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ON A HILBERT-TYPE INEQUALITY WITH A HOMOGENEOUS KERNEL OF REAL NUMBER DEGREE

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Abstract: By introducing a homogeneous kernel of real number degree with an independent parameter and estimating the weight coefficient, a bilateral form of the Hilbert-type series inequality with a best constant factor is established.

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1. Introduction

If $a_n, b_n \geq 0, 0 < \sum\limits_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum\limits_{n=1}^{\infty} b_n^2 < \infty$, then (see [2])

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \pi \left\{ \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 \right\}^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well known as Hilbert's inequality. Soon after, inequality (1.1) had been generalized by Hardy-Riesz as (see [2]): If $a_n b_n \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{1/q}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1.2) is named

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of Hardy-Hilbert's integral inequality (see [2]). It is important in analysis and its applications. It was studied extensively and refinements, generalizations and numerous variants appeared in the literature (see [2]- [8]). Under the same condition of (1.2), we have the Hardy-Hilbert's type inequality (see [2], Theorem 341, Theorem 342) as follows

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}; \tag{1.3}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\log(m/n)}{m-n}a_m b_n < \pi^2\csc^2\frac{\pi}{p}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}, \tag{1.4}$$

where the constant factors pq and $\pi^2\csc^2\frac{\pi}{p}$ are both the best possible.

In 2008, Yang (see [7]) gave a bilateral inequality as follows: If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1, 0 < \lambda \le 2, a, b, c \ge 0, a + bc > 0$, $a_n, b_n \ge 0$, such that $0 < \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1}a_n^p < \infty, 0 < \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1}b_n^q < \infty$. Then

$$H := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{a\max\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}$$

$$< C_\lambda(a,b,c)\left\{\sum_{n=1}^{\infty}n^{p(1-\frac{\lambda}{2})-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{2})-1}b_n^q\right\}^{1/q}, \tag{1.5}$$

where the constant factor $C_\lambda(a,b,c)$ is the best possible. In addition, for $0 < p < 1$, Yang got the reverse inequality as follows

$$H := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{a\max\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}$$

$$> C_\lambda(a,b,c)\left\{\sum_{n=1}^{\infty}[1-\theta_\lambda(a,b,c,n)]n^{p(1-\frac{\lambda}{2})-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{2})-1}b_n^q\right\}^{1/q}, \tag{1.6}$$

where $\theta_\lambda(a,b,c,m) := \frac{1}{C_1(a,b,c)}\int_0^{1/m^\lambda}\frac{1}{a+b+cu}u^{-1/2}\mathbf{d}u = O(\frac{1}{m^{\lambda/2}}) \in (0,1)$, and the constant factor $C_\lambda(a,b,c)$ is the best possible. By the way, in recent years, the reverse form of the Hardy-Hilbert's inequality has been studied by Zhong (see [10]), Zhao (see [9]), and so on.

So far, we only focus on the Hilbert's inequality with negative number homogeneous and non-homogeneous kernel, but we just take the first step on the study of the real number homogeneous kernel. The main purpose of this article is to establish the bilateral form of the Hilbert's type inequality concerning series with the homogeneous kernel of real number degree.

2. Main Results

Lemma 2.1. *Setting $\lambda, \mu \in \mathbb{R}, \lambda + \mu > 0$, define the weight function $\varpi_{\lambda,\mu}(m)$ as*

$$\varpi_{\lambda}, \mu(m) := \sum_{n=1}^{\infty} \frac{(\min\{m,n\})^{\lambda}}{(\max\{m,n\})^{\mu}} \cdot \frac{m^{-(\lambda-u)/2}}{n^{1+(\lambda-\mu)/2}} \ (m \in \mathbb{N}), \tag{2.1}$$

then we have the following inequality:

$$\frac{4}{\lambda+\mu}[1-\theta_{\lambda}(m)] < \varpi_{\lambda,\mu}(m) < \frac{4}{\lambda+\mu}, \tag{2.2}$$

where $0 < \theta_{\lambda}(m) := \frac{1}{2m^{(\lambda+\mu)/2}} = O\left(\frac{1}{m^{(\lambda+\mu)/2}}\right) \in (0,1) \ (m \to \infty)$.

Proof. On one hand, setting $t = y/m$, by monotonicity, we have

$$\begin{aligned}
\varpi_{\lambda,\mu}(m) &< \int_0^{\infty} \frac{(\min\{m,y\})^{\lambda}}{(\max\{m,y\})^{\mu}} \cdot \frac{m^{-(\lambda-u)/2}}{y^{1+(\lambda-\mu)/2}} \mathrm{d}y = \int_0^{\infty} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \cdot t^{-1-\frac{\lambda-\mu}{2}} \mathrm{d}t \\
&= \int_0^1 \frac{t^{\lambda}}{1} \cdot t^{-1-\frac{\lambda-\mu}{2}} \mathrm{d}t + \int_1^{\infty} \frac{1}{t^{\mu}} \cdot t^{-1-\frac{\lambda-\mu}{2}} \mathrm{d}t \\
&= \frac{4}{\lambda+\mu}.
\end{aligned} \tag{2.3}$$

On the other hand, similarly, setting $t = y/m$, we get

$$\begin{aligned}
\varpi_{\lambda,\mu}(m) &> \int_1^{\infty} \frac{(\min\{m,y\})^{\lambda}}{(\max\{m,y\})^{\mu}} \cdot \frac{m^{-(\lambda-u)/2}}{y^{1+(\lambda-\mu)/2}} \mathrm{d}y = \int_{\frac{1}{m}}^{\infty} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \cdot t^{-1-\frac{\lambda-\mu}{2}} \mathrm{d}t \\
&= \frac{4}{\lambda+\mu} - \int_0^{\frac{1}{m}} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \cdot t^{-1-\frac{\lambda-\mu}{2}} \mathrm{d}t \\
&= \frac{4}{\lambda+\mu} - \int_0^{\frac{1}{m}} t^{-1+\frac{\lambda+\mu}{2}} \mathrm{d}t \\
&= \frac{4}{\lambda+\mu}\left[1 - \frac{1}{2m^{(\lambda+\mu)/2}}\right] = \frac{4}{\lambda+\mu}[1-\theta_{\lambda}(m)],
\end{aligned}$$

where $0 < \theta_{\lambda}(m) := \frac{1}{2m^{(\lambda+\mu)/2}} = O(\frac{1}{m^{(\lambda+\mu)/2}}) < 1$. Hence (2.2) is valid. The lemma is proved. □

Lemma 2.2. *If $p > 0, p \neq 1$, $\frac{1}{p} + \frac{1}{q} = 1, \lambda, \mu \in \mathbb{R}, \lambda + \mu > 0$ and*

$0 < \varepsilon < \frac{p(\lambda+\mu)}{2}$, *setting*

$$J(\varepsilon) := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(\min\{m,n\})^{\lambda}}{(\max\{m,n\})^{\mu}} \cdot m^{-(1+\frac{\lambda+\mu}{2})-\frac{\varepsilon}{p}} n^{-(1+\frac{\lambda+\mu}{2})-\frac{\varepsilon}{q}}, \tag{2.4}$$

then for $\varepsilon \to 0^+$, *we have*

$$\left(\frac{4}{\lambda+\mu} - o(1)\right)\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} < J(\varepsilon) < \left(\frac{4}{\lambda+\mu} + \widetilde{o}(1)\right)\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}. \tag{2.5}$$

Proof. Setting $t = \frac{x}{n}$ in the following, in view of Lemma 2.1, we get

$$\begin{aligned}
J(\varepsilon) &< \sum_{n=1}^{\infty} n^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{q}} \left[\int_0^{\infty} \frac{(\min\{x,n\})^{\lambda}}{(\max\{x,n\})^{\mu}} \cdot x^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{p}} \mathrm{d}x\right] \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left[\int_0^{\infty} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \cdot t^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{p}} \mathrm{d}t\right] \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left[\int_0^{1} t^{\frac{\lambda+\mu}{2}-\frac{\varepsilon}{p}-1} \mathrm{d}t + \int_1^{\infty} t^{-\frac{\lambda+\mu}{2}-\frac{\varepsilon}{p}-1} \mathrm{d}t\right] \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left(\frac{1}{\frac{\lambda+\mu}{2}-\frac{\varepsilon}{p}} + \frac{1}{\frac{\lambda+\mu}{2}+\frac{\varepsilon}{p}}\right) = \left(\frac{4}{\lambda+\mu} + \widetilde{o}(1)\right)\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} (\varepsilon \to 0^+);
\end{aligned}$$

$$\begin{aligned}
J(\varepsilon) &> \sum_{n=1}^{\infty} n^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{q}} \left[\int_1^{\infty} \frac{(\min\{x,n\})^{\lambda}}{(\max\{x,n\})^{\mu}} \cdot x^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{p}} \mathrm{d}x\right] \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left(\int_{\frac{1}{n}}^{\infty} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \cdot t^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{p}} \mathrm{d}t\right) \\
&> \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left(\int_0^{\infty} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \cdot t^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{p}} \mathrm{d}t\right) \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{n} \left(\int_0^{\frac{1}{n}} t^{\frac{\lambda+\mu}{2}-\frac{\varepsilon}{p}-1} \mathrm{d}t\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left[\frac{4}{\lambda+\mu} + \widetilde{o}(1) - \frac{1}{\frac{\lambda+\mu}{2}-\frac{\varepsilon}{p}} \left(\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{\lambda+\mu}{2}-\frac{\varepsilon}{p}}}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right)^{-1}\right] \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left(\frac{4}{\lambda+\mu} - o(1)\right) \quad (\varepsilon \to 0^+).
\end{aligned}$$

The lemma is proved. □

Theorem 2.3. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda, \mu \in \mathbb{R}, \lambda + \mu > 0, a_n, b_n \geq 0$ *such that* $0 < \sum_{n=1}^{\infty} n^{p(1+\frac{\lambda-\mu}{2})-1} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{q(1+\frac{\lambda-\mu}{2})-1} b_n^q < \infty$. *Then we have the following inequality*

$$I := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^{\lambda}}{(\max\{m,n\})^{\mu}} a_m b_n$$

$$< \frac{4}{\lambda + \mu} \left\{ \sum_{n=1}^{\infty} n^{p(1+\frac{\lambda-\mu}{2})-1} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q(1+\frac{\lambda-\mu}{2})-1} b_n^q \right\}^{1/q}, \tag{2.6}$$

where the constant factor $\frac{4}{\lambda+\mu}$ *is the best possible.*

Proof. By Hölder's inequality with weight[4], we obtain

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^{\lambda}}{(\max\{m,n\})^{\mu}} a_m b_n \\
= & \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^{\lambda}}{(\max\{m,n\})^{\mu}} \left[\frac{m^{(1+\frac{\lambda-\mu}{2})/q}}{n^{(1+\frac{\lambda-\mu}{2})/p}} a_m \right] \left[\frac{n^{(1+\frac{\lambda-\mu}{2})/p}}{m^{(1+\frac{\lambda-\mu}{2})/q}} b_n \right] \\
\leq & \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^{\lambda}}{(\max\{m,n\})^{\mu}} \frac{m^{(p-1)(1+\frac{\lambda-\mu}{2})}}{n^{1+\frac{\lambda-\mu}{2}}} a_m^p \right\}^{1/p} \\
& \times \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^{\lambda}}{(\max\{m,n\})^{\mu}} \frac{n^{(q-1)(1+\frac{\lambda-\mu}{2})}}{m^{1+\frac{\lambda-\mu}{2}}} b_n^q \right\}^{1/q} \\
= & \left\{ \sum_{m=1}^{\infty} \varpi_{\lambda,\mu}(m) m^{p(1+\frac{\lambda-\mu}{2})-1} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \varpi_{\lambda,\mu}(n) n^{q(1+\frac{\lambda-\mu}{2})-1} b_n^q \right\}^{\frac{1}{q}}.
\end{aligned}$$

In view of (2.2), we have (2.6).

For $0 < \varepsilon < \frac{p(\lambda+\mu)}{2}$, setting $\widetilde{a}_m = m^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{p}}$, $\widetilde{b}_n = n^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{q}}$ $(m, n \in \mathbb{N})$, by (2.4), we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^{\lambda}}{(\max\{m,n\})^{\mu}} \cdot m^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{p}} n^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{q}} = J(\varepsilon).$$

Assume that the constant factor $\frac{4}{\lambda+\mu}$ in (2.6) is not the best possible, then there exists a positive number k with $0 < k \leq \frac{4}{\lambda+\mu}$, such that (2.6) is still correct by changing $\frac{4}{\lambda+\mu}$ to k, then, in particular, by (2.5), we have

$$\left(\frac{4}{\lambda + \mu} - o(1) \right) \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} < J(\varepsilon)$$

$$< k\left\{\sum_{n=1}^{\infty} n^{p(1+\frac{\lambda-\mu}{2})-1}\widetilde{a}_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} n^{q(1+\frac{\lambda-\mu}{2})-1}\widetilde{b}_n^q\right\}^{1/q} = k\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}.$$

It follows that $\frac{4}{\lambda+\mu} - o(1) < k$, so $\frac{4}{\lambda+\mu} \le k(\varepsilon \to 0^+)$. Hence the constant factor $\frac{4}{\lambda+\mu}$ in (2.6) is the best possible. This completes the proof. □

Theorem 2.4. *If* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda, \mu \in \mathbb{R}, \lambda + \mu > 0, a_n, b_n \ge 0$ *such that* $0 < \sum_{n=1}^{\infty} n^{p(1+\frac{\lambda-\mu}{2})-1} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{q(1+\frac{\lambda-\mu}{2})-1} b_n^q < \infty$. *Then we have the following inequality*

$$I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^{\lambda}}{(\max\{m,n\})^{\mu}} a_m b_n$$

$$> \frac{4}{\lambda+\mu}\left\{\sum_{n=1}^{\infty}[1-\theta_{\lambda}(n)] n^{p(1+\frac{\lambda-\mu}{2})-1} a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} n^{q(1+\frac{\lambda-\mu}{2})-1} b_n^q\right\}^{1/q}, \quad (2.7)$$

where $0 < \theta_{\lambda}(m) := \frac{1}{2m^{(\lambda+\mu)/2}} = O(\frac{1}{m^{(\lambda+\mu)/2}}) \in (0,1)$ $(m \to \infty)$, *and the constant factor* $\frac{4}{\lambda+\mu}$ *is the best possible.*

Proof. By the reverse Hölder's inequality with weight [4] and in view of (2.1), we obtain

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^{\lambda}}{(\max\{m,n\})^{\mu}}\left[\frac{m^{(1+\frac{\lambda-\mu}{2})/q}}{n^{(1+\frac{\lambda-\mu}{2})/p}} a_m\right]\left[\frac{n^{(1+\frac{\lambda-\mu}{2})/p}}{m^{(1+\frac{\lambda-\mu}{2})/q}} b_n\right] \\ &\ge \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^{\lambda}}{(\max\{m,n\})^{\mu}}\frac{m^{p(1+\frac{\lambda-\mu}{2})-1}}{n^{1+\frac{\lambda-\mu}{2}}} a_m^p\right\}^{1/p} \\ &\quad\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^{\lambda}}{(\max\{m,n\})^{\mu}}\frac{n^{q(1+\frac{\lambda-\mu}{2})-1}}{m^{1+\frac{\lambda-\mu}{2}}} b_n^q\right\}^{1/q} \\ &= \left\{\sum_{m=1}^{\infty}\varpi_{\lambda,\mu}(m) m^{p(1+\frac{\lambda-\mu}{2})-1} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\varpi_{\lambda,\mu}(n) n^{q(1+\frac{\lambda-\mu}{2})-1} b_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

By (2.2), in view of $q < 0$, we have (2.7).

For $0 < \varepsilon < \frac{p(\lambda+\mu)}{2}$, setting $\widetilde{a}_m = m^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{p}}$, $\widetilde{b}_n = n^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{q}}$ $(m, n \in \mathbb{N})$, by (2.4), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^{\lambda}}{(\max\{m,n\})^{\mu}}\cdot m^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{p}} n^{-(1+\frac{\lambda-\mu}{2})-\frac{\varepsilon}{q}} = J(\varepsilon).$$

Assume that the constant factor $\frac{4}{\lambda+\mu}$ in (2.7) is not the best possible, then there exists a positive number k with $k \geq \frac{4}{\lambda+\mu}$, such that (2.7) is still correct by changing $\frac{4}{\lambda+\mu}$ to k, then, in particular, by (2.5), we have

$$\left(\frac{4}{\lambda+\mu}+\widetilde{o}(1)\right)\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} > J(\varepsilon)$$

$$> k\left\{\sum_{n=1}^{\infty}[1-\theta_{\lambda}(n)]n^{p(1+\frac{\lambda-\mu}{2})-1}\widetilde{a}_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1+\frac{\lambda-\mu}{2})-1}\widetilde{b}_n^q\right\}^{1/q}$$

$$= k\left\{\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}-\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{(\lambda+\mu)/2}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right\}^{1/q}$$

$$= k\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left\{1-\left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right)^{-1}\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{(\lambda+\mu)/2}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p}.$$

It follows that

$$\frac{4}{\lambda+\mu}+\widetilde{o}(1) > k\left\{1-\left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right)^{-1}\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{(\lambda+\mu)/2}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p},$$

and then $\frac{4}{\lambda+\mu} \geq k(\varepsilon \to 0^+)$. Thus the constant factor $\frac{4}{\lambda+\mu}$ in (2.7) is the best possible. The theorem is proved. □

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References

[1] M. Gao, On Hilbert's inequality and its applications, *J. Math. Anal. Appl.*, **212** (1997), 316-323.

[2] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press (1934).

[3] J. Kuang, On new extensions of Hilbert's integral inequality, *Math. Anal. Appl.*, **235** (1999), 608-614.

[4] J. Kuang, *Applied Inequalities*, Shangdong Science Press, Jinan (2004).

[5] D.S. Mitrinovic, J.E. Pecaric, A.M. Fink, *Inequalities Involving Functions and their Integrals and Derivatives*, Kluwer Academic, Boston (1991).

[6] B.G. Pachpatte, Inequalities similar to the integral analogue of Hilbert's inequality, *Tamkang J. Math.*, **30** (1999), 139-146.

[7] B. Yang, On a relation between Hilbert's inequality and a Hilbert-type inequality, *Applied Mathematics Letters*, **21**, No. 5 (2008), 483-488.

[8] B. Yang, *The Norm of Operator and Hilbert-Type Inequalities*, Science Press, Beijing (2009).

[9] C. Zhao, L. Debnath, Some new inverse type Hilbert integral inequalities, *Journal of Mathematical Analysis and Applications*, **262**, No. 1 (2001), 411-418.

[10] W. Zhong, B. Yang, A reverse form of multiple Hardy-Hilbert's integral inequality, *International Journal of Mathematical Inequalities and Applications*, **1**, No. 1 (2007), 67-72.