will also say that $g \in VMO_{\text{loc}}(\Omega)$ if ζg belongs to $VMO(\Omega)$ for each $\zeta \in C^\infty_\circ(\Omega)$.

3. Auxiliary Results

Throughout this section, let $p \in \left]\frac{n}{2}, +\infty\right[$ and B an arbitrary open ball of $\mathbb{R}^n$, $n \geq 2$, of radius $d \in \mathbb{R}_+$. We consider in B the operator

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a \tag{3.1}$$

making the following assumption on its coefficients:

$$\left\{ \begin{array}{l} a_{ij} = a_{ji} \in L^\infty(B) \cap VMO(B)\,, \quad i,j = 1, \ldots, n\,, \\ \exists\, \nu \in \mathbb{R}_+ \quad : \quad \displaystyle\sum_{i,j=1}^{n} a_{ij}\, \xi_i\, \xi_j \;\geq\; \nu |\xi|^2 \quad \text{a.e. in } B\,,\ \forall \xi \in \mathbb{R}^n\,, \\ a_i \in L^r(B),\ i = 1, \ldots, n, \quad r > n \text{ if } \ p \leq n, \text{ and } \ r = p \text{ if } \ p > n, \\ a \in L^p(B), \quad a \leq 0 \quad \text{a.e. in } B\,. \end{array} \right. \tag{h_B}$$

Under (h_B), the operator L from $W^{2,p}(B)$ into $L^p(B)$ is bounded, and the following estimate holds

$$||Lu||_{L^p(B)} \leq c_1 ||u||_{W^{2,p}(B)} \quad \forall u \in W^{2,p}(B), \tag{3.2}$$

where $c_1 \in \mathbb{R}_+$ depends on $n, p, d, ||a_{ij}||_{L^\infty(B)}, ||a_i||_{L^r(B)}, ||a||_{L^p(B)}$.

We first provide the following "maximum principle":

Lemma 3.1. *Assume (h_B) and let $h \in W^{2,p}(B)$ such that $h_{|\partial B} \geq 0$. If $w \in W^{2,p}(B)$ satisfies*

$$\left\{ \begin{array}{l} Lw = 0 \quad \text{in } B, \\ w_{|\partial B} = h_{|\partial B}\,, \end{array} \right. \tag{3.3}$$

then $w \geq 0$ in B.

Proof. If $n \geq 3$, the result is already known (see Lemma 3.1 of [3]). In the case $n = 2$, we deduce the statement proceeding as in the proof of Lemma 3.1 of [3] but applying Theorem 3.5 of [4] whereas in [3] is used Theorem 2.1 of [14]. □

Moreover, we have the following existence and uniqueness result:

Lemma 3.2. *Assume that (h_B) holds. Then the problem*

$$\left\{ \begin{array}{l} u \in W^{2,p}(B) \cap \overset{\circ}{W}{}^{1,p}(B)\,, \\ Lu = f\,, \quad f \in L^p(B)\,, \end{array} \right. \tag{3.4}$$

is uniquely solvable and the solution u satisfies the a priori bound

$$||u||_{W^{2,p}(B)} \leq c ||f||_{L^p(B)}\,, \tag{3.5}$$

with $c \in \mathbb{R}_+$ depending on $n, p, d, \nu, ||a_{ij}||_{L^\infty(B)}, \eta[p(a_{ij})], ||a_i||_{L^r(B)}, ||a||_{L^p(B)}$,

$\omega_r[a_i, B], \omega_p[a, B]$, *and where* $p(a_{ij})$ *is an extension of* a_{ij} *to* $\mathbb{R}^n$ *of class* $L^\infty(\mathbb{R}^n) \cap VMO(\mathbb{R}^n)$.

Proof. It is enough to observe that an application of Theorem 5.1 of [12] yields that there exist extensions $p(a_{ij})$ of a_{ij} to $\mathbb{R}^n$ $(i, j = 1, ..., n)$ of class $L^\infty(\mathbb{R}^n) \cap VMO(\mathbb{R}^n)$. Then, if $n \geq 3$, the result is a consequence of Theorem 2.1 of [14], while it follows from Theorem 3.5 of [4] for $n = 2$. □

We are now in position to state the following result.

Lemma 3.3. *Suppose that condition* (h_B) *is satisfied, and consider problem (3.4). Then,* $f \geq 0$ *implies that* $u \leq 0$ *in* B.

Proof. If $p \geq n$, the statement is a consequence of Theorem III of [11]. Thus we confine ourselves to the case $\frac{n}{2} < p < n$ and we proceed in two steps.

Step 1. We first assume that $f \in L^n(B)$. Denote by $\dot{a}$ the extension of a to $\mathbb{R}^n$ with zero values out of B and by $(J_k)_{k \in \mathcal{N}}$ a sequence of mollifiers, for any k we define $a^k = J_k * \dot{a}$ and

$$L^k = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a^k \,.$$

The sequence $(a^k)_{k \in \mathcal{N}}$ satisfies the following properties:

$$a^k \longrightarrow a \text{ in } L^p(B)\,, \tag{3.6}$$

$$||a^k||_{L^p(B)} \leq ||a||_{L^p(B)}\,, \;\; \omega_p[a^k, B] \leq \omega_p[a, B]\,, \; \forall k \in \mathbb{N}\,, \tag{3.7}$$

$$a^k \leq 0 \text{ in } B\,, \quad \forall k \in \mathbb{N}\,. \tag{3.8}$$

For any $k \in \mathbb{N}$, consider now the Dirichlet problem

$$\begin{cases} u_k \in W^{2,p}(B) \cap \overset{\circ}{W}{}^{1,p}(B)\,, \\ L^k u_k = f \in L^n(B)\,. \end{cases} \tag{3.9}$$

By (3.7), (3.8) and Lemma 3.2, problem (3.9) is uniquely solvable and the solution u_k satisfies the a priori bound

$$||u_k||_{W^{2,p}(B)} \leq c||f||_{L^p(B)}\,, \tag{3.10}$$

with $c \in \mathbb{R}_+$ depending on $n, p, d, \nu, ||a_{ij}||_{L^\infty(B)}, \eta[p(a_{ij})], ||a_i||_{L^r(B)}, ||a||_{L^p(B)}$, $\omega_r[a_i, B], \omega_p[a, B]$. Moreover, from known regularity results (see Lemma 3.3 of [4] if $n = 2$ and Theorem 2.2 of [13] if $n \geq 3$) we deduce that $u_k \in W^{2,n}(B)$ and then, using again Theorem III of [11], we have that

$$u_k \leq 0 \text{ in } B\,. \tag{3.11}$$

On the other hand, by (3.10) we have that $(u^k)_{k \in \mathcal{N}}$ is a bounded sequence in $W^{2,p}(B)$; so there exists a subsequence, that we still denote as $(u^k)_{k \in \mathcal{N}}$, such that

$$\begin{cases} u_k \rightharpoonup u' \quad \text{in} \quad W^{2,p}(B)\,, \\ u_k \longrightarrow u' \quad \text{in} \quad W^{1,q}(B)\,,\ 1 < q < \dfrac{np}{n-p}\,, \\ u_k \longrightarrow u' \quad \text{in} \quad C(\bar{B})\,, \end{cases} \tag{3.12}$$

with $u' \in W^{2,p}(B) \cap W^{1,q}(B)$. Moreover, from (3.11), one has that $u' \leq 0$ in B.

We claim that the sequence $(L^k u_k)_{k\in\mathcal{N}}$ weakly converges to Lu' in $L^p(B)$. In fact, for each $\varphi \in L^{p'}(B)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\begin{aligned} \int_B |(L^k u_k - Lu')\varphi| dx &\leq \sum_{i,j=1}^{n} ||a_{ij}||_{L^\infty(B)} \int_B |((u_k)_{x_i x_j} - u'_{x_i x_j})\varphi| dx \\ &+ \sum_{i=1}^{n} ||a_i||_{L^r(B)} ||(u_k)_{x_i} - u'_{x_i}||_{L^q(B)} ||\varphi||_{L^{p'}(B)} \\ &+ \sup_{\bar{B}} |u_k - u'|\, ||a||_{L^p(B)} ||\varphi||_{L^{p'}(B)} + ||a^k - a||_{L^p(B)} \sup_{\bar{B}} |u_k|\, ||\varphi||_{L^{p'}(B)}\,, \end{aligned}$$

where $q = \dfrac{rp}{r-p}$. So, coming back to (3.6) and (3.12), the weak convergence of $(L^k u_k)_{k\in\mathcal{N}}$ to Lu' in $L^p(B)$ follows. Therefore $Lu' = f$ a.e. in B and $u' \in W^{2,p}(B) \cap \overset{\circ}{W}{}^{1,p}(B)$, and hence the uniqueness of the solution of problem (3.4) yields that $u = u'$, so that $u \leq 0$ in B.

Step 2. We now consider $f \in L^p(B)$. If we define $f^k = J_k * \dot{f}$, $k \in \mathbb{N}$, where $\dot{f}$ is the extension of f to $\mathbb{R}^n$ with zero values out of B, then the sequence $(f^k)_{k\in\mathcal{N}}$ satisfies the following properties:

$$f^k \longrightarrow f \text{ in } L^p(B)\,, \tag{3.13}$$

$$||f^k||_{L^p(B)} \leq ||f||_{L^p(B)}\,,\ \forall k \in \mathbb{N}\,, \tag{3.14}$$

$$f^k \geq 0 \text{ in } B\,, \quad \forall k \in \mathbb{N}\,. \tag{3.15}$$

If, for any $k \in \mathbb{N}$, we consider the Dirichlet problem

$$\begin{cases} u_k \in W^{2,p}(B) \cap \overset{\circ}{W}{}^{1,p}(B)\,, \\ Lu_k = f^k\,, \end{cases} \tag{3.16}$$

then Lemma 3.2 assures that (3.16) is uniquely solvable and the solution u_k satisfies the a priori bound

$$||u_k||_{W^{2,p}(B)} \leq c||f^k||_{L^p(B)} \leq c||f||_{L^p(B)}\,. \tag{3.17}$$

Moreover, from the previous step, we also have that $u_k \leq 0$ in B. Then, employing the same arguments of the proof of the previous step, we obtain again that $u \leq 0$ in B. □

Let $\mu_0, \mu_1, \mu_2 \in \mathbb{R}_+$ such that:

$$\sum_{i,j=1}^{n} ||a_{ij}||_{L^\infty(B)} \leq \mu_0 \,, \;\; d^{1-\frac{n}{r}} \sum_{i=1}^{n} ||a_i||_{L^r(B)} \leq \mu_1 \,, \;\; d^{2-\frac{n}{p}} ||a||_{L^p(B)} \leq \mu_2 .$$

Using Lemmas 3.1, 3.2 and 3.3, we are able to improve Lemma 3.2 of [3] and also to extend it to the case $n = 2$.

Lemma 3.4. *Suppose that condition (h_B) holds. If u is a solution of the problem*

$$\begin{cases} u \in W^{2,p}(B)\,, \\ Lu \geq \; f \in L^p(B)\,, \\ u_{|\partial B} \leq \; 0\,, \end{cases} \tag{3.18}$$

then there exists $C \in \mathbb{R}_+$ such that

$$\sup_B u \leq \; C\, d^{2-\frac{n}{p}} ||f^-||_{L^p(B)}\,, \tag{3.19}$$

with C dependent on $n, p, \nu, \mu_0, \mu_1, \mu_2, [p(a_{ij})]_{BMO(\mathcal{R}^n,\cdot)}, \omega_r[d^{1-\frac{n}{r}} a_i, B]$, $\omega_p[d^{2-\frac{n}{p}} a, B]$, and where $p(a_{ij})$ is an extension of a_{ij} to $\mathbb{R}^n$ in $L^\infty(\mathbb{R}^n) \cap VMO(\mathbb{R}^n)$.

Proof. Let $B = B(x_0, d)$, where x_0 is the centre of B, and put $B^* = B(x_0, 1)$. Consider the trasformation $T : B \longrightarrow B^*$ defined by the position

$$T(x) = x_0 + \frac{x - x_0}{d} = z\,,$$

and for each function g defined on B, put $g^* = g \circ T^{-1}$.

Observe that

$$L^* u^* = d^2 (Lu)^*\,,$$

where

$$L^* = \sum_{i,j=1}^{n} a_{ij}^* \frac{\partial^2}{\partial z_i \partial z_j} + d \sum_{i=1}^{n} a_i^* \frac{\partial}{\partial z_i} + d^2 a^*\,.$$

Denote by $p(a_{ij})$ an extension of a_{ij} to $\mathbb{R}^n$ such that

$$p(a_{ij}) \in L^\infty(\mathbb{R}^n) \cap VMO(\mathbb{R}^n),$$

$(i, j = 1, \ldots, n)$. Since

$$p(a_{ij})^* \in L^\infty(\mathbb{R}^n) \cap VMO(\mathbb{R}^n)\,, \;\; p(a_{ij})^*_{|B^*} = a_{ij}^*, \tag{3.20}$$

we also have

$$a_{ij}^* \in L^\infty(B^*) \cap VMO(B^*) \tag{3.21}$$

(see [12]). Moreover, the hypothesis (h_B) yields that

$$\begin{cases} a^*_{ij} = a^*_{ji}\,, \quad i,j = 1,\dots,n\,, \\ \displaystyle\sum_{i,j=1}^{n} a^*_{ij}\xi_i\xi_j \geq \nu|\xi|^2 \ \text{ a.e. in } B^*\,, \quad \forall \xi \in \mathbb{R}^n\,, \\ a^*_i \in L^r(B^*),\ i = 1,\dots,n,\ a^* \in L^p(B^*),\ a^* \leq 0 \text{ a.e. in } B^*\,. \end{cases} \tag{3.22}$$

Consider now the following problem:

$$\begin{cases} L^* v = h \in L^p(B^*)\,, \\ v \in W^{2,p}(B^*) \cap \overset{\circ}{W}{}^{1,p}(B^*)\,. \end{cases} \tag{3.23}$$

It follows from (3.20), (3.22) and from Lemma 3.2 that there exists a unique solution v of (3.23) satisfying the estimate

$$||v||_{W^{2,p}(B^*)} \leq \ K||h||_{L^p(B^*)}\,, \tag{3.24}$$

where K depends on $n, p, \nu, ||a^*_{ij}||_{L^\infty(B*)}, [p(a_{ij})^*]_{BMO(\mathcal{R}^n,\cdot)}, ||da^*_i||_{L^r(B^*)}$, $||d^2a^*||_{L^p(B^*)}, \omega_r[da^*_i, B^*], \omega_p[d^2a^*, B^*]$. Hence the constant K depends on n, p, $\nu, \mu_0, [p(a_{ij})]_{BMO(\mathcal{R}^n,\cdot)}, \mu_1, \mu_2, \omega_r[d^{1-\frac{n}{r}}a_i, B], \omega_p[d^{2-\frac{n}{p}}a, B]$.

Thus by (3.24) we obtain that there is $K_1 \in \mathbb{R}_+$, depending on the same parameters on which K does, such that

$$\max_{\bar{B}^*} |v| \leq \ K_1||h||_{L^p(B^*)}\,, \tag{3.25}$$

and hence for each $z \in B^*$ there is $g(z,\cdot) \in L^{p'}(B^*)$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ for which

$$v(z) = -\int_{B^*} g(z,y)h(y)dy\,. \tag{3.26}$$

The map $g(z,\cdot)$ is the Green function for the operator L^* in B^*. From (3.21), (3.22), (3.26) and from Lemma 3.3 we deduce that

$$\int_{B^*} g(z,y)\tilde{h}(y)dy \geq \ 0 \quad \forall \tilde{h} \in L^p(B^*)\,,\ \tilde{h} \geq 0\,; \tag{3.27}$$

moreover, by (3.25) and (3.26) we have that

$$||g(z,\cdot)||_{L^{p'}(B^*)} \leq \ K_1\,. \tag{3.28}$$

Setting $h = L^*u^* = d^2(Lu)^*$ in (3.23), we have that $v - u^* (\in W^{2,p}(B^*))$ is a solution of the problem

$$\begin{cases} L^*(v - u^*) = 0 \quad \text{in } B^*\,, \\ (v - u^*)_{|\partial B^*} = -u^*_{|\partial B^*} \geq \ 0\,, \end{cases}$$

and so it follows from (3.21), (3.22) and from Lemma 3.1 that $v - \ u^* \geq \ 0$ in

B^*. Thus, applying (3.26) with $h = L^*u^*$, it follows from (3.27) and (3.28) that

$$u^*(z) \leq -\int_{B^*} g(z,y) d^2 (Lu)^*(y) dy \tag{3.29}$$

$$\leq -d^2 \int_{B^*} g(z,y) f^*(y) dy \leq -2d^2 \int_{B^*} g(z,y)(f^*)^-(y) dy$$

$$\leq 2d^2 ||g(z,\cdot)||_{L^{p'}(B^*)} ||(f^*)^-||_{L^p(B^*)} \leq 2d^2 K_1 ||(f^-)^*||_{L^p(B^*)} \quad \forall z \in B^* .$$

It is now easy to deduce the statement from (3.29). □

4. Main Results

Let Ω be an open (bounded or not) subset of $\mathbb{R}^n$, $n \geq 2$, with the property (2.1), and let $p \in \left]\frac{n}{2}, +\infty\right[$. Consider in Ω the operator

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a \,, \tag{4.1}$$

and put

$$L_0 = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \,.$$

On the coefficients of L we make the following assumptions:

$$\begin{cases} a_{ij} = a_{ji} \in L^\infty_{\text{loc}}(\Omega) \cap VMO_{\text{loc}}(\Omega) \,, \quad i,j = 1, \ldots, n \,, \\ a_i \in L^r_{\text{loc}}(\Omega), i = 1, \ldots, n, \text{ where } r > n \text{ if } p \leq n, r = p \text{ if } p > n, \\ a \in L^p_{\text{loc}}(\Omega) \,, \\ \exists \nu \in L^\infty_{\text{loc}}(\Omega) : \nu(x) > 0 \quad \text{a.e. in } \Omega \,, \\ \displaystyle\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq \nu(x) |\xi|^2 \text{ a.e. in } \Omega \,, \ \forall \xi \in \mathbb{R}^n \,, \\ \text{for any open subset } E \subset\subset \Omega \,, \ \operatorname*{ess\,inf}_E \nu > 0 \,, \ \operatorname*{ess\,sup}_E a < 0. \end{cases} \tag{h}$$

Our aim is to prove the following result.

Theorem 4.1. *Assume that (h) holds. If u is a solution of the problem*

$$u \in W^{2,p}_{\text{loc}}(\Omega) \,, \quad Lu \geq 0 \,, \tag{4.2}$$

then u does not have any positive relative maximum in Ω.

Proof. By contradiction, suppose that there exists some point of positive relative maximum y in Ω for u. Since $u \in C^0(\Omega)$, there exists $d \in \mathbb{R}_+$ such that:

$$B(y,d) \subset\subset \Omega\,, \quad 0 < u(x) \le u(y) \quad \forall x \in B(y,d)\,. \tag{4.3}$$

Put

$$a_0 = -\operatorname*{ess\,sup}_{B(y,d)} a\,, \quad \mu_0 = \operatorname*{ess\,sup}_{B(y,d)} \sum_{i,j=1}^n a_{ij}\,, \quad \nu_0 = \operatorname*{ess\,inf}_{B(y,d)} \nu \tag{4.4}$$

and fix $\alpha \in \mathbb{R}_+$ such that

$$\alpha \ge \left(\frac{10\mu_0}{a_0}\right)^{\frac{1}{2}}. \tag{4.5}$$

For any $\lambda \in \left]0, \min\left(1, \frac{d}{\alpha}\right)\right]$ we have

$$B = B(y, \alpha\lambda) \subseteq B(y,d)\,. \tag{4.6}$$

Moreover, put

$$\varphi(x) = \varphi_\lambda(x) = 1 + \lambda^2 - \frac{|x-y|^2}{\alpha^2}\,, \tag{4.7}$$

$$x \in \overline{B(y,\alpha\lambda)} \text{ and } \lambda \in \left]0, \min\left(1, \frac{d}{\alpha}\right)\right],$$

and see that

$$1 \le \varphi \le 1 + \lambda^2\,. \tag{4.8}$$

Consider now the map v defined by the position

$$v(x) = \varphi(x)u(x) - u(y)\,, \quad x \in \overline{B(y,\alpha\lambda)}\,. \tag{4.9}$$

Note that

$$L_0(\varphi u) = \varphi L_0 u + u L_0 \varphi + 2 \sum_{i,j=1}^n a_{ij}\varphi_{x_j} u_{x_i}\,,$$

$$\sum_{i=1}^n a_i (\varphi u)_{x_i} = \varphi \sum_{i=1}^n a_i u_{x_i} + u \sum_{i=1}^n a_i \varphi_{x_i}\,,$$

so

$$\begin{aligned} \varphi L u &= \varphi L_0 u + \varphi \sum_{i=1}^n a_i u_{x_i} + \varphi a u \\ &= L_0(\varphi u) - u L_0 \varphi - 2 \sum_{i,j=1}^n a_{ij} \varphi_{x_j} u_{x_i} \\ &\quad + \sum_{i=1}^n a_i (\varphi u)_{x_i} - u \sum_{i=1}^n a_i \varphi_{x_i} + a\varphi u \ge 0 \text{ in } B. \end{aligned} \tag{4.10}$$

Since

$$u_{x_i}\varphi_{x_j} = \frac{\varphi_{x_j}!}{\varphi}(\varphi u)_{x_i} - \frac{\varphi_{x_i}\varphi_{x_j}}{\varphi^2}(\varphi u)\,,$$

from (4.10) it follows that

$$L_0(\varphi u) + \sum_{i=1}^n b_i(\varphi u)_{x_i} + c\varphi u \ \geq \ u\sum_{i=1}^n a_i\varphi_{x_i} \quad \text{in } B\,, \tag{4.11}$$

where

$$b_i = a_i - 2\sum_{j=1}^n a_{ij}\frac{\varphi_{x_j}}{\varphi}\,, \quad i = 1,\ldots,n\,, \tag{4.12}$$

and

$$c = a - \sum_{i,j=1}^n a_{ij}\frac{\varphi_{x_i x_j}}{\varphi} + 2\sum_{i,j=1}^n a_{ij}\frac{\varphi_{x_i}\varphi_{x_j}}{\varphi^2}\,. \tag{4.13}$$

Coming back to (4.9) and (4.11), we get

$$L_0 v + \sum_{i=1}^n b_i v_{x_i} + cv \geq \ u\sum_{i=1}^n a_i\varphi_{x_i} - cu(y) \ \text{ in } B\,. \tag{4.14}$$

Moreover, as $\varphi_{x_i} = -\frac{2(x_i - y_i)}{\alpha^2}$, $\varphi_{x_i x_j} = 0$ if $i \neq j$, $\varphi_{x_i x_j} = -\frac{2}{\alpha^2}$ if $i = j$, we have

$$\begin{aligned} c &\leq \ -a_0 + 2\frac{\sum_{i=1}^n a_{ii}}{\varphi}\cdot\frac{!1}{\alpha^2} + 8\lambda^2\frac{\sum_{i,j=1}^n a_{ij}}{\varphi^2}\cdot\frac{1}{\alpha^2} \\ &\leq \ -a_0 + (10\sum_{i,j=1}^n a_{ij})\cdot\frac{1}{\alpha^2} \ \text{ in } B \end{aligned}$$

and hence it follows from (4.5) that $c \leq 0$ in B.

Finally, since $\varphi_{|\partial B(y,\alpha\lambda)} = 1$, v satisfies the following condition:

$$v_{|\partial B} = u_{|\partial B} - u(y) \leq \ 0\,. \tag{4.15}$$

Therefore by (4.12)-(4.15) and by Lemma 3.4 we obtain:

$$\sup_{\Omega} v \ \leq \ C(\alpha\lambda)^{2-\frac{n}{p}}||(u\sum_{i=1}^n a_i\varphi_{x_i} - cu(y))^-||_{L^p(B)}\,, \tag{4.16}$$

where the constant C depends on $n, p, \nu_0, \mu_0, d^{1-\frac{n}{r}}\sum_{i=1}^n ||a_i||_{L^r(B(y,d))}$, $d^{2-\frac{n}{p}}||a||_{L^p(B(y,d))}, [p(a_{ij})]_{BMO(\mathcal{R}^n,\cdot)}, \omega_r[d^{1-\frac{n}{r}}a_i, B(y,d)], \omega_p[d^{2-\frac{n}{p}}a, B(y,d)]$. It follows from (4.16) that:

$$\sup_{\Omega} v \ \leq \ C_1\lambda^{2-\frac{n}{p}}||u\sum_{i=1}^n a_i\varphi_{x_i}||_{L^p(B)}\,, \tag{4.17}$$

with C_1 depending on the same parameters as C and on a_0. In particular, (4.17) yields

$$u(y) \le C_1 \lambda^{-\frac{n}{p}} u(y) || \sum_{i=1}^{n} a_i \varphi_{x_i} ||_{L^p(B)} . \tag{4.18}$$

Since $|\varphi_{x_i}| \le \frac{2}{\alpha} \cdot \lambda$, applying the Hölder inequality if $p \le n$, from (4.18) it follows that

$$u(y) \le C_2 u(y) \lambda^{1-\frac{n}{r}} \sum_{i=1}^{n} ||a_i||_{L^r(B)} \le C_2 u(y) \sum_{i=1}^{n} ||a_i||_{L^r(B)} , \tag{4.19}$$

with $C_2 \in \mathbb{R}_+$ depending on the same parameters as C_1. On the other hand, we can choose λ small enough, such that $\sum_{i=1}^{n} ||a_i||_{L^r(B)} \le \frac{1}{2C_2}$. Therefore it follows from (4.19) that

$$u(y) \le 0,$$

that is a contradiction. □

The following uniqueness result is an obvious consequence of Theorem 4.1.

Corollary 4.2. *If (h) is satisfied, the problem*

$$\begin{cases} u \in W^{2,p}_{\text{loc}}(\Omega) , \quad Lu = 0 , \\ \lim_{x \longrightarrow x_0} u(x) = 0 \quad \forall x_0 \in \partial\Omega , \\ \lim_{|x| \longrightarrow +\infty} u(x) = 0 \quad \textit{if } \Omega \textit{ is unbounded}, \end{cases} \tag{4.20}$$

has only the zero solution.

Proof. If we suppose $M = \sup_\Omega u > 0$, it results $M = u(y)$ for some $y \in \Omega$ and then u has a positive relative maximum in Ω. But this is not possible, by virtue of Theorem 4.1. Hence $\sup_\Omega u \le 0$. Since $-u$ verifies the same conditions of u, with the same arguments used in the above case, we have also that $\sup_\Omega(-u) \le 0$ and then we deduce that $u = 0$ in Ω. □

Remark 4.3. We point out that, if Ω is unbounded, the last assumption in (4.20) is needed, as the following simple example shows. Consider $w(x_1, x_2) = e^{x_1} \cos x_2$ in $\Omega = (-\infty, \infty) \times (-\frac{\pi}{2}, \frac{\pi}{2})$; it satisfies $\Delta w = 0$ in Ω and $w = 0$ on $\partial\Omega$.

References

[1] A.D. Aleksandrov, Majorization of solutions of second-order linear equations, *Vestnik Leningrad. Univ.*, **21**, No. 1 (1966), 5-25; English Translation in: *Amer. Math. Soc. Transl.*, **68** (1968), 120-143.

[2] X. Cabré, On the Alexandroff-Bakelman-Pucci estimates and reversed

Hölder inequality for solutions of elliptic and parabolis equations, *Comm. Pure Appl. Math.*, **48** (1995), 539-570.

[3] L. Caso, P. Cavaliere, M. Transirico, On the maximum principle for elliptic operators, *Math. Inequal. Appl.*, **7** (2004), 405-418.

[4] P. Cavaliere, M. Transirico, The Dirichlet problem for elliptic equations in the plane, *Comment. Math. Univ. Carolin.*, **46** (2005), 751-758.

[5] F. Chiarenza, M. Frasca, P. Longo, Interior $W^{2,p}$ estimates for non divergence elliptic equations with discontinuous coefficients, *Ricerche Mat.*, **40** (1991), 149-168.

[6] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Second Edition, Springer, Berlin-Heidelberg (1983).

[7] D. Greco, Nuove formole integrali di maggiorazione per le soluzioni di un'equazione lineare di tipo ellittico ed applicazioni alla teoria del potenziale, *Ricerche Mat.*, **5** (1956), 126-149.

[8] A.I. Koshelev, On boundedness in L^p of solutions of elliptic differential equations, *Mat. Sbornik*, **38** (1956), 359-372.

[9] C. Miranda, Sulle equazioni ellittiche del secondo ordine a coefficienti discontinui, *Ann. Mat. Pura Appl.*, **63**, No. 4 (1963), 353-386.

[10] C. Pucci, Operatori ellittici estremanti, *Ann. Mat. Pura Appl.*, **71**, No. 4 (1966), 141-170.

[11] C. Pucci, Limitazioni per soluzioni di equazioni ellittiche, *Ann. Mat. Pura Appl.*, **74**, No. 4 (1966), 15-30.

[12] M. Transirico, M. Troisi, A. Vitolo, BMO spaces on domains of $\mathbb{R}^n$, *Ricerche Mat.*, **45** (1996), 355-378.

[13] C. Vitanza, $W^{2,p}$-regularity for a class of elliptic second order equations with discontinuous coefficients, *Matematiche (Catania)*, **47** (1992), 177-186.

[14] C. Vitanza, A new contribution to the $W^{2,p}$-regularity for a class of elliptic second order equations with discontinuous coefficients, *Matematiche (Catania)*, **48** (1993), 287-296.