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ON ERROR ESTIMATION FOR APPROXIMATION METHODS INVOLVING DOMAIN DISCRETIZATION III: DETERMINISTIC PROBLEMS II. FULL DISCRETIZATION OF SEMI-DISCRETE FINITE DIFFERENCE SCHEMES FOR LINEAR EVOLUTIONARY PDES II: THE GENERAL, POSSIBLY NON-PARABOLIC, CASE

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Abstract: This is the third of a sequence of 12 papers, preceded by [16, 17] and followed by [18, 19, 20, 21, 22, 23, 24, 25, 26] (in this order), dedicated to the study of error estimates for approximation problems based on discretization of the domain of the approximated functions.

This paper, together with [17], presents a new approach to error estimation of the numerical solution of initial-boundary problems for linear differential equations by *full discretization of semi-discrete approximating problems*. The material in [17] and the present communication covers all previously unpublished results in Chapter 2 of [3], which extend, generalize and complement the results of [28, 10, 11, 12] and improve upon results of Bergh, Brenner, Löfström, Peetre, Thomée, Wahlbin, Widlund and others obtained in the case of semi-discrete approximation of a Cauchy problem for a general class of linear evolutionary partial differential equations. While [17] treated the parabolic case which features higher rate of approximation due to the smoothing properties of the parabolic resolving operator, the present paper considers the general, possibly non-parabolic, case which exhibits lower rate of approximation. The class of essentially non-parabolic partial differential equations (PDEs) included in the present consideration contains, e.g., the linear hyperbolic PDEs of first

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order *with variable coefficients*. This particular class of PDEs has been studied earlier in [8], for the case of semi-discrete approximation, and restricted only to the case of *constant coefficients*. The results of [8] have been upgraded in [11, 12] for the case of fully discrete approximation, based on the estimates obtained in [8] for the semi-discrete case. For this particular narrow subclass of PDEs, the convergence rate provided by the general error estimate proved in Theorem 2 of the present paper coincides with the respective convergence rates in the error estimates obtained in [8, 11, 12]; however, the present result extends also to the case of *variable coefficients*.

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1. Introduction

This is the third of a sequence of 12 papers, preceded by [16, 17] and followed by [18, 19, 20, 21, 22, 23, 24, 25, 26] (in this order), dedicated to the study of error estimates for approximation problems based on *discretization of the domain* of the approximated functions, and (in the concluding paper [26]) comparison of the similarities and differences with the error estimates derived by alternative approximation methods (typically, of projection type) based on finite-dimensional subspaces of functions having the same *continual domain* as the target function.

Within this sequence, this is the second of the two papers (the first one being [17]) dedicated to the technique of error estimation via full discretization of an intermediate class of semi-discrete approximating problems for which an advanced estimation technique is available in the literature.

In [17] we studied in detail the special class of evolutionary PDEs where the stationary component of the differential operator is of *parabolic* type. The solution of such *parabolic* problems smooths out very rapidly away from the initial condition, which results in high convergence rates. In [17] we presented our study *in order of 'decreasing parabolicity properties'*, whereby every next result is *increasingly more general*, while the respective convergence rate *decreases*. The results in the present paper constitute a natural continuation of this ordering: here we study the most general situation, without any assumptions about parabolicity, thus including in consideration, e.g., also some hyperbolic cases, see Remark 4 in Section 6; *the respective convergence rates are lower than in the parabolic case*.

For the reader's orientation in the necessary background, we refer to [16, Section 1, item 4.a; Section 4.1] and the references therein.

2. Semi-Discrete Approximating Problems

Here we shall study and solve the problem about full discretization of semi-discrete approximating problems on a concrete model example (see also [16, Section 4.1]): the Cauchy problem

Differential equation:

$$\frac{du}{dt}(x,t) = \left(P(x,\tfrac{\partial}{\partial x})u\right)(x,t) = \sum_{l=0}^{M} P_l(x)\frac{\partial^l}{\partial x^l}u(x,t),$$
$$M \in \mathbb{N},\ P_l \in C^\infty(\mathbb{R}),\ l = 0,1,\dots,M;\ x \in \mathbb{R},\ 0 < t \le T < \infty. \tag{1}$$

Initial condition:

$u(x,0) = f(x),\ x \in \mathbb{R};\ f \in L_p(\mathbb{R}),\ 1 \le p \le \infty.$

Here $C^\infty(A;B) = \{f : \mathrm{Dom} f = A, \mathrm{Cod} f \subset B, f - \text{infinitely smooth}\}$, for appropriate $A,\ B$; $C^\infty(A) = C^\infty(A;A)$.

The resolving operator of (1) is denoted by $G = G(t),\ t \in (0,T]$.

Consider the following *semi-discrete approximating problem*:

Finite difference scheme:

$$u_h(x,t+d) = \sum_{\alpha\in I} c_\alpha(x,h)u_h(x+\alpha h,t), \text{ where}$$
$$0 < h \le h_0,\ h_0 > 0;\ 0 < \alpha \le \lambda h^M,\ 0 < \lambda = \text{const};$$
$$x \in \mathbb{R},\ t \in \Sigma_d = \{0, d, \ldots, Nd = T\};\ |I| < \infty; \tag{2}$$
$$c_\alpha \in C^\infty(\mathbb{R}\times(0,h_0];\mathbb{R}),\ \alpha \in I,$$

Initial condition:

$$u_h(x,0) = f(x),\ x \in \mathbb{R}.$$

The resolving operator of (2) is denoted by $G_h = G_h(t),\ t \in \Sigma_d$.

Without loss of generality, assume $h_0 \le 1$.

This approximating problem is discrete only in t. It is assumed that the value of the approximate solution $u_h(x,t)$ are known μ - a.e. in $\mathbb{R}$ (μ - the customary Lebesgue measure: $\mu(dx) = dx$) In practice, when solving the problem numerically this assumption is unrealistic: it is necessary to discretize the values of x. The respective discrete problem has the same form as (2), but x belongs to a discrete mesh in $\mathbb{R}$, for example,

$$x \in \Sigma_h = \{x_\nu : x_\nu = \nu h,\ \nu \in \mathbb{Z}\}. \tag{3}$$

In practical problems Σ_h has to be finite, but this is going to be a partial case of our results (corresponding to f having compact support).

The main reason due to which in the literature (see [27], [31], [32], [29], [30], [8], [1]), the major emphasis has been put on the semi-discrete case of estimating the continual norm $\|u_h - u\|_{L_p}$ was discussed in [16, Section 4.1, formula (4)].

With fully discrete problems (3), the objective is to estimate in a similar way the respective discrete norm $\|u_h - u\|_{l^p(\Sigma_h)}$, see [16, Section 4.1, formula (5)], where, as customary,

$$\|f\|_{l^p(\Sigma_h)} = \left(\sum_{\nu\in\mathbb{Z}} h|f(x_\nu)|^p\right)^{1/p}$$

is the p-th root of a Riemann sum over the uniform mesh Σ_h, see (3), approximating $\|f\|_{L_p(\mathbb{R})}$ with step h.

The kind of information about the error provided by each one of the estimates in the above-said continual and discrete norm was analyzed and compared in [16, Section 4.1, Lemma 1], see also the references in [16].

3. Preliminaries

3.1. Functional Moduli and Function Spaces

Here we present the notation for the functional characteristics needed in the sequel. As a general reference about the functional characteristics introduced in this section concerning the notation, definitions, meaning and range of parameters, relevant properties and interrelations, as well as further details, we cite [14] together with [15], and the references therein. Another concise reference source on the material in this section is [16, Section 2] and the references therein.

1. The sequence space $l^p(\Sigma_h)$ defined over the mesh Σ_h (see [28, 10, 11, 12, 3, 14]).
2. The inhomogeneous Sobolev space $W_p^m(\mathbb{R})$, with $W_p^0 = L_p$ (see, e.g., [1, 4, 14]).
3. The integral modulus of smoothness (ω-modulus) $\omega_m(f;h)_{L_p}$ (short: $\omega_m(f;h)_p$) (see [28, 10, 11, 12, 6, 3, 14], cf. also [4, 1] where the notation is modified).
4. The averaged modulus of smoothness (τ-modulus) $\tau_m(f;h)_{L_p}$ (short: $\tau_m(f;h)_p$) (see [6, 28, 10, 11, 12, 13, 3, 14]).
5. The Steklov-means $f_{k,t}$ (see [2, 5, 6, 14]).
6. The Wiener amalgam space $A_{p,h}(\mathbb{R})$, with norm $\|.\|_{A_{p,h}} = \tau_0(f;h)_{L_p}$ (see [12, 13, 3, 14, 15]).
7. The inhomogeneous Besov space $B_{pq}^s(\mathbb{R})$ (see [1, 28, 10, 11, 12, 13, 3, 14]).
8. The inhomogeneous A-space $A_{pq}^s(\mathbb{R})$, an analogue of the respective Besov space where the ω-modulus in the definition of the norm in B_{pq}^s is replaced by the respective τ-modulus (with the same parameters) in the definition of the norm in A_{pq}^s (see [28, 10, 11, 12, 13, 3, 14]).
9. The inhomogeneous Triebel–Lizorkin space $F_{pq}^s(\mathbb{R})$ (see [1, 3, 14]).
10. The Wiener–Young p-variation $\bigvee_{-\infty}^{\infty}{}_p\, g$ of g, the case $p = 1$ corresponding to the customary Jordan variation (see [33, 34]).

11. The Bessel potential $J^s f$ of f:
$$J^s f = \mathfrak{F}^{-1}\left((1+|.|^2)^{\frac{s}{2}}\,\mathfrak{F}f\right),$$
where $\mathfrak{F}, \mathfrak{F}^{-1}$ are the direct and inverse Fourier transform, respectively (cf. [4, 1]).

[16, Section 2.3] contains an important comparison between the properties of the integral and averaged moduli of smoothness.

Following our practice in [3] and [14], in order to distinguish between previously known results and the new ones obtained here, we shall add the additional marker '**(K.)**' (abbreviated from '(K)nown') to the enumeration of every statement in the sequel which has been previously known (with respective reference to available relevant literature).

3.2. Correct Posedness, Stability and Accuracy

Definition 1. **(K.)** (cf. [7], [29, p. 154].) We say that the Cauchy problem is correctly posed in L_p, if the resolving operators $G(t)$ of (1), $t \in [0,T]$ form a strongly continuous operator semi-group (see, e.g. [9]) in L_p with infinitesimal generating operator $P(x, \frac{\partial}{\partial x})$ (considered as an operator with dense definition domain $\mathfrak{D}(P) \subset L_p$):

$$(G(t)f)(x) = u(x,t),\ x \in \mathbb{R},\ t \in [0,T],$$

$$G(0)f = f$$

$$G(t_1+t_2) = G(t_1)G(t_2),\ t_1, t_2 \geq 0$$

$\|G(t)f\|_{L_p} \leq c\|f\|_{L_p},\ \forall f \in L_p;\ c = c(T), c \neq c(t,f)$ (stability of $G(t)$ in L_p).

$$\lim_{t\to+0} \|t^{-1}(G(t)f - f) - Pf\|_{L_p} = 0,\ f \in \mathfrak{D}(P).$$

Definition 2. **(K.)** We say that the problem (1) is strongly correctly posed, if it is correctly posed in W_p^m, $\forall m \in \mathbb{N} \cup \{0\}$.

In the case when f is a scalar-valued function (here we limit the considerations to only this case), a "correctly posed" problem and a "strongly correctly posed" problem are equivalent concepts (cf., e.g., [30]). This is why in the sequel we shall only be referring of "correctly posed" problems.

Definition 3. **(K.)** We say that the resolving operator $G_h(t) : L_p \to L_p$, $t \in \Sigma_d$ of (2) is *stable* in the normed space A, if it is bounded in A uniformly in $t \in \Sigma_d$.

Definition 4. **(K.)** (cf., e.g., [29], [30].) We say that G_h approximates G with convergence rate of order $\mu > 0$ (short: G_h is accurate of order μ), if for sufficiently smooth solutions of (1)

$$u(x, t + d) = G_h(d)u(x, t) + dO(h^\mu),\ h \to +0.$$

In [27] it is proved that, if (1) be correctly posed and G_h be accurate of order μ, then, $\| (G_h(d) - G(d))\, f\|_{L_p} \leq cdh^\mu \|f\|_{W_p^{M+\mu}},\ \forall f \in W_p^{M+\mu}$. (For some other equivalent definitions in the case of equations with constant coefficients, see [8], [1], [29].)

Definition 5. **(K.)** We say that the discrete problem (2) is correctly posed in L_p if G_h is stable in $W_p^m,\ m = 0, 1, \ldots, M + \mu$.

All problems (2) with c_α – constant in $x,\ \alpha \in I$, are correctly posed, because $G_h(t)f,\ \forall t \in \Sigma_d$, is a finite linear combination of values of f. There is also a large class of correctly posed problems (2) with coefficients c_α which are variable in x.

For the error operator we shall be using the notation $E_h = E_h(t) : E_h(t) = G_h(t) - G(t),\ t \in \Sigma_d$.

4. Cauchy Problem for Linear PDEs of Arbitrary Order with C^∞–Variable Coefficients: The General, Possibly Non-Parabolic, Case

In this section we formulate and prove the main new result in this paper, Theorem 2.

Similarly to the parabolic case, considered in [17], we begin by introducing the available result for the semi-discrete approximating problem which we shall use as an intermediate result needed in the proof of Theorem 2.

Theorem 1. **(K.)** (see [27]) *Let*

a. (1) be correctly posed in $L_p,\ 1 \leq p \leq \infty$*;*

b. G_h *be stable in* L_p *and accurate of order* $\mu \in \mathbb{N}$.

Then,

$\forall s \in (0, M+\mu),\ \forall T > 0 \Rightarrow\ \exists c > 0, c_s > 0:\ jd \le T \Rightarrow$

$$\|E_h(jd)f\|_{L_p} \le \begin{cases} ch^{\mu}\|f\|_{W_p^{M+\mu}} & \forall f \in W_p^{M+\mu}, \\ c_s h^{s\frac{\mu}{M+\mu}}\|f\|_{B^s_{p\infty}} & \forall f \in B^s_{p\infty}. \end{cases}$$

It is possible, analogously to [18], to use Theorem 1, valid for the semi-discrete case, in order to obtain its analogue for the fully discrete case, one of the conditions for this being (cf. section 5) that problem (2) be correctly posed. It turns out that we can get rid of this condition (which is essential only in the case of variable coefficients) by deriving a direct proof of a more general statement, as follows.

Theorem 2. *Let*

a. (1) be correctly posed in L_p;

b. G_h be accurate of order $\mu \in \mathbb{N}$;

c. G, G_h be stable in $A_{p,h}$.

Then,

$\forall T > 0 \Rightarrow\ \exists c > 0:$
$\|E_h(jd)f\|_{A_{p,h}} \le c\left(h^{\mu}\|f\|_{L_p} + \tau_{M+\mu}(f; h^{\frac{\mu}{M+\mu}})_{L_p}\right),\ \forall f \in A_{p,h}{+}W_p^{M+\mu},\ jd \le T.$
(Here, with no loss of generality, we assume $h \le 1$).

In the proof of Theorem 2 we shall need the following known lemma.

Lemma 1. **(K.)** (see [29, p. 155]) *Let*

a. (1) be correctly posed;

b. G_h be accurate of order μ.

Then,

$\forall m \in \mathbb{N} \cap \{0\}\ \exists c > 0$*:*

$$\|E_h(d)f\|_{W_p^m} \le cdh^{\mu}\|f\|_{W_p^{M+\mu+m}},\ \forall f \in W_p^{M+\mu+m}$$

Now we proceed to the proof of Theorem 2, as follows.

Proof. (Proof of Theorem 2.) We shall use an approach analogous to the one applied in the proof of Theorem 1 in [27]).

Consider $f_{r,h_1} : r \in \mathbb{N},\ r \ge M+\mu+1,\ h_1 > 0$. Then,

$$\|E_h(jd)f\|_{A_{p,h}} \leq \|E_h(f - f_{r,h_1})\|_{A_{p,h}} + \|E_h f_{r,h_1}\|_{A_{p,h}} \leq c\tau_r(f;\max\{h,h_1\})_{L_p} + \|E_h f_{r,h_1}\|_{A_{p,h}} \quad (4)$$

holds true. Here we used the stability of G G_h in $A_{p,h}$.

Let us estimate $\|E_h f_{r,h_1}\|_{A_{p,h}}$.

The following representation is known:

$$E_h(jd) = G_h(jd) - G(jd) = G_h(d)^j - G(d)^j = \sum_{\nu=0}^{j-1} G_h(d)^{j-\nu-1}(G_h(d) - G(d))G(d)^\nu \quad (5)$$

(see, e.g., [29, p. 156]).

$f_{r,h_1} \in W_p^{M+\mu+1}$, G_h – stable in $A_{p,h}$, "correctly posed problem" $\Leftrightarrow$ "strongly correctly posed problem" (see Section 2), [14, Lemma 8], Lemma 1 $\Rightarrow$

$$\|E_h(jd)f_{r,h_1}\|_{A_{p,h}} = \|\sum_{\nu=0}^{j-1} G_h((j-\nu-1)d)(G_h(d) - G(d))G(\nu d)f_{r,h_1}\|_{A_{p,h}} \leq$$

$$\leq \sum_{\nu=0}^{j-1} \|G_h((j-\nu-1)d)(G_h(d) - G(d))G(\nu d)f_{r,h_1}\|_{A_{p,h}} \leq$$

$$\leq c\sum_{\nu=0}^{j-1} \|(G_h(d) - G(d))G(\nu d)f_{r,h_1}\|_{A_{p,h}} \leq$$

$$\leq c\sum_{\nu=0}^{j-1} (\|(G_h(d) - G(d))G(\nu d)f_{r,h_1}\|_{A_{p,h}} + c\tau_1((G_h(d) - G(d))G(\nu d)f_{r,h_1}; h)_{L_p}) \leq$$

$$\leq c\sum_{\nu=0}^{j-1} (\|(G_h(d) - G(d))G(\nu d)f_{r,h_1}\|_{A_{p,h}} + ch\|(G_h(d) - G(d))G(\nu d)f_{r,h_1}\|_{W_p^1}) \leq$$

$$\leq c\sum_{\nu=0}^{j-1} \left(cdh^\mu\|G(\nu d)f_{r,h_1}\|_{W_p^{M+\mu}} + cdh^{\mu+1}\|G(\nu d)f_{r,h_1}\|_{W_p^{M+\mu+1}}\right) \leq$$

$$\leq c\sum_{\nu=0}^{j-1} \left(cdh^\mu c\|f_{r,h_1}\|_{W_p^{M+\mu}} + cdh^{\mu+1}c\|f_{r,h_1}\|_{W_p^{M+\mu+1}}\right) \leq$$

$$\leq cTh^\mu \left(\|f_{r,h_1}\|_{W_p^{M+\mu}} + h\|f_{r,h_1}\|_{W_p^{M+\mu+1}}\right) \leq$$

$$\leq cTh^\mu \left(\|f\|_{L_p} + h_1^{-M-\mu}\omega_{M+\mu}(f;h_1)_{L_p} + hh_1^{-M-\mu-1}\omega_{M+\mu+1}(f;h_1)_{L_p}\right)$$

Select $h_1 = h^{\frac{\mu}{M+\mu}}$.

$h \leq 1 \Rightarrow$

$\max\{h, h^{\frac{\mu}{M+\mu}}\} = h^{\frac{\mu}{M+\mu}} \Rightarrow$

$h^{1+\frac{\mu}{M+\mu}(-M-\mu-1)} = h^{1-\frac{\mu}{M+\mu}(M+\mu)-\frac{\mu}{M+\mu}} = h^{1-\frac{\mu}{M+\mu}}.h^{-\mu} \leq h^{-\mu} \Rightarrow$

$$\|E_h f_{r,h_1}\|_{A_{p,h}} \leq cT\left(h^{\mu}\|f\|_{L_p} + \omega_{M+\mu}(f; h^{\frac{\mu}{M+\mu}})_{L_p}\right). \tag{6}$$

(4), (6) $\Rightarrow$ the statement of the theorem. □

5. Some Implications

From Theorem 2 and the properties of the τ-moduli we can obtain many new corollaries. Here are some model examples:

Corollary 1. *Let*

a. $f \in L_p$;

b. $\bigvee_{-\infty}^{\infty} {}_p\, f < \infty$;

c. $1 \leq p < \infty$.

Then,

$$\|E_h f\|_{A_{p,h}} \leq ch^{\frac{1}{p(M+\mu)}}\left(\|f\|_{L_p} + \bigvee_{-\infty}^{\infty} {}_p\, f\right).$$

Corollary 2. *Let* $f \in A^s_{p\infty}$, $0 < s < M+\mu, 1 \leq p \leq \infty$.

Then,

$$\|E_h f\|_{A_{p,h}} \leq ch^{s\frac{\mu}{M+\mu}}\|f\|_{A^s_{p\infty}}.$$

Corollary 3. *Let*

a. $\frac{1}{p} < s < M+\mu-1$;

b. $f \in F^s_{p2}$;

c. $\bigvee\limits_{-\infty}^{\infty} {}_p\, J^s f < \infty$;

d. $1 \le p < \infty$.

Then,

$$\|E_h f\|_{A_{p,h}} \le c h^{(s+\frac{1}{p})\frac{\mu}{M+\mu}} \left(\|f\|_{L_p} + \bigvee_{-\infty}^{\infty} {}_p\, J^s f \right).$$

The proofs of Corollaries 1, 2 are derived by straightforward argument based on the properties of the averaged moduli. The proof of Corollary 3 is analogous to that of [17, Corollary 1].

6. Concluding Remarks

Remark 1. The method proposed here can also be applied for implicit schemes, since the theory of implicit semi-discrete difference schemes is analogous.

Remark 2. The method developed and the results obtained in this paper and the subsequent paper are an application of the general theory developed in [14], [15].

Remark 3. The material in this paper covers the part of the previously unpublished results obtained in [3, Chapter 2] and complementary to the results about the parabolic case published in [17].

Remark 4. In [11, 12] we studied the convergence of the approximate solution of (2) to the solution of (1) for a linear hyperbolic equation of first order *with constant coefficients*. There we obtained error estimates based on respective results for the semi-discrete problem (see [8]). We can discuss the present result in Theorem 2 in comparison with the results in [8, 11, 12], as follows.

1. The present result is ***more general*** than the results in [11, 12].
   (a) The requirement in [11, 12] about the coefficients in the PDE being ***constant*** was a consequence of the essential use of the results for

the semi-discrete case obtained in [8]. In [8] constraining the considerations to only constant coefficients was essential, due to to the fact that the technique developed in [8] for derivation of error estimates in the semi-discrete case was based on the theory of Fourier multipliers in L_p.

(b) In the present result of Theorem 2 the coefficients in the PDE can be ***variable***.

2. For the particular case of constant coefficients in the PDE, the new general error estimate in Theorem 2 ***exactly coincides*** with the respective error estimates obtained in [11, 12].

3. The study of the semi-discrete case in [8] was conducted in a more general setting than the one in [11, 12]. In [8] the case of ***systems of linear hyperbolic PDEs of first order with constant coefficients*** was treated in essentially the same way as the case of an individual linear hyperbolic PDE of first order with constant coefficients, by considering vector-valued, rather than scalar-valued, Fourier multipliers in L_p. In [8] it was also shown that this upgrade was not entirely straightforward; for example, it was shown that ***non-commutativity*** of matrices plays a relevant role, thus also necessitating the introduction of the concept of a ***strongly correctly posed problem*** (1) (see Definition 2 in Section 3.2) which in the case of systems of PDEs is ***distinct*** from the concept of a ***correctly posed problem*** (1) (see Definition 1 in Section 3.2).

4. Similar to [11, 12], the study conducted in [17] and the present paper (and originating in [3]) does not consider the vector-valued case of systems of PDEs in problem (1). However, this limitation is not essential; at the time of writing [3] it was imposed to avoid expanding the volume of the study due to the distinctness of the concepts of a correctly posed and a strongly correctly posed problem (1) in this case, which necessitates also the consideration of new distinct cases for the approximating problem (2). ***Future work*** on the topic includes extending the results of [17] and the present paper to the more general case of ***fully discrete approximate solution of systems of linear, not necessarily parabolic, evolutionary PDEs with variable coefficients***. The particular case of systems of linear hyperbolic equations of first order with constant coefficients would then be an upgrade of the results of [8] from the the semi-discrete to the fully discrete case, while the new results would also be extending the results of [8] to the case of variable coefficients. The

main tool to be used in this more general situation is again ***the identity (5)*** which is specially designed to be used in non-commutative cases, and extends fairly straightforwardly to the case of linear system of PDEs with variable coefficients, with respective upgrading of the approximating problem (2).

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