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ON ERROR ESTIMATION FOR APPROXIMATION METHODS INVOLVING DOMAIN DISCRETIZATION IV: DETERMINISTIC PROBLEMS III. FULLY DISCRETE FINITE DIFFERENCE SCHEMES FOR LINEAR ODES I: ESTIMATES IN TERMS OF PROPERTIES OF THE SOLUTION

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Abstract: This is the fourth of a sequence of 12 papers, preceded by [16, 17, 18] and followed by [19, 20, 21, 22, 23, 24, 25, 26] (in this order), dedicated to the study of error estimates for approximation problems based on discretization of the domain of the approximated functions. Within this sequence, in [17] and [18], for a model example of a Cauchy problem for a linear differential equation with variable coefficients, we applied an indirect method for error estimation based on results known in advance for the respective semi-discrete approximating problems.

In a follow-up subsequence of six of these papers, of which this is the first one, followed by [19, 20, 21, 22, 23] (in this order) we develop a direct discrete method for error estimation based on an *extended Lax principle*, essentially proposed first in [6] but explicitly formulated for the first time in the present paper (section 2). Here and in the next papers [19, 20] we apply the proposed method to obtain sharp error estimates for a model boundary problem for a linear ordinary differential equation of second order with variable coefficients and right-hand side. This study is being continued in the remaining three papers [21, 22, 23] of the subsequence by an analogous application of the proposed method to obtain sharp error estimates for a model initial-boundary problem

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for a linear parabolic partial differential equation of second order.

In the present paper we discuss the first 4 stages of the extended Lax principle for the model problem in consideration. These stages essentially correspond to the classical Lax principle, but the rather coarse classical error estimates obtained via this principle have been essentially sharpened in [1] using more advanced tools for error estimation, such as integral and averaged moduli of smoothness. Here we provide a systematic exposition of the results of [1], sharpen, generalize, and upgrade these results, and complement them with some additional ones, in order to prepare the error estimates at Stage 4 for use in the derivation of appropriate *a priori* estimates at Stage 5 and the final results of the new method at Stage 6 of the extended Lax principle, related to ordinary differential equations, in the subsequent papers [19, 20].

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1. Introduction

This is the fourth of a sequence of 12 papers, preceded by [16, 17, 18] and followed by [19, 20, 21, 22, 23, 24, 25, 26] (in this order), dedicated to the study of error estimates for approximation problems based on *discretization of the domain* of the approximated functions, and (in the concluding paper [26]) comparison of the similarities and differences with the error estimates derived by alternative approximation methods (typically, of projection type) based on finite-dimensional subspaces of functions having the same *continual domain* as the target function.

Within this sequence, in [17] and [18], for model examples of Cauchy problems, we applied an indirect method for error estimation based on results known in advance for the respective semi-discrete approximating problems.

In a sequence of six papers, of which this is the first one, followed by [19, 20, 21, 22, 23] (in this order) we develop a direct discrete method for error estimation based on an *extended Lax principle*, essentially proposed first in [6] but explicitly formulated for the first time in the present paper (section 2). Here and in the next papers [19, 20, 21, 22, 23] we apply the proposed method to obtain sharp error estimates for the following two model initial-boundary problems (one for an ordinary differential equation (ODE), and one for a partial differential equation (PDE)):

1. for a model linear ODE with variable coefficients and right-hand side (RHS) (see the present paper and [19, 20]);

2. for a model linear parabolic PDE with constant coefficients and a variable RHS (see [21, 22, 23]).

2. A Fully Discrete Method for Error Estimation

The *fully discrete method for error estimation directly in terms of properties of the problem's data*, proposed in [6, Chapter 3], is comprised of a sequence of stages, the entirety of which we choose to term as *an* ***extended*** *Lax principle*, as follows.

1. Derivation of estimates for the *local approximation error of the residual* (i.e., the difference on the mesh nodes between the left-hand side (LHS) and the RHS of the approximating linear finite difference equation when applied on the exact solution of the target differential equation).

2. Using the linearity of the continual target problem and the discrete approximating problem involved, derivation of a discrete approximating problem whose solution is *the error on the nodes of the mesh* (termed *discrete approximating problem for the error*).

3. Derivation of *a priori estimates* for the solution of the discrete approximating problem for the error.

4. Combining the results obtained at Stages 1 and 3 to derive *error estimates in terms of properties of the solution* (of *the exact target problem*). (The properties of the solution are measured quantitatively in terms of certain functional characteristics, such as functional norms, quasi-norms or metrics, or functional moduli (for examples, see section 3 below, [14] and the references therein).

5. Derivation of ***consistent*** *a priori* estimates for the solution of the exact target problem. (*Consistency* is meant here in the sense that the LHS of the *a priori* estimate should be in terms of the functional characteristics appearing at Stage 4).

6. Combining the results obtained at Stages 4 and 5 to derive *error estimates directly in terms of properties of the data functions and/or the data scalar parameters* (of the exact target problem).

The sub-sequence of Stages 1–4 constitutes *the classical Lax principle: local convergence of the residual* (to zero) (Stage 1) *and stability* (of the discrete approximating problem for the error) (Stages 2 and 3) *imply convergence of the error* (to zero) (Stage 4).

This classical qualitative formulation of the Lax principle was upgraded to quantitative relatively very early (see, e.g., a collection of classical references in [7]) but the quantitative error estimates derived at Stage 4 in this early period were rather coarse, in the sense that they required too high order of smoothness of the exact solution to ensure a given rate of approximation (see, e.g., [7]), thus unnecessarily limiting the range of applicability of the error estimates at Stage 4. This was due mainly to the rather limited choice (at that time) of available functional characteristics measuring regularity.

With the appearance of newer functional characteristics, notably, functional moduli such as the integral and the averaged moduli of smoothness, and the intensive development of their theory, in particular, by relating them to Peetre K-functionals (see, e.g., [14, 15] and the relevant references therein) it became possible to derive sharp error estimates at Stage 4 under minimal, or very close to minimal, assumptions about regularity of the solution. Such early results were obtained, notably, in [1] for ODEs, and in [27, 6] for PDEs. In [6] we proposed the upgrading and extending of the 'Lax principle paradigm' with Stages 5 and 6 (although the term 'extended Lax principle' is used here for the first time!).

The purpose of the present paper is to provide a systematic exposition of the results of [1], to sharpen, generalize, and upgrade these results, and to complement them with some additional ones, in order to prepare the error estimates at Stage 4 for use in the derivation of consistent *a priori* estimates at Stage 5 and the final results of the new method at Stage 6 related to ODEs, see the subsequent papers [19, 20].

The same organization will be repeated for PDEs, where [21] will be dedicated to sharp error estimates at Stage 4, in preparation for the derivation of the results at Stages 5 and 6 in the subsequent papers [22, 23].

2.1. Discrete approximating problems for ODEs

Consider as a model exact target problem the following boundary-value problem for the *stationary ODE of heat conductivity and diffusion* (see, e.g., [7]).

$$\begin{aligned} &(k(x)u'(x))' - q(x)u(x) = -f(x),\ 0 < x < 1, \\ &u(0) = \alpha,\ u(1) = \beta, \end{aligned} \tag{1}$$

where:

- $\alpha,\ \beta \in \mathbb{R}$;
- $k(x) \geq c_0 > 0,\ \ q(x) \geq 0,\ \ \forall x \in [0,1]$;
- $k,\ q,\ f \in BM[0,1]$;
- $k,\ q,\ f$ have (eventually) only discontinuities of the first kind, forming a set with zero measure;
- *"conjugation conditions"* are fulfilled, as follows: u – continuous, ku' – continuous, $\forall x \in [0,1]$ (including the (eventual) points of discontinuity of $k,\ q,\ f$).

Solve (1) numerically via the following *homogeneous conservative finite difference scheme* (see [7]) on the $(N+1)$-node, $N \in \mathbb{N}$, uniform mesh $\Sigma_h = \{x_i = i/N,\ i = 0,\ldots,N\},\ h = 1/N$:

$$\begin{aligned} &(ay_{\bar{x}})_x - dy = -\varphi,\ x \in \mathring{\Sigma}_h, \\ &y_0 = \alpha, y_N = \beta, \end{aligned} \tag{2}$$

where, as customary (see, e.g., [7]), $y_x = y_{x,i}$ and $y_{\bar{x}} = y_{\bar{x},i}$ are the *forward* and *backward divided difference operators*, respectively:

$$y_{x,i} = \frac{y_{i+1} - y_i}{h},\quad i = 0,\ldots,N-1; \qquad y_{\bar{x},i} = \frac{y_i - y_{i-1}}{h},\quad i = 1,\ldots,N.$$

Thanks to the linearity of the continual and discrete approximating problems, for the error $z_i,\ i = 0,1,\ldots,N$ we get the uniquely defined linear discrete problem

$$\begin{aligned} &(az_{\bar{x}})_x - dz = -\psi, \\ &z_0 = z_N = 0. \end{aligned} \tag{3}$$

Here ψ is the *local approximation error of the residual*:

$$\psi = (au_{\bar{x}})_x - du + \varphi$$

Our purpose is to estimate z.

In view of the homogeneity, a_i, d_i, φ_i are determined via *template functionals*, as follows:

$$\begin{aligned} a_i &= \mathrm{A}[k(x_i + sh], \\ d_i &= \mathrm{F}[q(x_i + sh)], \\ \varphi_i &= \mathrm{F}[q(x_i + sh)]. \end{aligned} \tag{4}$$

Here F is a linear template functional defined for $\bar{f}(s) \in BM\left[-\frac{1}{2}, \frac{1}{2}\right]$, where

$$\begin{aligned} BM(\Omega) = \{ \quad & f : \mathrm{Dom} f = \Omega, \mathrm{Cod} f \in \mathbb{R}, \\ & f - \text{measurable and bounded everywhere on } \Omega, \\ & \|f\|_{BM(\Omega)} = \sup_{x \in \Omega} |f(x)| < \infty\} \end{aligned}$$

(Ω – an open, closed or semi-open interval in $\mathbb{R}$; Dom g and Cod g – the domain and codomain of a function g, respectively).

The functionals F and A have the following properties:

- the template functional F:
  - is linear over $BM\left[-\frac{1}{2}, \frac{1}{2}\right]$;
  - is exact over the constants: $\mathrm{F}[1] = 1$;
  - is positive: $\mathrm{F}[\bar{f}(s)] \geq 0$ for $\bar{f}(s) \geq 0, \ s \in \left[-\frac{1}{2}, \frac{1}{2}\right]$;
- for the (possibly, *non-linear*) template functional A, see [7, p. 116].

Additional conditions are being imposed on the template functionals A and F, in order to ensure rate $O(h^2)$ of the local approximation error of the residual:

- in the case of F: $\mathrm{F}[s] = 0$;
- in the case of A: see [7, p. 118].

In particular, when

- $\mathrm{F}[\bar{f}(s)] = \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} f(\cdot + sh) ds,$

- $\mathrm{A}[\bar{K}(s)] = \left(\int\limits_{-1}^{0} \frac{ds}{K(\cdot + sh)} \right)^{-1}$,

- $a,\ d,\ \varphi$ are obtained via (4),

the respective scheme is termed *"the best"* (cf. [7], [9]).

In our argumentation in the sequel of this paper and in [19, 20], we shall be assuming that $k,\ q,\ f$ have the minimum of properties needed in the course of the exposition.

In particular, all arguments in the exposition are valid, if $k,\ q,\ f$ are piecewise continuous in $[0, 1]$ (not necessarily having bounded variation).

In the concluding remarks to [19] and [20] we shall show that our results continue to hold true also under assumptions on $k,\ q,\ f$ which are much more general than piecewise continuity, and are close to those in the formulation of (1).

3. Preliminaries: Functional Moduli and Function Spaces

Here we present the notation for the functional characteristics needed in the sequel. As a general reference about the functional characteristics introduced in this section concerning the notation, definitions, meaning and range of parameters, relevant properties and interrelations, as well as further details, we cite [14] together with [15], and the references therein. Another concise reference source on the material in this section is [16, section 2] and the references therein.

1. The function space $C(\Omega) \subset BM(\Omega)$ of all (real-valued) continuous functions on Ω, where Ω and $BM(\Omega)$ were defined in section 2.1. The restriction of the norm $\| \cdot \|_{BM}$ on C will be denoted, as usual, by $\| \cdot \|_C$; recall that with respect to this norm C is a Banach space which is a closed subspace of the Banach space BM (see, e.g., [4]).

2. For a function $f \in BM(\Omega)$,
$$S(t, f; x) = sup\,\{f(y) : y \in [x - t, x + t] \cap \Omega\}$$
is the upper Baire's function of f at $x \in \Omega$ with step $t > 0$ (see [14] and the references therein).

3. The local modulus of smoothness of order $k \in \mathbb{N}$ at $x \in \Omega$ (Ω as in the definition of the upper Baire's function) with $t > 0$
$$\omega_k(f, x; t) = \sup \left\{ \|\Delta_h^k f(y)\| : y, y + kh \in \left[x - \frac{kt}{2}, x + \frac{kt}{2} \right] \cap \Omega \right\}$$
(see [14] and the references therein); it is natural to define also $\omega_0(f, x; t) := S(t, f; x)$.

4. The sequence space $l^p(\Sigma_h)$ defined over the mesh Σ_h (see [28, 10, 11, 12, 6, 14]).

5. The integral modulus of smoothness (ω-modulus) $\omega_m(f; h)_{L_p}$ (short: $\omega_m(f; h)_p$) (see [28, 10, 11, 12, 9, 6, 14], cf. also [5, 2] where the notation is modified).

6. The averaged modulus of smoothness (τ-modulus) $\tau_m(f; h)_{L_p}$ (short: $\tau_m(f; h)_p$) (see [9, 28, 10, 11, 12, 13, 6, 14]).

7. The Steklov-means $f_{k,t}$ (see [3, 8, 9, 14]).

8. The Wiener amalgam space $A_{p,h}(\Omega)$, with norm $\|.\|_{A_{p,h}} = \tau_0(f; h)_{L_p}$ (see [12, 13, 6, 14, 15]); here Ω is as in section 2.1.

[16, Section 2.3] contains an important comparison between the properties of the integral and averaged moduli of smoothness.

Following our practice in [6] and [14], in order to distinguish between previously known results and the new ones obtained here, we shall add the additional marker '**(K.)**' (abbreviated from '(K)nown') to the enumeration of every statement in the sequel which has been previously known (with respective reference to available relevant literature).

4. Stationary Heat Conductivity and Diffusion ODE with Variable Coefficients and Right-Hand Side

4.1. Estimates of the local approximation error of the residual

In [1] the following representation is used of the error of local approximation of the residual ψ, i.e., the RHS in (3):

Lemma 1. **(K.)** *(See [7, p. 126].)* $\psi = \eta_x + \psi^*$, *where*
$\eta_i = (au_{\bar{x}})_i - (ku')_{i-\frac{1}{2}}$,

$$\psi_i^* = \left(\varphi_i - \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x_i + sh)ds\right) - \left(d_i u_i - \int_{-\frac{1}{2}}^{\frac{1}{2}} q(x_i + sh)u(x_i + sh)ds\right).$$

By using the properties of the local moduli of smoothness, in [1] the following estimate of the local approximation of the residual has been obtained.

Lemma 2. **(K.)** *(See [1], [9, pp. 207, 209].)*

$$|\eta_i| \leq \frac{\|k\|_{BM}}{2}\omega_2(u', x_{i-\frac{1}{2}}; \frac{h}{2}) + \frac{3\|u'\|_{BM}}{2}\omega_2(k, x_{i-\frac{1}{2}}; \frac{h}{2})$$

$$|\psi_i^*| \leq 2\|u\|_c \omega_2(q, x_i; \frac{h}{2}) + \frac{\|q\|_{BM}}{2}\omega_2(u, x_i; \frac{h}{2}) + 2\omega_2(f, x_i; \frac{h}{2}).$$

In the particular case of "the best" scheme, Lemma 2 is sharpened in [1] in the following way.

Lemma 3. **(K.)** *(See [1], [9, pp. 213, 215].) For "the best" scheme,*

$$|\eta_i| \leq \omega_2(u', x_{i-\frac{1}{2}}; \frac{h}{2}) + \frac{1}{2c_0}\left(\|f\|_{BM} + \|q\|_{BM}\|u\|_c\right) \int_{x_{i-1}}^{x_i} \omega_1(k, x; h)dx$$

$$|\psi_i^*| \leq \|u\|_c \omega_2(q, x_i; \frac{h}{2}) + \|q\|_{BM}\omega_2(u, x_i; \frac{h}{2}).$$

We show here that Lemma 3 can be additionally sharpened. Namely, the following new result holds true.

Lemma 4. *For "the best" scheme,*

$$|\eta_i| \leq \omega_2(u', x_{i-\frac{1}{2}}; \frac{h}{2}) + \frac{1}{2c_0}\left(S(h, |f|; x_i) + S(h, |q|; x_i)S(h, |u|; x_i)\right) \int_{x_{i-1}}^{x_i} \omega_1(k, x; h)dx$$

$$|\psi_i^*| \leq \frac{1}{2}S(h, |q|; x_i)\omega_2(u, x_i; \frac{h}{2}) + \frac{1}{2}\omega_2(q, x_i; h)\omega_2(u, x_i; h).$$

Proof. (Outline.) The proof of the statement about $|\eta_i|$ is an evident sharpening of the proof of the respective statement in Lemma 3.

Now we shall outline the proof of the second statement, about $|\psi_i^*|$, as follows.

$$\psi_i^* = d_i u_i - \int_{-\frac{1}{2}}^{+\frac{1}{2}} q(x_i+sh)u(x_i+sh)ds$$

$$= u(x_i)\int_{-\frac{1}{2}}^{+\frac{1}{2}} q(x_i+sh)ds - \int_{-\frac{1}{2}}^{+\frac{1}{2}} q(x_i+sh)u(x_i+sh)ds$$

$$= -\int_{-\frac{1}{2}}^{+\frac{1}{2}} q(x_i+sh)(u(x_i+sh)-u(x_i))ds$$

$$= -\left(\int_{0}^{\frac{1}{2}} q(x_i+sh)(u(x_i+sh)-u(x_i))ds \right.$$

$$\left. + \int_{-\frac{1}{2}}^{0} q(x_i+sh)(u(x_i+sh)-u(x_i))ds \right)$$

$$= -\int_{0}^{\frac{1}{2}} (q(x_i+sh)(u(x_i+sh)-u(x_i)) + q(x_i-sh)(u(x_i-sh)-u(x_i)))ds$$

$$- \int_{0}^{\frac{1}{2}} (q(x_i+sh)(u(x_i-sh)-u(x_i)) - q(x_i+sh)(u(x_i-sh)-u(x_i)))ds$$

$$= -\int_{0}^{\frac{1}{2}} (q(x_i+sh)(u(x_i+sh)-2u(x_i)+u(x_i-sh)))$$

$$+ (q(x_i-sh)-q(x_i+sh))(u(x_i-sh)-u(x_i))ds.$$

From here, the statement about $|\psi_i^*|$ follows easily, applying the triangle inequality and using the definitions of the upper Baire's function and the local moduli of smoothness. □

The following lemma is also new.

Lemma 5. *Under the conditions of Lemma 2*
$|\eta_i| \le \frac{1}{2}S(2h,k;x_i)\omega_2(u',x_{i-\frac{1}{2}};\frac{h}{2}) + \frac{3}{2}S(\frac{h}{2},u';x_{i-\frac{1}{2}})\omega_2(k,x_{i-\frac{1}{2}};\frac{h}{2})$
$|\psi_i^*| \le 2S(\frac{h}{2},u;x_i)\omega_2(q,x_i;\frac{h}{2}) + \frac{1}{2}S(h,q;x_i)\omega_2(u,x_i;\frac{h}{2}) + 2\omega_2(f,x_i;\frac{h}{2})$.

Proof. (Outline.) This lemma is a sharpening of Lemma 2 in the spirit of Lemma 4, and has analogous proof. Let us note that the inequality $a_i = \mathrm{A}[k(x_i+sh)] \le S(2h,k;x_i)$ follows easily from the positivity and normalization of the operator A. □

4.2. An a priori estimate for the solution of the discrete approximating problem

The error estimates in [1] and in this paper are based on the following *a priori* estimate in [7] for the discrete problem (3).

Lemma 6. **(K.)** *(See [7], [1].)*

$$\|z\|_{l^\infty(\Sigma_h)} \le \frac{2}{c_0}\sum_{i=1}^{N} h(|\eta_i| + |\psi_i^*|).$$

This *a priori* estimate was obtained in [7] by studying the properties of the discrete Green's function for (3).

4.3. Error estimates in terms of properties of the solution

Now we are in a position to apply the Lax principle: each of the summands in the RHS of the *a priori* estimate in Lemma 6 in section 4.2 will be bounded from above using the estimates for the local approximation of the residual in each of Lemmata 1–5 in section 4.1; the resulting upper estimate of the error z will be further bounded from above by averaged moduli of smoothness by a standard estimation argument using the properties of the upper Baire's function and the local and averaged moduli of smoothness.

Lemma 6 in section 4.2 and Lemmata 1, 2 in section 4.1 imply

Theorem 1. **(K.)** *(See [1], [9, p. 211].)*

$$\|z\|_{l^\infty(\Sigma_h)} \le \frac{2}{c_0}\Big(\frac{\|k\|_{BM}}{2}\tau_2(u';h)_{L_1} + \frac{3\|u'\|_{BM}}{2}\tau_2(k;h)_{L_1} + 2\|u\|_c\tau_2(q;h)_{L_1} + \frac{\|q\|_{BM}}{2}\tau_2(u;h)_{L_1} + 2\tau_2(f;h)_{L_1}\Big).$$

Lemma 6 in section 4.2 and Lemmata 1, 3 in section 4.1 imply

Theorem 2. **(K.)** *(See [1], [9, p. 216].) For "the best" scheme,*

$$\|z\|_{l^\infty(\Sigma_h)} \le \frac{2}{c_0}\Big(\tau_2(ku';h)_{L_1} + \|u\|_c\tau_2(q;h)_{L_1} + \|q\|_{BM}\tau_2(u;h)_{L_1} + \frac{h}{2c_0}(\|f\|_{BM} + \|q\|_{BM}\|u\|_c)\tau_1(k;h)_{L_1}\Big).$$

Using Lemma 4 instead of Lemma 3, as well as the properties of the spaces $A_{p,h}$ and Hölder's inequality, we obtain the following new sharpening of Theorem 2.

Theorem 3. *For "the best" scheme,*

$$\|z\|_{l^\infty(\Sigma_h)} \le \frac{2}{c_0}\Big(\tau_2(ku';h)_{L_1} + \frac{1}{2}\omega_1(u;h)_{L_\infty}\tau_1(q;h)_{L_1} + \frac{1}{2}\|q\|_{A_{p',h}}\tau_2(u;h)_{L_p} + \frac{h}{2c_0}(\|f\|_{A_{r',h}} + \|q\|_{A_{r',h}}\|u\|_c)\tau_1(k;h)_{L_r}\Big).$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{r} + \frac{1}{r'} = 1,\ 1 \le p, r \le \infty$.

From Lemma 5 we derive the following new generalization and sharpening of Theorem 1.

Theorem 4. *Under the conditions of Theorem 1*

$$\|z\|_{l^\infty(\Sigma_h)} \leq \frac{2}{c_0}\Big(\frac{3}{2}\|u'\|_{A_{r_1',h}}\tau_2(k;h)_{L_{r_1}} + \|k\|_{A_{r_2',h}}\tau_2(u';h)_{L_{r_2}} + \|u\|_{A_{r_3',h}}\tau_2(q;h)_{L_{r_3}} + \frac{c}{2}\|q\|_{A_{r_4',h}} h\omega_1(u';h)_{L_{r_4}} + 2\tau_2(f;h)_{L_{r_1}} \Big),$$

where $0 < c < 16, 1 < r_j \leq \infty, \frac{1}{r_j} + \frac{1}{r_j'} = 1, j = 1,2,3,4.$

4.4. Some implications

From Theorem 3 and the properties of $\omega_1(u;h)_{L_\infty}$ and $\tau_2(u;h)_{L_p}$ we straightforwardly obtain

Corollary 1. *Under the conditions of Theorem 3*

$$\|z\|_{l^\infty(\Sigma_h)} \leq \frac{2}{c_0}\Big(\tau_2(ku';h)_{L_1} + \frac{h}{2}\|u'\|_{BM}\tau_1(q;h)_{L_1} + \frac{ch}{2}\|q\|_{A_{p',h}}\omega_1(u';h)_{L_p} + \frac{h}{2c_0}(\|f\|_{A_{r',h}} + \|q\|_{A_{r',h}}\|u\|_c)\tau_1(k;h)_{L_r} \Big).$$

where $0 < c < 16$.

In view of the properties of the integral and average moduli of smoothness, Theorems 1–4 and Corollary 1 yield a rich diversity of corollaries on approximation rates under respective regularity assumptions about the exact solution u of problem (1) and its derivatives involved in the differential equation in problem (1). In [1] the reader can find several instances of such corollaries, as implied by Theorems 1 and 2 proved in [1], but we note that using the new results in Theorems 3, 4 and Corollary 1 above, the corollaries in [1] can be sharpened, generalized and complemented by a variety of additional new results. For more about this type of corollaries following from the properties of the ω- and τ-moduli, we refer the reader to the next two papers in this sequence, [19] and [20].

5. Concluding Remarks

Remark 1. The type of error estimates in terms of regularity properties of the solution provides only an estimate about the convergence rate but not an estimate of the multiplicative factor to this rate, since it is assumed that there is available *a priori* information about the solution belonging to a certain function space corresponding to a certain convergence rate, but the solution itself is not *a priori* known. In other words, if the solution belongs to a certain function class A_σ where σ is the convergence order, and if $h > 0$ is the approximation step, then the respective error estimate is either of the qualitative type $o(h^\sigma)$ or of the qualitative type $O(h^\sigma)$ with the constant factor in the O-estimate known to exist, but without a quantitative bound on its size. In contrast to this, the error estimates in [19, 20] in terms directly of properties of the functional and parametric data of problem (1) (variable coefficients, variable RHS and scalar parameters of the boundary conditions) provide also a quantitative bound for the constant factor to the rate.

Remark 2. The importance of error estimates in terms of regularity properties of the solution increases in *a posteriori* error analysis when the solution is assumed to be already known, and this type of error estimates provides not only qualitative but also quantitative information about the multiplicative factors to the rates.

Remark 3. The method developed and the results obtained in this paper and the subsequent papers [19, 20] on this topic are an application of the general theory developed in [14], [15].

Remark 4. The material in this paper and the subsequent papers [19, 20] covers the part of the previously unpublished results in [6, Chapter 3] related to ODEs and complementary to the unpublished results in [6, Chapter 3] about PDEs which are scheduled to appear in [21, 22, 23].

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References

[1] А. С. Андреев, В. А. Попов, Бл. Сендов. Оценки погрешности численного решения обыкновенных дифференциальных уравнений. *ЖВМ и МФ*, 21:635–650, 1981. (In Russian.)

[2] Й. Берг, Й. Лефстрем. *Интерполяционные пространства. Введение.* Мир, Москва, 1980. (In Russian.)

[3] Ю. А. Брудный. Приближение функций N-переменных квазимногочленами. *Изв. АН СССР, сер.мат.*, 34:564–583, 1970. (In Russian.)

[4] Н. Данфорд, Дж. Т. Шварц. *Линейные операторы, Т. 1: Общая теория.* Изд. иностр. лит., Москва, 1962. (In Russian.)

[5] С. М. Никольский. *Приближение функций многих переменных и теоремы вложения.* Наука, Москва, 1969. (In Russian.)

[6] Л. Т. Дечевски. *Някои приложения на теорията на функционалните пространства в числения анализ.* Ph. D. dissertation, Факултет по математика и механика, Софийски Университет, София, 1989. (In Bulgarian.)

[7] А. А. Самарский. *Введение в теорию разностных схем.* Наука, Москва, 1971. (In Russian.)

[8] Бл. Сендов. Модифицированная функция Стеклова. *Докл. БАН*, 36(3):315–317, 1983. (In Russian.)

[9] Бл. Сендов, В. А. Попов. *Усреднени модули на гладкост.* Бълг. мат. моногр., 4. БАН, София, 1983. (In Bulgarian.)

[10] L. T. Dechevski. Network-norm error estimation using interpolation of spaces and application to differential equations. Proc. Conf. on Constructive Theory of Functions, Varna'1984. Bulg. Acad. Sci., Sofia, 1984, pp. 260–265.

[11] L. T. Dechevski. Network-norm and L_∞-norm error estimates for the numerical solutions of evolutionary equations. Proc. Conf. on Numerical Methods and Applications, Sofia'1984. Bulg. Acad. Sci., Sofia, 1985, pp. 224–231.

[12] L. T. Dechevski. Network-norm error estimates for the numerical solutions of evolutionary equations. *Serdica*, 12:53–64, 1986.

[13] L. T. Dechevski. τ-moduli and interpolation. Proc. US-Swedish Seminar on Function Spaces and Applications, Lund'1986. Lecture Notes in Math. 1302. Springer, Berlin–Heidelberg–New York, 1988, pp. 177–190.

[14] L.T. Dechevsky. Properties of function spaces generated by the averaged moduli of smoothness. *Int. J. Pure Appl. Math.*, 41(9):1305–1375, 2007.

[15] L.T. Dechevsky. Concluding remarks to paper "Properties of function spaces generated by the averaged moduli of smoothness". *Int. J. Pure Appl. Math.*, 49(1):147–152, 2008.

[16] L. T. Dechevsky. On error estimation for approximation methods involving domain discretization I: problem setting. *Int. J. Pure Appl. Math.*, **57**, No. 5 (2009), 605-627.

[17] L. T. Dechevsky. On error estimation for approximation methods involving domain discretization II. Deterministic problems I. Full discretization of semi-discrete finite difference schemes for linear evolutionary PDEs I: the parabolic case. *Int. J. Pure Appl. Math.*, **57**, No. 5 (2009), 629-651.

[18] L. T. Dechevsky. On error estimation for approximation methods involving domain discretization III. Deterministic problems II. Full discretization of semi-discrete finite difference schemes for linear evolutionary PDEs II: the general, possibly non-parabolic, case. *Int. J. Pure Appl. Math.*, **57**, No. 5 (2009), 653-668.

[19] L. T. Dechevsky. On error estimation for approximation methods involving domain discretization V. Deterministic problems IV. Fully discrete finite difference schemes for linear ODEs II. Estimates in terms of properties of the problem's data functions I: homogeneous boundary conditions. *Int. J. Pure Appl. Math.*, **57**, No. 5 (2009), 687-732.

[20] L. T. Dechevsky. On error estimation for approximation methods involving domain discretization VI. Deterministic problems V. Fully discrete finite difference schemes for linear ODEs III. Estimates in terms of properties of the problem's data functions II: inhomogeneous boundary conditions. *Int. J. Pure Appl. Math.*, **57**, No. 5 (2009), 733-759.

[21] L. T. Dechevsky. On error estimation for approximation methods involving domain discretization VII. Deterministic problems VI. Fully discrete finite difference schemes for linear PDEs I: estimates in terms of properties of the solution. *Int. J. Pure Appl. Math.*, **57**, No. 5 (2009), 761-787.

[22] L. T. Dechevsky. On error estimation for approximation methods involving domain discretization VIII. Deterministic problems VII. Fully discrete finite difference schemes for linear PDEs II. Estimates in terms of properties of the problem's data functions I: homogeneous boundary conditions. *Int. J. Pure Appl. Math.*, To appear.

[23] L. T. Dechevsky. On error estimation for approximation methods involving domain discretization IX. Deterministic problems VIII. Fully discrete finite difference schemes for linear PDEs III. Estimates in terms of properties of the problem's data functions II: inhomogeneous boundary conditions. *Int. J. Pure Appl. Math.*, To appear.

[24] L. T. Dechevsky. On error estimation for approximation methods involving domain discretization X. Indeterministic problems I: risk estimates for non-parametric regression with random design. *Int. J. Pure Appl. Math.*, To appear.

[25] L. T. Dechevsky. On error estimation for approximation methods involving domain discretization XI. Indeterministic problems II: risk estimates for non-parametric regression with deterministic design. *Int. J. Pure Appl. Math.*, To appear.

[26] L. T. Dechevsky. On error estimation for approximation methods involving domain discretization XII: comparison between estimates in continual and discrete norms. (To appear.)

[27] V. A. Popov, A. S. Andreev. On the error estimation in numerical methods. Banach Center Publ., PWN, Warsaw, 1984, pp. 647–658.

[28] V. A. Popov, L. T. Dechevski. On the error of numerical solution of the parabolic equation in network norms. *Compt. Rend. Acad. Bulg. Sci.*, 36(4):429–432, 1985.