International Journal of Pure and Applied Mathematics
Volume 57 No. 5 2009, 605-627

ON ERROR ESTIMATION FOR APPROXIMATION METHODS INVOLVING DOMAIN DISCRETIZATION I: PROBLEM SETTING

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Abstract: This is the first of a sequence of 12 papers, including also [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] (in this order), dedicated to the study of error estimates for approximation problems based on discretization of the domain of the approximated functions, and (in the concluding paper [32]) comparison of the similarities and differences with the error estimates derived by alternative approximation methods (typically, of projection type) based on finite-dimensional subspaces of functions having the same continual domain as the target function. This paper is dedicated to the following:

- In Section 1: indicating general classes of problems in numerical analysis divided into respective pairs, depending on whether the definition domain is being discretized into a mesh, or this domain is continual.
- In Section 2: introducing tools for measuring the error, such as integral and averaged moduli of smoothness, related concepts, and comparisons of properties.
- In Section 3: specifying the research objectives of the research in the entire sequence of papers [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].
- In Section 4: detailed specification of the model problems to be considered in [22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

Received: May 8, 2009

In Section 5 we discuss past and future work on some of the classes of problems listed in Section 1 and not included in the study in [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32], as well as future extension of the results about the linear case in the latter study to nonlinear problems.

AMS Subject Classification: 65L12, 65L70, 65M15, 65M22, 26A15, 26A16, 34A30, 34G05, 34G10, 35G16, 35K05, 35K20, 39A06, 39A14, 39A70, 41A05, 41A25, 41A55, 41A63, 46E35, 46E39, 46N20, 46N30, 46N40, 47B38, 47B39, 47D03, 47D06, 47E05, 47F05, 47N20, 47N30, 47N40, 62D99, 62G08, 65C60, 65D05, 65D10, 65D15, 65D25, 65D30, 65D32, 65J10, 65L10, 65L20, 65L60, 65M05, 65M06, 65M10, 65M12, 65M60, 65N05, 65N06, 65N08, 65N10, 65N12, 65N15, 65N22, 65U05
Key Words: error, estimate, approximation, step, convergence, rate, order, domain, mesh, discretization, discrete, semi-discrete, continual, numerical, analysis, differentiation, integration, pointwise interpolation, quasi-interpolation, fitting, finite difference scheme, finite element, non-parametric, regression, deterministic, indeterministic, random, independent, identically distributed, bias, variance, sample, knot, risk, modulus of smoothness, integral, averaged, Riemann sum, uniform, non-uniform, K-functional, Lebesgue space, sequence space, Sobolev space, Triebel-Lizorkin space, Besov space, interpolation space, real, complex, A-space, Wiener amalgam, metric, norm, quasi-norm, isomorphism, embedding, bound, equivalence constant, initial, boundary, initial-boundary, differential equation, ordinary, partial, stationary, non-stationary, template, functional, linear, non-linear, positive, univariate, multivariate, multidimensional

1. Introduction

This is the first of a sequence of 12 papers, including also [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] (in this order), dedicated to the study of error estimates for approximation problems based on *discretization of the domain* of the approximated functions, and (in the concluding paper [32]) comparison of the similarities and differences with the error estimates derived by alternative approximation methods (typically, of projection type) based on finite-dimensional subspaces of functions having the same *continual domain* as the target function. There follows a (very inexhaustive) list of important general types of problems in numerical analysis for which this study is of interest. Each of these types is divided into respective pairs, depending on whether (a) the definition domain is being discretized into a mesh, or (b) this domain is continual:

1. interpolation or fitting problems involving a finite number of functionals which are of type
   (a) pointwise interpolation: interpolated are values of functions, or, which is the same, 'integral' functionals with a kernel which is a Dirac δ-function (singular distribution), so that only a finite discrete subset (mesh) of the domain of the approximated function is being covered;
   (b) quasi-interpolation: interpolated are integrals of functions with a kernel which is a regular (locally integrable) distribution, and so that the whole domain of the approximated function is being covered;
2. solving operator equations by
   (a) collocation methods for finding 'classical' solutions;
   (b) Galerkin-Petrov methods for finding 'weak' solutions;
3. numerical methods involving computations of inner products
   (a) by numerical quadratures;
   (b) by exact computation of the integrals (e.g., for spline functions);
4. numerical methods for solving initial and boundary problems for ordinary and partial differential equations
   (a) by finite difference schemes;
   (b) by finite/boundary elements;
5. in anisotropic problems where the properties of the solution are very different in different variables: for example, for solving time-dependent problems (in, possibly, time-dependent domains)
   (a) by finite difference schemes in the time variable;
   (b) by finite/boundary elements, finite volumes or semi-discrete finite difference schemes (finite difference schemes which are defined over a continual domain);
6. in statistics, for risk estimation in regression problems where the estimated quantity is not a finite-dimensional vector of parameters, but a function belonging to an infinitely dimensional function space – the so-called non-parametric regression problems

(a) in the case of deterministic design of the knot-vector, i.e., when the noisy observations of the values of the estimated function are registered at some a priori determined values of the argument (constituting the knot-vector of the sample);

(b) in the case of random design of the knot-vector, i.e., when the the values of the argument are random, e.g., uniformly distributed.

2. Functional Characteristics Used in Error Estimation

An important research topic in approximation theory since the 1960-s is to study and develop tools for error estimation which sharpen the assumptions of classical constructive theory of functions under which a certain convergence rate is proved. This has lead to the introduction of various functional characteristics measuring regularity, such as *functional moduli of continuity / regularity / smoothness*.

2.1. Integral Moduli of Smoothness

One of the important types of functional moduli of regularity is *the integral modulus (or ω-modulus) of smoothness*:

$$\omega_k(f;t)_{L_p(\Omega)} = \sup_{0<\|h\|\leq t} \|\Delta_h^k f\|_{L_p(\Omega_{k,t})}, \quad 0<p<\infty,$$

$$\omega_k(f;t)_{L_\infty(\Omega)} = \sup_{0<\|h\|\leq t} \sup_{x\in\Omega_{k,h}} |\Delta_h^k f(x)|, \quad \text{where } k\in\mathbb{N},\ t>0,\ h\in\mathbb{R},$$

$$\Delta_h^k f(y) = \sum_{n=0}^{k}(-1)^{k+n}\binom{k}{n} f(y+nh),$$

$$y\in\Omega_{k,h},\ \Omega_{k,h} = \{x : x, x+kh\in\Omega\}\,.$$

Here $L_p(\Omega) = L_p(\Omega, dx)$ are the customary real-valued Lebesgue spaces, here and in the sequel Ω is an interval, f is measurable respective to the usual Lebesgue measure. Here we give in detail only the definition of the integral moduli in the case of one-dimensional domain Ω; for their detailed definition in the multi-dimensional case, see, e.g., [43].

Another important and much more general characteristics used to measure regularity in abstract spaces is the Peetre K-functional

$$K(t,f;A,B) = K(t,f) = \inf_{f=f_0+f_1}(\|f_0\|_A + t\|f_1\|_B),$$

where A, B are normed spaces (or, most generally, quasi-seminormed abelian groups).

There was a new, considerable advance in the estimation technique via integral moduli after the observation was made in the 1960s (see, e.g., [11]) that for $p : 1 \leq p \leq \infty$ the integral modulus is equivalent to a respective K-functional, i.e.,

$$\exists c_0, c_1 > 0 : c_0 K(t^k, f; A, B) \leq \omega_k(f;t)_{L_p} \leq c_1 K(t^k, f; A, B), \forall f \in A + B,$$

for appropriate A, B. (Here $A + B$ is the usual sum of A, B, where $\|.\|_{A+B} = K(1, .; A, B)$.) In the case of $\omega_k(f;t)_{L_p}$, $p : 1 \leq p \leq \infty,\ k \in \mathbb{N}$, the appropriate function spaces are: for A, the Lebesgue space L_p; for B, the homogeneous Sobolev space

$$\dot{W}_p^k(\Omega) = \Big\{ f : \mathrm{Dom} f = \Omega,\ \mathrm{Cod} f \subset \mathbb{R},\ f^{(k)} \in L_p(\Omega), \\ \|f\|_{\dot{W}_p^k(\Omega)} = \|f^{(k)}\|_{L_p(\Omega)} < \infty \Big\}, \\ 1 \leq p \leq \infty,\ k \in \mathbb{N},$$

where $\mathrm{Dom} f$ and $\mathrm{Cod} f$ are the domain and codomain of the function f, respectively.

We shall also use the denotation

$$\omega_k(f;t)_{L_p} \sim K(t, f; A, B),\ 0 < t < \infty,$$

where the equivalence constants are *independent* of the step t.

This equivalence made it possible to study properties of integral moduli from the point of view of function-space theory using powerful functional-analytic apparata and, in the first place, the theory of interpolation spaces. The outcome was a precise, convenient and general estimation technique via integral moduli (see, e.g., [48, 49, 46, 10, 51, 52, 2]) in which Sobolev, Besov and Triebel-Lizorkin funciton spaces play an important role (see, e.g., [5, 2, 45, 50]). In particular, a space scale which proves to be important for error estimation in terms of integral moduli of smoothness is the scale of *inhomogeneous Besov spaces*

$$B_{pq}^s(\Omega) = \Big\{ f : f, f^{(k)} \in L_p(\Omega), \|f\|_{B_{pq}^s(\Omega)} \\ = \|f\|_{L_p(\Omega)} + \left(\int_0^1 \left(t^{-s} \omega_k(f,t)_{L_p(\Omega)} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \Big\}, \\ 0 < p, q \leq \infty,\ k \in \mathbb{N},\ \frac{1}{\min\{p,1\}} - 1 < s < k.$$

2.2. Averaged Moduli of Smoothness

As noted in [3, 19, 20], integral moduli proved to have their limitations as an error estimation tool. In our present framework of the afore-mentioned types of problems 1–6 it is possible to identify the range of their applicability, as follows: *error estimates for approximation methods of type (b), i.e., when information about the approximating functions is assumed available over the entire continual definition domain of the approximated function.*

Another type of functional modulus of contnuity/smoothness is *the averaged modulus (or τ-modulus) of smoothness*

$$\tau_k(f;t)_{L_p(\Omega)} = \|\omega_k(f,.;t)\|_{L_p(\Omega)}, \quad 0 < p \leq \infty$$

where t, k, Ω are as with $\omega_k(f;t)_{L_p}$, and the local moduli of smoothness are defined by

$$\omega_k(f,x;t) = \sup\left\{\|\Delta_h^k f(y)\| : y, y+kh \in \left[x - \frac{kt}{2}, x + \frac{kt}{2}\right] \cap \Omega\right\}, x \in \Omega.$$

Averaged moduli were introduced later than integral moduli (see, e. g., [4], [7]) and have so far been finding an ever-increasing range of applications in numerical analysis and approximation theory in cases where integral moduli do not suffice (see, e. g., [1, 8, 9, 3, 21]). For this type of modulus of smoothness an equivalent K-functional has also been found (see [15, 3, 19, 20]). Typical for the K-functional equivalent to the averaged modulus, and essentially diverse from the K-functional equivalent to the integral modulus, is that in the former case *the spaces in the K-functional depend on the step of this K-functional*:

$$\tau_k(f;t)_{L_p} \sim K(t, f; A(t), B(t)),$$

while the equivalence constants are still *independent* of t, like in the case of $\omega_k(f;t)_{L_p}$. Here $A(t) = A_{p,t}$, $B(t) = \dot{W}^k_{p,t}$, where the spaces $A_{p,t}$, $\dot{W}^k_{p,t}$ are defined, as follows.

For $0 < p \leq \infty,\ 0 < t < \infty$

$$A_{p,t}(\Omega) = \left\{ f : f \in BM(\Omega),\ \|f\|_{A_{p,t}(\Omega)} = \left(\int_\Omega S(t,|f|;x)^p dx\right)^{\frac{1}{p}} \right\},$$

$$0 < p < \infty,$$

$A_{\infty,t}(\Omega) = BM(\Omega)$, where $S(t,f;x) = sup\{f(y) : y \in [x-t, x+t] \cap \Omega\}$ is the upper Baire's function,

$$BM(\Omega) = \{ \quad f : \text{Dom} f = \Omega, \text{Cod} f \in \mathbb{R}, \\ f - \text{measurable and bounded everywhere on } \Omega, \\ \|f\|_{BM(\Omega)} = \sup_{x\in\Omega} |f(x)| < \infty\}.$$

$$\dot{W}^k_{p,t}(\Omega) = \Big\{ f : \text{Dom} f = \Omega,\ \text{Cod} f \subset \mathbb{R},\ f^{(k)} \in A_{p,t}(\Omega), \\ \|f\|_{\dot{W}^k_{p,t}(\Omega)} = \|f^{(k)}\|_{A_{p,t}(\Omega)} < \infty \Big\},$$

$$0 < p \le \infty,\ 0 < t < \infty,\ k \in \mathbb{N}.$$

The analogue of Besov spaces $B^s_{pq}(\Omega)$ generated by the integral moduli of smooothness are A-spaces, see [47],

$$A^s_{pq}(\Omega) = \Big\{ f : f \in L_p(\Omega),\ \|f\|_{A^s_{pq}(\Omega)} = \|f\|_{L_p(\Omega)} \\ + \left(\int_0^1 \left(t^{-s} \tau_k(f,t)_{L_p(\Omega)}\right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \Big\},$$

$$0 < p, q \le \infty,\ 0 < s < k,\ k \in \mathbb{N},$$

whose (quasi-)norms are generated in the same way as with Besov spaces, but with the integral modulus $\omega_k(f,t)_{L_p(\Omega)}$ (short notation $\omega_k(f,t)_p$) replaced by the averaged modulus $\tau_k(f,t)_{L_p(\Omega)}$ (short notation $\tau_k(f,t)_p$).

2.3. Comparison between the Properties of the Integral and the Averaged Moduli of Smoothness

The properties of the integral moduli of smoothness have been studied in detail in both the univariate and multivariate case, while the properties of the averaged moduli of smoothness have so far been studied in comparable detail only in the univariate case, see [8, 19] and the references therein for a detailed list of their properties. The spaces $A_{p,t}$ were first introduced in [15, 3], see also [19, 20], and by now it is clear that they should be considered as an important component of the theory of the averaged moduli of smoothness, the (quasi-) norm in $A_{p,t}$ essentially corresponding to $\tau_0(f,t)_p$, just as the (quasi-)norm in L_p essentially corresponds to $\omega_0(f,t)_p$. Here we shall mention only the most important conclusions drawn in [3, 19, 20] about the diversity of the properties of the two types of moduli of smoothness:

1. In all cases, $\|f\|_{L_p} \leq \|f\|_{A_{p,t}}$, $\omega_k(f,t)_p \leq \tau_k(f,t)_p$, and these inequalities prove to be very essential for the range of applicability of the two types of moduli to error estimation, because

   (a) the right-hand sides of these inequalities are appropriate as error measures (in the case of the first inequality) and as measures of the convergence rate in the step t (in the case of the second inequality) for approximation methods based on discretization of the domain (of type (a), according to the classification in Section 1);

   (b) the left-hand sides of these inequalities are *not* appropriate for these purposes.

2. While the (quasi-)norm $\|f\|_{L_p}$ is independent of the approximation step t, the corresponding (quasi-)norm $\|f\|_{A_{p,t}}$ essentially depends on t, and this proves to be relevant to error estimation in several important ways: for instance,

   (a) while the space L_p is closed under the complex interpolation method $\mathcal{C}_\theta$ and the real interpolation method $\mathcal{K}_{\theta,p}$, $0 < \theta < 1$, when acting between any $p_0, p_1 : 0 < p_0, p_1 < \infty$, such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ (for notation and definition of $\mathcal{C}_\theta$ and $\mathcal{K}_{\theta,p}$, see [2, 19]), the space $A_{p,t}$ is closed under these conditions if, and only if, in A_{p_0,t_0}, A_{p_1,t_1} we have $t_0 \sim t_1 \sim t$, with equivalence constants independent of t, as the approximation step $t > 0$ converges to zero (see [3, 19]);

   (b) Comparing the Marchaud-type inequalities for integral and averaged moduli of smoothness, it is seen that in the case of the averaged moduli, an additional smoothness index $1/p$ appears in the right-hand side of the Marchaud type inequality, which is a Sobolev-embedding type of effect, due to $A_{p,t}$ being a Wiener amalgam space (see [20] and the references therein) which is locally L_∞-regular and globally L_p-regular, while L_p can be considered also as a Wiener amalgam space which is L_p-regular both locally and globally (see [19, 20]).

3. When for the smoothness index s it is fulfilled that $s > 1/p$, or if $s = 1/p$ and for the logarithmic metric index q of the spaces B^s_{pq} and A^s_{pq} it is fulfilled $0 < q \leq \min\{1, p\}$ (which essentially coincides with the range for which B^s_{pq} and A^s_{pq}, $0 < p \leq \infty$, contain *only continuous functions*, the spaces A^s_{pq} and B^s_{pq} are isomorphic (see [19] and the references therein). In the complement of this range for s, p, q to the maximal range $s > 0$, $0 < p, q \leq \infty$ of these parameters where the A-spaces are defined

(which is a subset of the maximal range where Besov spaces are defined), both A-spaces and Besov spaces contain discontinuous functions, but are otherwise essentially diverse. These diversities include the following.

(a) While the functions in A^s_{pq} all belong to the space BM and are, therefore, 'well behaved, possibly discontinuous, functions', it is well-known that the essentially larger space B^s_{pq} contains 'monstrous' unbounded fractal functions.

(b) There exist examples of functions $f \in BM$, for which $\omega_k(f,t)_p$ and $\tau_k(f,t)_p$, $p < \infty$, have essentially different rate of convergence to zero as the approximation step $t > 0$ tends to zero (see [42]).

(c) The scale of the Besov spaces is closed under real complex interpolation $\mathcal{C}_\theta$, $0 < \theta < 1$, and real interpolation $\mathcal{K}_{\theta,q(\theta)}$, $0 < \theta < 1$, $0 < q_0, q_1 \leq \infty$, when p_0, p_1 and $q(\theta)$ are such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q(\theta)} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ hold (see [2]), while it is known that the scale of A-spaces is not closed under such interpolation (see [19, 20, 44]).

3. Research Objectives

The objectives of the research in the sequence of papers [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] are, as follows.

1. To show that for the general classes of problems 1–6 considered in Section 1 the error of the approximation methods of type (a) (based on discretization of the domain) can be efficiently and sharply estimated, if in the place of the integral moduli of smoothness we use the averaged moduli and/or their equivalent K-functionals. As will be seen, the new technique produces results, which

   (a) essentially, coincide in the cases when integral moduli are also applicable;

   (b) provide essential new information when being out of the range of applicability of the integral moduli.

2. To continue the work initiated in [3, 17, 12, 13, 14, 16] towards developing, on the basis of the averaged and integral moduli of smoothness and elements of the theory of function spaces, a precise, convenient and very general method for error estimation which handles in an essentially

uniform way the derivation of error bounds for the classes of problems of type 4 and 5, (a) and (b), as defined in Section 1, for initial, boundary and initial-boundary problems for *linear* (ordinary and partial) differential equations

(a) in terms of regularity properties of the exact solution and those of its derivatives which are present in the respective differential equation;

(b) directly in terms of regularity properties of the data functions of the exact initial / boundary/ initial-boundary problem: boundary / initial values and/or coefficients / right-hand side of the linear differential equation.

3. To continue the research in [38, 39, 40, 34, 21, 35, 37, 36] on statistical non-parametric estimation of regression functions and densities. The task here will be to derive and compare analogous risk estimates for the cases of

   (a) deterministic design of the knots, leading to estimates of type (a) (i.e., for discrete meshes);

   (b) random design of the knots, leading to estimates of type (b) (i.e., for continual domains).

4. To compare the estimates obtained for the approximation methods of type (a) and (b), according to their classification in Section 1 (i.e., with, or without, discretization of the domain). To this end, we shall use results obtained in [19, 20] on the study of the properties of the averaged moduli of smoothness, their respective equivalent K-functionals, and the function spaces generated by them – the A-spaces; these results will be compared to the respective results readily available for the integral moduli of smoothness.

4. Model Problems

This section contains the setting of the model problems which will be considered in the sequence of papers [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32], in the context of which we shall seek to attain our objectives 1–4, as described in Section 3.

4.1. Semi-Discrete Approximating Problems for Linear Evolutionary PDEs

In [22, 23] we shall study and solve the problem about full discretization of semi-discrete approximating problems on a concrete model example: the following Cauchy problem.

Differential equation:

$$\frac{du}{dt}(x,t) = \left(P(x, \tfrac{\partial}{\partial x})u\right)(x,t) = \sum_{l=0}^{M} P_l \frac{\partial^l}{\partial x^l} u(x,t),$$
$$M \in \mathbb{N},\ P_l \in C^\infty(\mathbb{R}),\ l = 0, 1, \ldots, M;\ x \in \mathbb{R},\ 0 < t \le T < \infty. \tag{1}$$

Initial condition:

$u(x,0) = f(x),\ x \in \mathbb{R};\ f \in L_p(\mathbb{R}),\ 1 \le p \le \infty.$

Consider the following semi-discrete approximating problem:

Finite difference scheme:

$$u_h(x, t+d) = \sum_{\alpha \in I} c_\alpha(x,h) u_h(x + \alpha h, t),\ \text{where}$$
$$0 < h \le h_0,\ h_0 > 0;\ 0 < \alpha \le \lambda h^M,\ 0 < \lambda = \text{const};$$
$$x \in \mathbb{R},\ t \in \Sigma_d = \{0, d, \ldots, Nd = T\};\ |I| < \infty; \tag{2}$$
$$c_\alpha \in C^\infty(\mathbb{R} \times (0, h_0]; \mathbb{R}),\ \alpha \in I.$$

Initial condition:

$u_h(x,0) = f(x),\ x \in \mathbb{R}.$

This problem is discrete only in t. It is assumed that the value of the approximate solution $u_h(x,t)$ are known μ - a.e. in $\mathbb{R}$ (μ - the customary Lebesgue measure: $\mu(dx) = dx$). In practice, when solving the problem numerically, this assumption is unrealistic: it is necessary to discretize the values of x. The respective discrete problem has the same form as (2), but x belongs to a discrete mesh in $\mathbb{R}$, for example,

$$x \in \Sigma_h = \{x_\nu : x_\nu = \nu h,\ \nu \in \mathbb{Z}\}. \tag{3}$$

In practical problems Σ_h has to be finite, but this is going to be a partial case of our results (corresponding to f having compact support).

The main reason due to which in the literature (see [46], [51], [52], [48], [49], [10], [2]), the major emphasis has been put on the semi-discrete case is, as follows: when studying the rate of convergence of u_h to u when $h \to +0$, in the semi-discrete case it is possible to apply a diversity of tools of functional analysis and operator theory, such as, for example, harmonic analysis, theory of Fourier–multipliers in L_p, theory of strongly continuous uniformly bounded operator (semi-)groups, interpolation of Sobolev and Besov spaces, etc., which allows the derivation of very precise estimates of $\|u_h - u\|_{L_p}$. Most often, these estimates have the form:

$$\|u_h - u\|_{L_p} \le c h^{\varphi(s)} \|f\|_{p,s}, \tag{4}$$

where $0 \le s \le \mu,\ \mu > 0,\ \varphi(s)$ is monotonously increasing, $\|\cdot\|_{p,s}$ is the norm in some function space, depending on p, s (e.g., $B^s_{p\infty}$).

With fully discrete problems (3), the objective is to estimate in a similar way the respective discrete norm corresponding to (4):

$$\|u_h - u\|_{l^p(\Sigma_h)}, \tag{5}$$

where, as customary,

$$\|f\|_{l^p(\Sigma_h)} = \left(\sum_{\nu \in \mathbb{Z}} h |f(x_\nu)|^p \right)^{1/p}$$

is the p-th root of a Riemann sum over the uniform mesh Σ_h, see (3), approximating $\|f\|_{L_p(\mathbb{R})}$ with step h.

Let us analyze and compare the kind of information about the error provided by each one of the error norms of the two types (4) and (5):

Lemma 1. *(cf. also [14]) Let $g \in L_p,\ 1 \le p < \infty$. Then,*

$$\|g\|_{L_p(\mathbb{R})} = \left(\frac{1}{h} \int_{-\frac{h}{2}}^{+\frac{h}{2}} \|g\|^p_{l^p(\Sigma_h + \xi)} d\xi \right)^{\frac{1}{p}}.$$

Proof.

$$\|g\|_{L_p(\mathbb{R})} = \int_{-\infty}^{+\infty} |g(x)|^p dx = \sum_{k=-\infty}^{\infty} \int_{x_k - \frac{h}{2}}^{x_k + \frac{h}{2}} |g(x)|^p dx.$$

$\forall k \in \mathbb{Z}$ we make a change of variable $\xi : x = x_k + \xi$. Furthermore,

$$g \in L_p \Rightarrow$$

$$\|g\|^p_{L_p} < \infty \Rightarrow$$

$$\sum_{k=-\infty}^{\infty} \int\limits_{x_k-\frac{h}{2}}^{x_k+\frac{h}{2}} |g(x)|^p dx = \sum_{k=-\infty}^{\infty} \int\limits_{-\frac{h}{2}}^{+\frac{h}{2}} |g(x_k+\xi)|^p d\xi =$$

$$= \int\limits_{-\frac{h}{2}}^{+\frac{h}{2}} \sum_{k=-\infty}^{\infty} |g(x_k+\xi)|^p d\xi =$$

$$= \frac{1}{h} \int\limits_{-\frac{h}{2}}^{+\frac{h}{2}} \|g\|^p_{l^p(\Sigma_h+\xi)} d\xi. \qquad \square$$

Lemma 1 indicates that the estimates of the type (4) are an integral average of estimates of the type (5) for a family of difference schemes depending on the parameter $\xi \in \left[-\frac{h}{2}, +\frac{h}{2}\right]$. Therefore, the information provided by (4) has, in a certain sense, probabilistic nature, and does not refer to any concrete difference scheme. On the contrary, estimates in the norm $\|\cdot\|_{l^p(\Sigma_h)}$ provide information for a concrete difference scheme, defined over a concrete mesh Σ_h. Furthermore, estimates in the norm $\|\cdot\|_{A_{p,h}}$ (if it is possible to derive one), provide information about the error for "the worst" (for the concrete initial-value function) (non-uniform) mesh

$$\tilde{\Sigma}_h = \left\{ \tilde{x}_\nu : \tilde{x}_\nu \in \left[(\nu - \frac{1}{2})h, (\nu + \frac{1}{2})h \right), \ \nu \in \mathbb{Z} \right\}.$$

Based on results from [3, 19], in [22, 23] we shall propose a general method, by way of which estimates of the type (5), $1 \le p < \infty$, can be stereotypically derived from estimates of the type (4). (For $p = \infty$, $A_{p,h}$ and BM are isomorphic, with equivalence constants independent of t, hence, the estimates of the two types (4), (5) coincide.)

The method proposed here can also be applied for implicit schemes, since the theory of implicit semi-discrete difference schemes is analogous.

4.2. Fully Discrete Approximating Problems

In [22, 23], see Section 4.2, for a general model example of an initial-value problem, we apply an indirect method for error estimation based on results known in advance for the respective semi-discrete approximating problems.

In [24, 25, 26, 27, 28, 29] we shall apply a direct discrete method for error estimation for two model initial-boundary problems – one for an ordinary differential equation (ODE), and one for a partial differential equation (PDE).

4.2.1. Boundary-Value Problem for the Stationary ODE of Heat Conductivity and Gas Diffusion

In [24, 25, 26] we shall consider the following model problem:

$$\begin{aligned}&(k(x)h'(x))' - q(x)u(x) = -f(x),\ 0 < x < 1,\\ &u(0) = \alpha,\ u(1) = \beta,\end{aligned} \tag{6}$$

where: $\alpha, \beta \in \mathbb{R}$; $k(x) \geq c_0 > 0$, $q(x) \geq 0$, $\forall x \in [0,1]$; $k, q, f \in BM[0,1]$; $k,\ q,\ f$ have (eventually) only discontinuities of the first kind, forming a set with zero measure; "conjugation conditions" are fulfilled, as follows: u – continuous, ku' – continuous, $\forall x \in [0,1]$ (including the (eventual) points of discontinuity of $k,\ q,\ f$).

Solve (6) numerically via the following homogeneous conservative finite difference scheme (see [6]):

$$\begin{aligned}&(ay_{\bar{x}})_x - dy = -\varphi,\ x \in \mathring{\Sigma}_h,\\ &y_0 = \alpha, y_N = \beta,\end{aligned} \tag{7}$$

where $N \in \mathbb{N}$, $h = 1/N$, $\Sigma_h = \{x_\nu,\ \nu = 0, \ldots, N\}$, and $\mathring{\Sigma}_h = \{x_\nu,\ \nu = 1, \ldots, N-1\}$ is the sub-mesh of interior knots in Σ_h. Here and in the sequel we use the standard notation for forward and backward finite differences, see [6] and Section 4.2.2.

For the error $z_i,\ i = 0, 1, \ldots, N$, we get the uniquely defined linear problem

$$\begin{aligned}&(az_{\bar{x}})_x - dz = -\psi,\\ &z_0 = z_N = 0.\end{aligned} \tag{8}$$

Again ψ is the error of local approximation of the residual:

$$\psi = (au_{\bar{x}})_x - du + \varphi$$

Our purpose is to estimate the error z.

In view of the homogeneity, $a_i,\ d_i,\ \varphi_i$ are determined from

$$\begin{aligned}&a_i = A[k(x_i + sh],\\ &d_i = \mathrm{F}[q(x_i + sh)],\\ &\varphi_i = \mathrm{F}[q(x_i + sh)],\end{aligned} \tag{9}$$

where the template functional F is defined for $\bar{f}(s) \in BM\left[-\frac{1}{2}, \frac{1}{2}\right]$ and has the following properties

1. $\mathrm{F}[1] = 1$;

2. F is linear over $BM\left[-\frac{1}{2}, \frac{1}{2}\right]$;

3. F is positive: $\mathrm{F}[\bar{f}(s)] \geq 0$ for $\bar{f}(s) \geq 0,\ s \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

For the (possibly, non-linear) template functional A, see [6, p. 116]. Additional conditions are being imposed on A and F, in order to ensure rate of local approximation of the residual $O(h^2)$; in the case of F, this is $\mathrm{F}[s] = 0$; in the case of A, see [6, p. 118]. In particular, when $\mathrm{F}[\bar{f}(s)] = \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} f(\cdot + sh)ds,\ A[\bar{k}(s)] = \left(\int\limits_{-1}^{0} \frac{ds}{k(\cdot + sh)}\right)^{-1}$, and $a,\ d,\ \varphi$ are obtained via (9), the respective scheme is termed "the best" (cf. [6], [8]).

In our argumentation in [24, 25, 26], we shall be assuming that $k,\ q,\ f$ have the properties needed in the course of exposition. In particular, all arguments in the exposition are valid, if $k,\ q,\ f$ are piecewise continuous in $[0, 1]$ (not necessarily having bounded variation). In some remarks in [24, 25, 26], we shall show that our results continue to hold true also under assumptions on $k,\ q,\ f$ which are much more general than piecewise continuity, and are close to those in the formulation of (6).

4.2.2. Dirichlet Boundary-Value (Cauchy–Dirichlet Initial-Boundary) Problem for the Heat Equation

In [27, 28, 29] we shall consider the following model problem:

$$\begin{array}{l} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t), \quad x \in [0,1],\ \ t \in (0,T],\ \ T > 0, \\ u(x,0) = g(x), \quad x \in [0,1], \\ u(j,t) = g_j(t), \quad t \in [0,T],\ j = 0,1, \\ g(\cdot) \in BM[0,1],\ \ g_j(\cdot) \in BM[0,T],\ j = 0,1, \\ f \text{ -- defined and bounded everywhere on } \Omega = [0,1] \times [0,T] \text{ and measurable.} \end{array} \tag{10}$$

(Measurability is with respect to the customary one- and two-dimensional Lebesgue measures for $[0, 1], [0, T]$ and Ω, respectively.)

We assume that a "conjugation condition" is also fulfilled: u – continuous on the interior of Ω (cf. [6], p. 71)

Approximate (10) by the following homogeneous conservative finite difference scheme (see [6, pp. 110-119, 185-192]):

$$\begin{aligned}
&\tfrac{v_{i,j+1}-v_{ij}}{d} - \left(\sigma \tfrac{v_{i+1,j+1}-2v_{i,j+1}+v_{i-1,j+1}}{h^2} + (1-\sigma)\tfrac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{h^2}\right) = \varphi_{ij},\\
&\qquad x \in \mathring{\Sigma}_h = \{\xi_i : \xi_i = ih,\ i=1,\dots,N-1;\ Nh=1\},\\
&\qquad t \in \mathring{\Sigma}_d = \{\eta_j : \eta_j = jd,\ j=0,1,\dots,M-1;\ Md=T\},\\
&v_{i0} = g_i,\ x_i \in \{ih : i=0,1,\dots,N\} = \Sigma_h,\\
&v_{0j} = (g_0)_j,\ v_{1j} = (g_1)_j,\ j=0,1,\dots,M,\ t_j \in \{jd : j=0,1,\dots,M\} = \Sigma_d,
\end{aligned} \tag{11}$$

The error in quest $z_{ij} = v_{ij} - u_{ij}$ in (10, 11) is uniquely determined as the solution of the following finite difference problem (cf. [6], p. 189):

$$\begin{aligned}
&\tfrac{z_{i,j+1}-z_{ij}}{d} - \left(\sigma \tfrac{z_{i+1,j+1}-2z_{i,j+1}+z_{i-1,j+1}}{h^2} + (1-\sigma)\tfrac{z_{i+1,j}-2z_{i,j}+z_{i-1,j}}{h^2}\right) = \psi_{ij},\\
&\qquad i=1,2,\dots,N-1;\ j=0,1,\dots,M-1\\
&z_{i0} = 0,\ i=0,1,\dots,N\\
&z_{0j} = z_{1j} = 0,\ j=0,1,\dots,M.
\end{aligned} \tag{12}$$

(For simplicity of exposition, the approximation on the boundary is assumed to be exact.) Estimation of the RHS in (12) $\psi = \{\psi_{ij}\}$ yields the convergence rate and conditions for this rate concerning the local approximation of the residual; estimation of the solution of (12) $z = \{z_{ij}\}$ yields the rate and conditions for convergence to zero of the error of the considered finite difference method.

ψ is defined only for $i = 1, 2, \dots, N-1$. We extend its definition to $i = 0, N$, as follows: $\psi_0 = \psi_N = 0$.

φ_{ij} in (11) is an approximation of f_{ij}. More precisely, in view of the homogeneity of the finite difference scheme (see [6, pp. 116-119, 187])

$$\varphi_{ij} = \mathrm{F}[f(x+sh, t_{j+\frac{1}{2}})], \tag{13}$$

where the template functional F was defined in section 4.2.1.

4.3. Non-Parametric Regression

In [30, 31] we shall be deriving risk estimates for wavelet estimators for the standard non-parametric regression-function estimation problem: estimate an unknown function $f(x)$ on the basis of a sample of N noisy observations

$$Y_i = f(t_i) + \sigma\varepsilon_i, \quad i = 1, \dots, N, \tag{14}$$

where σ and ε_i represent the variance and error terms respectively.

The sample of random errors $\{\varepsilon_i\}$ will be assumed to consist of independent, identically distributed (i.i.d.) random variables.

In [30] we shall consider the case of deterministic design ($t_i = x_i$, $i = 1, \dots, N$), where the x_i-s are a priori selected knots forming a mesh. In this case the values Y_i are also i.i.d.

In [31] we shall consider the case of random design ($t_i = X_i$, $i = 1, \dots, N$), where X_i, $i = 1, \dots, N$, are random numbers from a cumulative distribution function (c.d.f.) $F^N : \mathbb{R}^N \to [0, 1]$. We shall be assuming that the sample X_i, $i = 1, \dots, N$ consists of i.i.d. random variables, under which assumption the Y_i's are independent.

5. Concluding Remarks and Future Work

In Section 4 we considered model problems corresponding to items 4, 5 and 6 in Section 1.

In [41], instead of using the integral or averaged moduli of smoothness, we used the Bramble–Hilbert Lemma to derive error estimates about quasi-interpolation (i.e., where integral moduli can be used) and pointwise interpolation (i.e., where averaged moduli can be used). However, our error estimates for the case of pointwise interpolation were only limited to the case of L_∞-metric, due to considerations about degenerating of the finite elements considered there. Future work in this direction includes, in the first place, *full-scale development of the theory of the averaged moduli of smoothness in the multivariate case* (so far this theory has been developed in detail only in the univariate case). This topic essentially covers item 1 in Section 1.

For error estimates about quadrature formulae on cartesian-product domains, it is needed to apply essentially only the univariate version of the averaged moduli of smoothness, similar to the case of the model problem considered in Section 4.2.2 (for the respective estimates, see [27, 28, 29]). This is why there are numerous results about error estimation for quadrature formulae in the multivariate case on cartesian-product domains (we shall not be providing references about this here). Much less is known about error estimates for quadratures on general domains (one of the ways being to reduce the problem again to cartesian-product domains by appropriate changes of variables). We intend to return on this topic again in the future, in relation with the so-called *Euler Beta-function B-splines* (an easily computable simplified form of generalized *expo-rational B-splines*, see [33]). This topic essentially covers item 3 in Section 1.

About item 2 in Section 1, we refer to the short announcement [18] and the references therein. Forthcoming is the complete publication of all proofs and details related to the results announced in [18]. These can be used for extending results of the types obtained in [24, 25, 26, 27, 28, 29] to a variety of nonlinear cases.

Acknowledgments

This work was partially supported by the 2007 and 2008 Annual Research Grants of the R&D Group for Mathematical Modeling, Numerical Simulation and Computer Visualization at Narvik University College, Norway.

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