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AN INTERIOR-CONJUGATE GRADIENT MIX-METHOD FOR POSITIVE GEOMETRIC PROGRAMMING

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Abstract: The geometric programming is a special kind of nonlinear programming, and is widely used in the engineering design, production management, chemical process and so on. A new type interior-conjugate gradient mix-method is proposed for constrained positive geometric programming. First, the constrained positive geometric programming is transformed into an equally unconstrained geometric programming problem using interior penalty function method. Then particular parameters β_k are constructed based on conjugate gradient method. It is proved that the method is globally convergent.

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1. Introduction

The geometric programming constitutes an important class between the nonlinear

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optimization problems. Since its original development by Duffin, Peterson and Zener [1] at the Westinghouse *R&D* center, it has been studied extensively and has spawned a wide variety of applications, especially in the area of engineering design [2].

In this paper, it considers the constrained positive geometric programming and the constrained positive geometric programming is stated as follows:

$$\begin{cases} \min \quad g_0(t) := \sum_{j=1}^{m_0} c_{0j} \prod_{i=1}^{n} t_i^{\alpha_{0ji}}, \\ \text{s.t.} \quad g_l(t) := \sum_{j=1}^{m_l} c_{lj} \prod_{i=1}^{n} t_i^{\alpha_{lji}} \leq 1,\ l = 1, 2, \ldots, L, \\ \qquad t = (t_1, t_2, \ldots, t_n)^T > 0, \end{cases} \tag{1.1}$$

where $c_{lj>0}$, $l = 1, 2, \ldots, L$, α_{lji} is arbitrary real number and $m_0, m_1, \ldots, m_l$ are positive integers for $\sum_{l=0}^{L} m_l = m$.

Let $t_i = e^{x_i}$, $i = 1, 2, \ldots, n$, then, the top programming can transform it to the following form

$$\text{(GP)} \qquad \begin{cases} \min \quad f_0(x) := \sum_{j=1}^{m_0} c_{0j} e^{\sum_{i=1}^{n} \alpha_{0ji} x_i}, \\ \text{s.t.} \quad f_l(x) := \sum_{j=1}^{m_l} c_{lj} e^{\sum_{i=1}^{n} \alpha_{lji} x_i} \leq 1,\ l = 1, 2, \ldots, L, \end{cases} \tag{1.2}$$

where $x = (x_1, x_2, \ldots, x_n)^T$.

We can prove that (GP) is a convex programming [3]-[5] and if we find $x = (x_1^*, x_2^*, \ldots, x_n^*)^T$ which is the minimum point of (GP), then the minimum point of original programming is $t^* = (e^{x_1^*}, e^{x_2^*}, \ldots, e^{x_n^*})^T$.

Problem (GP) simplifies the objective function and constrained functions of original programming to the algebraic sum of index functions whose index are new variables. Then index function has the feature that easily to find the gradient vector and second-order derivative matrix. So it can reduce the computational workload and improve convergence rate of algorithm.

Inner penalty function method [6] transforms a constrained problem into a sequence of unconstrained problems. The optimal solution of the unconstrained problem can be made arbitrarily close to the optimal solution of the constrained problem. In this paper, using interior penalty function method to transform the constrained geometric programming problem into the unconstrained geometric programming problem, and because conjugate gradient method is one of the useful methods for unconstrained problem. We shall construct an interior-conjugate gradient mix-method and prove global convergence of the algorithm.

2. Constructive Algorithm

First, let $D = \{x \mid f_l(x) \leq 1,\ l = 1, 2 \ldots, L\}$, then (GP) can be simply recorded as:

$$\min_{x \in D} f_0(x).$$

Constructive penalty function $P(x, \rho) = s(x)r(\rho)$, where $r(\rho)$ satisfies the conditions: there is a descent sequence $\{\rho_k\}$ such that $\lim_{k \to +\infty} \rho_k = 0$ and $\lim_{k \to +\infty} r(\rho_k) = 0$ satisfies: $s(x) \geq 0$ for $x \in \mathrm{int} D$ and $s(x) \to +\infty$ for $x \to \partial D$. For example, we can select

$$r(\rho) = \rho^2,\ r(\rho_k) = \rho^k \quad (0 < \rho < 1),$$

$$s(x) = (\sum_{l=1}^{m_l} (\ln(-f_l(x) + 1))^2.$$

Let $F(x, \rho) = f(x) + p(x, \rho)$, then the construction of penalty function can guarantee that the minimum point of unconstrained programming problem $\min_{x \in R^n} F(x, \rho_k) = \min_{x \in R^n} (f(x) + p(x, \rho_k))$ always does not beyond constrained regional D and $\lim_{k \to \infty} p(x, \rho_k) = 0$. So $\min_{x \in R^n} f(x) = \lim_{k \to \infty} \min_{x \in R^n} \{f(x) + p(x, \rho_k)\}$. Thus, the solution of (GP) is transformed into finding the solution of

$$\min_{x \in R^n} F(x, \rho_k) = \min_{x \in R^n} (f(x) + p(x, \rho_k)).$$

Conjugate gradient method is one of the useful algorithms for unconstrained programming problem [7]. So we construct conjugate gradient algorithm and its iteration is:

$$x_{k+1} = x_k + \alpha_k d_k, \tag{2.3}$$

$$d_k = \begin{cases} -g_k & k = 1, \\ -g_k + \beta_k d_{k-1} & k \geq 2, \end{cases} \tag{2.4}$$

where $d_k = \nabla f(x_k)$ and the different selections of β_k represent different conjugate gradient algorithm [8]-[9]. In this paper, we select

$$\beta_k = \frac{\| g_k \|^2}{(1 - \mu) d_{k-1}^T g_k - d_{k-1}^T g_{k-1}}, \tag{2.5}$$

where $\mu \in [0,1]$. It is the mixture of CD method and DY method, that is, $\beta_k = \beta_k^{DY}$ for $\mu = 0$ and $\beta_k = \beta_k^{CD}$ for $\mu = 1$.

Assumption H for $F(x)$:

(i) $F(x)$ is a continuously differentiable function and has lower bound on R^n.

(ii) The gradient function $g(x)$ of $F(x)$ satisfies *Lipschitz* condition, that is, there exists a constant $L > 0$ such that

$$\| g(x) - g(y) \| \leq L \| x - y \|, \forall\, x,\, y \in R^n.$$

Step length α_k is determined by the generalized *Wolfe* inexact line search:

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k, \tag{2.6}$$

$$\sigma_1 g_k^T d_k \leq g(x_k + \alpha_k d_k)^T d_k \leq -\sigma_2 g_k^T d_k, \tag{2.7}$$

where σ_1 and σ_2 are constants, with $0 < \delta \leq \sigma_1 < 1$ and $\sigma_2 \geq 0$ and $\sigma_1 + \sigma_2 = 1$.

3. Convergence Properties

Lemma 3.1. (Descent Condition) *Suppose that objective function $F(x)$ satisfies assumption H, and consider iterative methods (2.3) and (2.4), where we select β_k which satisfies (2.5) and step length α_k is determined by (2.6) and (2.7), then we get $d_k^T g_k < 0$ for all $k \geq 1$.*

Proof. When $k = 1$, we get $d_1^T g_1 = - \| g_1 \|^2 < 0$.

Suppose $k > 1$, we have $d_{k-1}^T g_{k-1<0}$, hence

$$\begin{aligned} d_k^T g_k &= - \| g_k \|^2 + \beta_k d_{k-1}^T g_k \\ &= - \| g_k \|^2 + \frac{\| g_k \|^2 \, d_{k-1}^T g_k}{(1-\mu) d_{k-1}^T g_k - d_{k-1}^T g_{k-1}} \\ &\leq - \| g_k \|^2 + \frac{\| g_k \|^2 \, (-\sigma_2 d_{k-1}^T g_{k-1})}{\sigma_1 (1-\mu) d_{k-1}^T g_{k-1} - d_{k-1}^T g_{k-1}} \\ &= - \| g_k \|^2 + \frac{-\sigma_2}{\sigma_1(1-\mu) - 1} \| g_k \|^2 \\ &= (-1 + \frac{-\sigma_2}{\sigma_1(1-\mu)-1}) \| g_k \|^2 = \frac{\sigma_1 \mu}{\sigma_1(1-\mu)-1} \| g_k \|^2 . \end{aligned}$$

Since $0 < \sigma_1 < 1$, $\sigma_1 + \sigma_2 = 1$, $\mu \in [0,1]$, we get $\sigma_1(1-\mu) - 1 < 0$. That is $d_k^T g_k < 0$. This completes the proof. □

Lemma 3.2. *Suppose that objective function $F(x)$ satisfies assumption H, and consider the general method $x_{k+1} = x_k + \alpha_k d_k$, where $d_k^T g_k < 0$ and* *step length α_k is determined by (2.6) and (2.7). Then*

$$\sum_{k\geq 1}\frac{(d_k^T g_k)^2}{\| g_k \|^2} < +\infty.$$

This relationship is called Zoutendijk condition (see [10]).

Theorem 3.1. *Suppose that objective function satisfies assumption H, and considering iterative methods (2.3) and (2.4). Step length α_k is determined by (2.6) and (2.7) and $\sigma_2 = 0$. When β_k satisfies (2.5) then or $g_k = 0$ for some k, or $\lim_{k\to\infty} \inf \| g_k \|^2 = 0$.*

Proof. Suppose, by contradiction, that there exists a constant $c > 0$ such that $\| g_k \|^2 \geq c$, $k = 1, 2 \ldots$. By equation (2.5) and $\sigma_2 = 0$, we get

$$\beta_k^{CD} \leq \beta_k \leq \beta_k^{DY},$$

noting $\beta_k^{DY} = \frac{g_k^T d_k}{g_{k-1}^T d_{k-1}}$, according to $d_k + g_k = \beta_k d_{k-1}$, we have by taking square of the modulus of its both sides that

$$\| d_k \|^2 = -\| g_k \|^2 - 2g_k^T d_k + \beta_k^2 \| d_{k-1} \|^2 .$$

By dividing the same quantity $(g_k^T d_k)^2$ in both sides, we have

$$\begin{aligned}
\frac{\| d_k \|^2}{(g_k^T d_k)^2} &= \frac{\beta_k^2 \| d_{k-1} \|^2}{(g_k^T d_k)^2} - \frac{2}{g_k^T d_k} - \frac{\| g_k \|^2}{(g_k^T d_k)^2} \\
&\leq \frac{\| d_{k-1} \|^2}{(g_{k-1}^T d_{k-1})^2} - \left(\frac{\| g_k \|^2}{(g_k^T d_k)^2} + \frac{2}{g_k^T d_k} + \frac{1}{\| g_k \|^2}\right) + \frac{1}{\| g_k \|^2} \\
&= \frac{\| d_{k-1} \|^2}{(g_{k-1}^T d_{k-1})^2} - \left(\frac{\| g_k \|}{g_k^T d_k} + \frac{1}{\| g_k \|}\right)^2 + \frac{1}{\| g_k \|^2} \\
&\leq \frac{\| d_{k-1} \|^2}{(g_{k-1}^T d_{k-1})^2} + \frac{1}{\| g_k \|^2}.
\end{aligned}$$

Noting

$$\frac{\| d_1 \|^2}{(g_1^T d_1)^2} = \frac{1}{\| g_1 \|^2},$$

so from the above we have

$$\frac{\| d_k \|^2}{(g_k^T d_k)^2} \leq \sum_{i=1}^{k} \frac{1}{\| g_k \|^2} \leq \frac{k}{c}.$$

Hence

$$\frac{(g_k^T d_k)^2}{\| d_k \|^2} \geq \frac{c}{k},$$

that is

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\| d_k \|^2} = +\infty,$$

violating Lemma 3.2. This completes the proof. □

References

[1] R.J. Duffin, E.L. Peterson, C. Zener, *Geometric Programming Theory and Application*, Wiley, New York (1967).

[2] Zhang Kecun, *Geometric Programming and Optimal Design*, Xi'an Jiaotong University (1991).

[3] Zhang Kecun, Zhao Yingliang, *Algorithm and Analysis of Numerical Calculation*, Science Press (2003).

[4] Zhang Kecun, Wang Yanjun, A new algorithm for signomial geometric programming without constraints, *Numerical Mathematics: A Journal of Chinese Universities*, **4** (2001), 318-323.

[5] Zhang Kecun, Yang Boting, The new development of geometric programming, *Journal of Xi'an Jiaotong University*, **6** (1998), 90-94.

[6] Chengxian Xu, Chen Zhiping, Li Naicheng, *Modern Optimization Methods*, Science Press (2002).

[7] Dai Yuhong, Yuan Yaxiang, *Nonlinear Conjugate Gradient Methods*, Shanghai Science and Technology Press (2000).

[8] Gaohang Yu-Yutai Guan, A conjugate gradient method with sufficient descent property for large-scale optimization problems, *Journal on Numerical Method and Computer Application*, **3** (2006), 183-190.

[9] Shi Zhenjun, A class of globally convergent algorithms for nonlinear programming problem, *Chinese Journal of Engineering Mathematics*, **14**, No. 1 (1997), 26-32.

[10] G. Zoutendijk, Nonlinear programming, computational method, In: *Integer and Nonlinear Programming* (Ed. J. Abadie J), Amsterdam, North-Holland (1970), 37-86.