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ON ERROR ESTIMATION FOR APPROXIMATION METHODS INVOLVING DOMAIN DISCRETIZATION V: DETERMINISTIC PROBLEMS IV. FULLY DISCRETE FINITE DIFFERENCE SCHEMES FOR LINEAR ODES II: ESTIMATES IN TERMS OF PROPERTIES OF THE PROBLEM'S DATA FUNCTIONS I: HOMOGENEOUS BOUNDARY CONDITIONS

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Abstract: This is the fifth of a sequence of 12 papers, preceded by [19, 20, 21, 22] and followed by [23, 24, 25, 26, 27, 28, 29] (in this order), dedicated to the study of error estimates for approximation problems based on discretization of the domain of the approximated functions.

Within this sequence, in [20] and [21], for a model example of a Cauchy problem for a linear differential equation with variable coefficients, we applied an indirect method for error estimation based on results known in advance for the respective semi-discrete approximating problems.

In a follow-up subsequence of six of these papers, of which this is the second one, preceded by [22] and followed by [23, 24, 25, 26] (in this order) we develop a direct discrete method for error estimation based on an *extended Lax principle*, essentially proposed first in [5] but explicitly formulated for the first time in [22, section 2].

In [22], here and in the next paper [23] we apply the proposed method to obtain sharp error estimates for a model boundary problem for a linear ordinary

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differential equation of second order with variable coefficients and right-hand side. This study is being continued in the remaining three papers [24, 25, 26] of the subsequence by an analogous application of the proposed method to obtain sharp error estimates for a model initial-boundary problem for a linear parabolic partial differential equation of second order.

In [22] we discussed the first 4 stages of the extended Lax principle for the model problem in consideration. These stages essentially correspond to the classical Lax principle, but the rather coarse classical error estimates obtained via this principle have been essentially sharpened in [1] using more advanced tools for error estimation, such as integral and averaged moduli of smoothness. In [22] we provided a systematic exposition of the results of [1], sharpened, generalized, and upgraded these results, and complemented them with some additional ones, in order to prepare the error estimates at Stage 4 for use in the derivation of appropriate *a priori* estimates at Stage 5 and the final results of the new method at Stage 6 of the extended Lax principle, related to ordinary differential equations, in the present paper and the subsequent paper [23].

In the present paper we address two major topics (corresponding to Stages 5 and 6 of the extended Lax principle), as follows.

- Stage 5: We develop *a priori* estimates for the continual problem (1) which are *consistent* with the error estimates in terms of properties of the solution from [22].

- Stage 6: Based on the results obtained on Stage 4 in [22] and the results obtained here on Stage 5, we derive sharp error estimates directly in terms of the data (variable coefficients and variable right-hand side) of the continual boundary-value problem (1) for the case of *homogeneous boundary conditions*. These new error estimates imply a diversity of corollaries providing certain approximation rates under minimal assumptions about regularity of the data.

In [23] we shall extend the results obtained here for homogeneous boundary conditions to the general case of inhomogeneous boundary conditions.

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norm, bound, boundary, differential equation, ordinary, stationary, template, functional, linear, non-linear, positive, univariate

1. Introduction

This is the fifth of a sequence of 12 papers, preceded by [19, 20, 21, 22] and followed by [23, 24, 25, 26, 27, 28, 29] (in this order), dedicated to the study of error estimates for approximation problems based on *discretization of the domain* of the approximated functions, and (in the concluding paper [29]) comparison of the similarities and differences with the error estimates derived by alternative approximation methods (typically, of projection type) based on finite-dimensional subspaces of functions having the same *continual domain* as the target function.

Within this sequence, in [20] and [21], for model examples of Cauchy problems, we applied an indirect method for error estimation based on results known in advance for the respective semi-discrete approximating problems.

In a sequence of six papers, of which this is the second one, preceded by [22] and followed by [23, 24, 25, 26] (in this order), we develop a direct discrete method for error estimation based on an *extended Lax principle*, essentially proposed first in [5] but explicitly formulated for the first time in [22, section 2].

In [22], here and in the next papers [23, 24, 25, 26] we apply the proposed method to obtain sharp error estimates for the following two model initial-boundary problems (one for an ordinary differential equation (ODE), and one for a partial differential equation (PDE)):

1. for a model linear ODE with variable coefficients and right-hand side (RHS) (see the present paper and [22, 23]);
2. for a model linear parabolic PDE with constant coefficients and a variable RHS (see [24, 25, 26]).

In [22] we discussed the first 4 stages of the extended Lax principle for the model problem in consideration. These stages essentially correspond to the classical Lax principle, but the rather coarse classical error estimates obtained via this principle have been essentially sharpened in [1] using more advanced tools for error estimation, such as integral and averaged moduli of smoothness. In [22] we provided a systematic exposition of the results of [1], sharpened, generalized, and upgraded these results, and complemented them with some additional ones, in order to prepare the error estimates at Stage 4 for use in the derivation of appropriate *a priori* estimates at Stage 5 and the final results of the new method at Stage 6 of the extended Lax principle, related to ordinary differential equations, in the present paper and the subsequent paper [23].

In the present paper we address two major topics (corresponding to Stages 5 and 6 of the extended Lax principle), as follows.

- Stage 5: We develop *a priori* estimates for the continual problem (1) which are *consistent* with the error estimates in terms of properties of the solution from [22].
- Stage 6: Based on the results obtained on Stage 4 in [22] and the results obtained here on Stage 5, we derive sharp error estimates directly in terms of the data (variable coefficients and variable right-hand side) of the continual boundary-value problem (1) for the case of *homogeneous boundary conditions*. These new error estimates imply a diversity of corollaries providing certain approximation rates under minimal assumptions about regularity of the data.

In [23] we shall extend the results obtained here for homogeneous boundary conditions to the general case of inhomogeneous boundary conditions.

2. A Fully Discrete Method for Error Estimation

The *fully discrete method for error estimation directly in terms of properties of the problem's data*, proposed in [5, Chapter 3], is comprised of a sequence of stages, the entirety of which we chose in [22] to term as *an* ***extended*** *Lax principle*. In brief, the main stages of the proposed method are, as follows.

A. Classical Lax principle

- Stage 1. Derivation of estimates for the *local approximation error of the residual*.
- Stage 2. Derivation of a discrete approximating problem whose solution is *the error on the nodes of the mesh* (termed *discrete approximating problem for the error*).
- Stage 3. Derivation of *a priori estimates* for the solution of the discrete approximating problem for the error.

- Stage 4. Combining the results obtained at Stages 1 and 3 to derive *error estimates in terms of properties of the solution* (of *the exact target problem*).

B. Extension of the Lax principle

- Stage 5. Derivation of ***consistent*** *a priori* estimates for the solution of the exact target problem.
- Stage 6. Combining the results obtained at Stages 4 and 5 to derive *error estimates directly in terms of properties of the data functions and/or the data scalar parameters* (of the exact target problem).

For more details, see [22, section 2].

Within the above framework, the organization of the exposition in the current sequence of relevant papers is, as follows.

- Part A.
  - ODEs: [22]
  - PDEs: [24]
- Part B.
  - ODEs:
    - Homogeneous boundary conditions: the present paper
    - Inhomogeneous boundary conditions: [23]
  - PDEs:
    - Homogeneous boundary conditions: [25]
    - Inhomogeneous boundary conditions: [26]

2.1. Discrete Approximating Problems for ODEs

Consider as a model exact target problem the following boundary-value problem for the *stationary ODE of heat conductivity and diffusion* (see, e.g., [9]).

$$\begin{aligned} &(k(x)u'(x))' - q(x)u(x) = -f(x),\ 0 < x < 1, \\ &u(0) = \alpha,\ u(1) = \beta, \end{aligned} \tag{1}$$

where:

- $\alpha,\ \beta \in \mathbb{R}$;

- $k(x) \geq c_0 > 0, \;\; q(x) \geq 0, \;\; \forall x \in [0,1];$
- $k, \; q, \; f \in BM[0,1];$
- $k, \; q, \; f$ have (eventually) only discontinuities of the first kind, forming a set with zero measure;
- *"conjugation conditions"* are fulfilled, as follows: u – continuous, ku' – continuous, $\forall x \in [0,1]$ (including the (eventual) points of discontinuity of $k, \; q, \; f$).

Solve (1) numerically via the following *homogeneous conservative finite difference scheme* (see [9]) on the $(N+1)$-node, $N \in \mathbb{N}$, uniform mesh $\Sigma_h = \{x_i = i/N, \; i = 0, \ldots, N\}, \; h = 1/N$:

$$\begin{aligned} &(ay_{\bar{x}})_x - dy = -\varphi, \; x \in \mathring{\Sigma}_h, \\ &y_0 = \alpha, y_N = \beta, \end{aligned} \tag{2}$$

where, as customary (see, e.g., [9]), $y_x = y_{x,i}$ and $y_{\bar{x}} = y_{\bar{x},i}$ are the *forward* and *backward divided difference operators*, respectively:

$$y_{x,i} = \frac{y_{i+1} - y_i}{h}, \quad i = 0, \ldots, N-1; \qquad y_{\bar{x},i} = \frac{y_i - y_{i-1}}{h}, \quad i = 1, \ldots, N.$$

Thanks to the linearity of the continual and discrete approximating problems, for the error $z_i, \; i = 0, 1, \ldots, N$ we get the uniquely defined linear discrete problem

$$\begin{aligned} &(az_{\bar{x}})_x - dz = -\psi, \\ &z_0 = z_N = 0. \end{aligned} \tag{3}$$

Here ψ is the *local approximation error of the residual*:

$$\psi = (au_{\bar{x}})_x - du + \varphi$$

Our purpose is to estimate z.

In view of the homogeneity, a_i, d_i, φ_i are determined via *template functionals*, as follows:

$$\begin{aligned} a_i &= \mathrm{A}[k(x_i + sh], \\ d_i &= \mathrm{F}[q(x_i + sh)], \\ \varphi_i &= \mathrm{F}[q(x_i + sh)]. \end{aligned} \tag{4}$$

Here F is a linear template functional defined for $\bar{f}(s) \in BM\left[-\frac{1}{2}, \frac{1}{2}\right]$, where

$$\begin{aligned} BM(\Omega) = \{ \quad & f : \mathrm{Dom} f = \Omega, \mathrm{Cod} f \in \mathbb{R}, \\ & f - \text{measurable and bounded everywhere on } \Omega, \\ & \|f\|_{BM(\Omega)} = \sup_{x \in \Omega} |f(x)| < \infty \} \end{aligned}$$

(Ω – an open, closed or semi-open interval in $\mathbb{R}$; Dom g and Cod g – the domain and codomain of a function g, respectively).

The functionals F and A have the following properties:

- the template functional F:
  - is linear over $BM\left[-\frac{1}{2}, \frac{1}{2}\right]$;
  - is exact over the constants: $\mathrm{F}[1] = 1$;
  - is positive: $\mathrm{F}[\bar{f}(s)] \geq 0$ for $\bar{f}(s) \geq 0,\ s \in \left[-\frac{1}{2}, \frac{1}{2}\right]$;
- for the (possibly, *non-linear*) template functional A, see [9, p. 116].

Additional conditions are being imposed on the template functionals A and F, in order to ensure rate $O(h^2)$ of the local approximation error of the residual:

- in the case of F: $\mathrm{F}[s] = 0$;
- in the case of A: see [9, p. 118].

In particular, when

- $\mathrm{F}[\bar{f}(s)] = \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} f(\cdot + sh) ds$,
- $\mathrm{A}[\bar{K}(s)] = \left(\int\limits_{-1}^{0} \frac{ds}{K(\cdot + sh)} \right)^{-1}$,
- $a,\ d,\ \varphi$ are obtained via (4),

the respective scheme is termed *"the best"* (cf. [9], [1]).

In our argumentation in [22], the sequel of the present paper and in [23], we are assuming that $k,\ q,\ f$ have the minimum of properties needed in the course of the exposition.

In particular, all arguments in the exposition are valid, if k, q, f are piecewise continuous in $[0,1]$ (not necessarily having bounded variation).

In the concluding remarks in the present paper and in [23] we shall show that our results continue to hold true also under assumptions on k, q, f which are much more general than piecewise continuity, and are close to those in the formulation of (1).

3. Preliminaries: Functional Moduli and Function Spaces

Here we present the notation for the functional characteristics needed in the sequel. As a general reference about the functional characteristics introduced in this section concerning the notation, definitions, meaning and range of parameters, relevant properties and interrelations, as well as further details, we cite [17] together with [18], and the references therein. Another concise reference source on the material in this section is [19, section 2] and the references therein.

1. The function space $C(\Omega) \subset BM(\Omega)$ of all (real-valued) continuous functions on Ω, where Ω and $BM(\Omega)$ were defined in section 2.1. The restriction of the norm $\|\cdot\|_{BM}$ on C will be denoted, as usual, by $\|\cdot\|_C$; recall that with respect to this norm C is a Banach space which is a closed subspace of the Banach space BM (see, e.g., [4]).

2. For a function $f \in BM(\Omega)$,
$$S(t,f;x) = sup\,\{f(y) : y \in [x-t, x+t] \cap \Omega\}$$
is the upper Baire's function of f at $x \in \Omega$ with step $t > 0$ (see [17] and the references therein).

3. The local modulus of smoothness of order $k \in \mathbb{N}$ at $x \in \Omega$ (Ω as in the definition of the upper Baire's function) with $t > 0$
$$\omega_k(f,x;t) = \sup\left\{\|\Delta_h^k f(y)\| : y, y+kh \in \left[x - \frac{kt}{2}, x + \frac{kt}{2}\right] \cap \Omega\right\}$$
(see [17] and the references therein); it is natural to define also $\omega_0(f,x;t) := S(t,f;x)$.

4. The sequence space $l^p(\Sigma_h)$ defined over the mesh Σ_h (see [32, 13, 14, 15, 5, 17]).

5. The inhomogeneous Sobolev space $W_p^m(\Omega)$, with $W_p^0 = L_p$ (see, e.g., [2, 8, 17]); here and in the remaining items in this list Ω is as in section 2.1.

6. The integral modulus of smoothness (ω-modulus) $\omega_m(f;h)_{L_p}$ (short: $\omega_m(f;h)_p$) (see [32, 13, 14, 15, 11, 5, 17], cf. also [8, 2] where the notation is modified).

7. The averaged modulus of smoothness (τ-modulus) $\tau_m(f;h)_{L_p}$ (short: $\tau_m(f;h)_p$) (see [11, 32, 13, 14, 15, 16, 5, 17]).

8. The Steklov-means $f_{k,t}$ (see [3, 10, 11, 17]).

9. The Wiener amalgam space $A_{p,h}(\Omega)$, with norm $\|.\|_{A_{p,h}} = \tau_0(f;h)_{L_p}$ (see [15, 16, 5, 17, 18]).

10. The inhomogeneous Besov space $B_{pq}^s(\Omega)$ (see [2, 32, 13, 14, 15, 16, 5, 17]).

11. The inhomogeneous A-space $A_{pq}^s(\Omega)$, an analogue of the respective Besov space where the ω-modulus in the definition of the norm in B_{pq}^s is replaced by the respective τ-modulus (with the same parameters) in the definition of the norm in A_{pq}^s (see [32, 13, 14, 15, 16, 5, 17]).

12. The inhomogeneous Triebel–Lizorkin space $F_{pq}^s(\Omega)$ (see [2, 5, 17]).

13. The Wiener–Young p-variation $\bigvee_{\Omega}{}_p\, g$ of g, the case $p = 1$ corresponding to the customary Jordan variation (see [33, 34]), with $\bigvee_{\Omega}{}_p = \bigvee_a^b{}_p$ for $\Omega = [a,b]$.

14. The space $AC(\Omega)$ of all absolutely continuous functions on $\Omega \subset \mathbb{R}$.

[19, Section 2.3] contains an important comparison between the properties of the integral and averaged moduli of smoothness.

Following our practice in [5] and [17], in order to distinguish between previously known results and the new ones obtained here, we shall add the additional marker '**(K.)**' (abbreviated from '(K)nown') to the enumeration of every statement in the sequel which has been previously known (with respective reference to available relevant literature).

4. Stationary Heat Conductivity and Diffusion ODE with Variable Coefficients and Right-Hand Side

4.1. Error Estimates in Terms of Properties of the Solution

The previous paper in this sequence, [22], was dedicated to the derivation of sharp error estimates in terms of properties of the solution of the continual problem (1), i.e., to Stages 1–4 of the extended Lax principle. It contained an exposition of the main relevant results in [1]: [22, Theorem 1] for the general formulation of the discrete approximating problem (2), and [22, Theorem 2] for the special case of the "best" finite difference scheme in (2). Furthermore, in view of the need for developing new *a priori estimates* and error estimates in terms of the data for the continual problem (1) (see sections 4.2 and 4.3 below) in [22] we upgraded the above-said results of [1] with the following three new results: [22, Theorem 4] (an upgrade of [22, Theorem 1] for the general choice of finite difference scheme in (2)), [22, Theorem 3] (an upgrade of [22, Theorem 1] for the special case of the "best" finite difference scheme in (2)), and [22, Corollary 1] (a corollary of [22, Theorem 3], and a further upgrade of [22, Theorem 1] for the "best" scheme in (2)). As we shall see in sections 4.2 and 4.3 below, it is these three new upgraded results in [22] that prove to be appropriate for the derivation of consistent *a priori* estimates and respective sharp error estimates in terms of the data for the continual problem (1); more precisely, we shall be explicitly using [22, Theorem 4] and [22, Corollary 1]. For the reader's convenience, we present explicitly these results here, as follows.

Theorem 1. **(K.)** *(See [22, Theorem 4].) For an arbitrary admissible selection of the homogeneous conservative finite difference scheme (i.e., for arbitrary admissible template operators* A, F, *see section 2.1 in problem (2)), it holds true that*

$$
\begin{aligned}
\|z\|_{l^\infty(\Sigma_h)} \le \frac{2}{c_0} \Big(\quad & \frac{3}{2}\|u'\|_{A_{r_1',h}} \tau_2(k;h)_{L_{r_1}} \\
& + \|k\|_{A_{r_2',h}} \tau_2(u';h)_{L_{r_2}} \\
& + \|u\|_{A_{r_3',h}} \tau_2(q;h)_{L_{r_3}} \\
& + \frac{c}{2}\|q\|_{A_{r_4',h}} h\omega_1(u';h)_{L_{r_4}} \\
& + 2\tau_2(f;h)_{L_{r_1}} \ \Big)\,,
\end{aligned}
$$

where $0 < c < 16, 1 < r_j \le \infty, \frac{1}{r_j} + \frac{1}{r_j'} = 1, j = 1, 2, 3, 4.$

Theorem 2. (K.) *(See [22, Corollary 1].) Under the conditions of Theorem 1, for the special case of the "best" scheme in (2) (see section 2.1 for the definitions of the template operators* A, F *in this special case) it holds true that*

$$\begin{aligned}\|z\|_{l^\infty(\Sigma_h)} \leq \frac{2}{c_0}\Big(\quad & \tau_2(ku';h)_{L_1} \\ & +\frac{h}{2}\|u'\|_{BM}\tau_1(q;h)_{L_1} \\ & +\frac{ch}{2}\|q\|_{A_{p',h}}\omega_1(u';h)_{L_p} \\ & +\frac{h}{2c_0}(\|f\|_{A_{r',h}}+\|q\|_{A_{r',h}}\|u\|_c)\tau_1(k;h)_{L_r}\ \Big)\end{aligned}$$

where $0 < c < 16$, $\frac{1}{p}+\frac{1}{p'}=\frac{1}{r}+\frac{1}{r'}=1,\ 1 \leq p,r \leq \infty$.

4.2. A Priori Estimates for the Continual Problem

The estimates in [9] for the discrete problem are obtained by exploiting the properties of the discrete Green's function for (3). Since [9] studies only error estimates in terms of properties of the solution (i.e., estimates up to Stage 4 of the extended Lax principle), [9] does not discuss *a priori* estimates for the continual problem (1), nor does it cite any relevant literature.

The purpose of the present section is to derive *a priori* estimates for the continual problem (1) needed for Stage 5 of the extended Lax principle. To this end, here we shall consider the respective properties of the continual Green's function $G = G(x,\xi);\ x,\xi \in [0,1]$ for problem (1). For conciseness of the exposition, we shall consider the case of homogeneous boundary conditions $\alpha = \beta = 0$ in (1). In the general case of inhomogeneous boundary conditions it is possible to apply analogous arguments (see Remarks 2, 6 below and [23]).

4.2.1. A Priori Estimates

For "sufficiently nice" $k,\ q,\ f$, the function G has the following properties:

P1. The solution $u(x)$ of (1) exists, is unique, and

$$u(x) = \int_0^1 G(x,\xi)f(\xi)d\xi,\ \forall\xi \in [0,1].$$

P2.

$$\frac{\partial}{\partial x}\left(k(x)\frac{\partial}{\partial x}G(x,\xi)\right) - q(x)G(x,\xi) = -\delta(x,\xi),\ x,\xi \in [0,1],$$
$$G(0,\xi) = G(1,\xi) = 0,\ \xi \in [0,1];$$

δ is Dirac's δ-function. Besides, it is fulfilled

P2.1 $G(x,\xi)$ is continuous in $[0,1]^2$.

P2.2 $\lim_{\varepsilon\to+0} k(x)\frac{\partial}{\partial x}G(x,\xi)\mid_{x=\xi-\varepsilon}^{x=\xi+\varepsilon} = -1$.

P3.

$$G(x,\xi) = \begin{cases} \dfrac{A(x)B(\xi)}{A(1)}, & 0 \le x \le \xi, \\ \dfrac{A(\xi)B(x)}{A(1)}, & \xi \le x \le 1, \end{cases}$$

where A, B are solutions, respectively, of:

$$\begin{aligned} &(k(x)A'(x))' - q(x)A(x) = 0, \quad x \in [0,1], \\ &A(0) = 0, k(0)A'(0) = 1, \end{aligned} \tag{5}$$

$$\begin{aligned} &(k(x)A'(x))' - q(x)B(x) = 0, \quad x \in [0,1], \\ &B(1) = 0, k(1)B'(1) = -1, \end{aligned} \tag{6}$$

A, B, kA', kB' – continuous in $[0,1]$, while A and B have the properties:

P3.1 $[0,1]$ A monotonously increases, B monotonously decreases,

$$A(x) \le A(1), B(x) \le B(0), x \in [0,1]; A(1) = B(0);$$

P3.2 $k(x)\,(B(x)A'(x) - A(x)B'(x)) = \text{const} = A(1),\ 0 \le x \le 1$.

P4. $G(x,\xi) = G(\xi,x), x,\xi \in [0,1];\ G(x,\xi) > 0, x,\xi \in [0,1]$.

It is well known that **P1–4.** hold true, e.g., in the classical case, when k are q piecewise continuous, f is continuous (cf. [9, pp. 105–106], [6, pp. 297–300]). It can be shown (see Remark 1) that **P1–4.** are fulfilled (with minor modifications) also for much more general assumptions for $k,\ q,\ f$, close to the conditions, under which problem (1) in section 2.1 was formulated (cf. Remark 4).

Our main task here will be to estimate

$$|G(x,\xi)| = G(x,\xi), |\frac{\partial}{\partial x}G(x,\xi)|, |\frac{\partial}{\partial \xi}G(x,\xi)|$$

uniformly in $x,\xi \in [0,1]$.

In the argumentation we shall often be relying on the fact that if φ' is bounded and measurable in $[0,1]$, then, it is summable there and, therefore, φ is absolutely continuous in $[0,1]$, modulo a set of measure zero. (We shall always be selecting the absolutely continuous representative within the equivalence class.) We shall also be using the fact that if φ is absolutely continuous on $[0,\alpha)$ and $(\alpha,1]$, $\alpha \in (0,1)$, and if φ is continuous at α, then, φ is absolutely continuous on $[0,1]$.

Lemma 1. *Assume that the following is fulfilled:*

1. $$(k(x)u'(x))' - q(x)u(x) = \psi(x);$$
2. ψ *– bounded and measurable;*
3. $\psi \geq 0$ ($\psi \leq 0$) *a.e. in* $[0,1]$*;*
4. $k,\ q$ *– bounded and measurable;*
5. $c_0 > 0,\ k \geq c_0$ *everywhere in* $[0,1]$*;*
6. $q \geq 0$ *a.e. in* $[0,1]$*;*
7. k *may have discontinuities of only the first kind;*
8. *"conjugation conditions" hold:* $u,\ ku'$ *– continuous in* $(0,1)$.

Then,

a. *either the function* u *is constant on* $(0,1)$,

b. *or, depending on the sign in condition 3 above, it can not attain a strictly positive global maximum (strictly negative global minimum) for* $x_0 \in (0,1)$.

Proof. Assume that the opposite is true: without loss of generality,

$\exists x_0 : x_0 \in (0,1), u(x_0) = \max\limits_{x\in[0,1]} u(x) = M_0 > 0,\ \exists\delta > 0$:

$$(\forall x : x_0 - \delta < x < x_0 \Rightarrow u(x) < u(x_0),\ \forall x \in (x_0, x_0+\delta) \Rightarrow u(x_0) \geq u(x)).$$

(It is always possible to find global maximum in $(0,1)$, which is strict at least from one of the sides, because, by assumption, $u \neq \text{const}$. The functions q and ψ are bounded and measurable, u is continuous, therefore, $(ku')'$ is bounded and measurable, therefore, ku' is absolutely continuous in $[0,1]$. Hence, since $\frac{1}{k}$

is measurable and bounded, ($k \geq c_0$ in $[0,1]$), it follows that u' is measurable, therefore, u is absolutely continuous. Hence,

$$u(x) = \int_{x_0}^{x} u'(t)dt + u(x_0) = \int_{x_0}^{x} u'(t)dt + M_0,\ x \in [0,1].$$

$\forall x \in (x_0 - \delta, x_0)$ it is fulfilled that

$$u(x) = \int_{x_0}^{x} u'(t)dt + M_0 < M_0,$$

Therefore,

$$u(x) = \int_{x_0}^{x} u'(t)dt < 0,$$

i.e.,

$$u(x) = \int_{x}^{x_0} u'(t)dt > 0,$$

therefore,

$$\exists S_x : S_x \subset (x, x_0), \mu(S_x) > 0, \xi \in S \Rightarrow u'(\xi) > 0$$

(S_x can be chosen to be measurable, since u' is measurable. This implies

$$k(\xi)u'(\xi) > 0, \xi \in S_x, x \in (x_0 - \delta, x_0).$$

Furthermore,

$$k(x)u'(x) = \int_{x_0}^{x} (k(t)u'(t))'dt + k(x_0)u'(x_0),\ x \in [0,1].$$

Since ku' is continuous, $k \neq 0$ in $[0,1]$ and, by assumption, k may have only discontinuities of the first kind, then also the function u' may have only discontinuities of the first kind. It is clear that $u'(x_0 - 0) \geq 0, u'(x_0 + 0) \leq 0$, wherefrom $k(x_0 - 0)u'(x_0 - 0) \geq 0, k(x_0 + 0)u'(x_0 + 0) \leq 0$ and, by continuity, $k(x_0)u'(x_0) = 0$. Therefore,

$$k(x)u'(x) = \int_{x_0}^{x} (k(t)u'(t))'dt,\ x \in [0,1].$$

Hence, $\forall x \in (x_0 - \delta, x_0 + \delta),\ \forall \xi \in S_x$ it holds true that

$$\int_{x_0}^{\xi} (k(t)u'(t))' dt > 0,$$

i.e.,

$$\int_{\xi}^{x_0} (k(t)u'(t))' dt < 0,$$

therefore, $\exists S_{x,\xi} : S_{x,\xi} \subset (\xi, x_0),\ \mu(S_{x,\xi}) > 0 :\ \forall \zeta \in S_{x,\xi}$ it is fulfilled that $(k(\zeta)u'(\zeta))' < 0$.

The continuity of u implies $\exists d > 0 :\ \forall x \in (x_0 - d, x_0)$ it holds true that $u(x) \geq \frac{M_0}{2} > 0$.

Let $x \in (x_0 - d, x_0) \cap (x_0 - \delta, x_0), \xi \in S_x$. From $q \geq 0$ a.e. in $[0,1]$ it follows that for a.e. $\zeta \in S_{x,\xi}$ it is fulfilled that

$$(k(\zeta)u'(\zeta))' - q(\zeta)u(\zeta) \leq (k(\zeta)u'(\zeta))' - q(\zeta)\frac{M_0}{2} \leq (k(\zeta)u'(\zeta))' < 0,$$

which leads to contradiction with $\psi \geq 0$ a.e. in $[0,1]$, because $\mu(S_{x,\xi}) > 0$. $\square$

Lemma 2. *For the Green's function for (1) with $\alpha = \beta = 0$, under the the condition that **P1–4.** are fulfilled, the following estimates hold true:*

$$0 < G(x,\xi) \leq \frac{1}{c_0},\ x, \xi \in [0,1]$$

$$|\frac{\partial}{\partial x} G(x,\xi)| \leq \frac{1}{c_0}(1 + \frac{\|q\|_{L_1}}{c_0}), |\frac{\partial}{\partial \xi} G(x,\xi)| \leq \frac{1}{c_0}(1 + \frac{\|q\|_{L_1}}{c_0}),$$
for $x, \xi \in [0,1], x \neq \xi$.

Proof. The proof follows the analogy with [9, Lemma 1, p. 123]. For the Green's function for (1) with $\alpha = \beta = 0, q \equiv 0$ a.e. with $[0,1]$ we can find an explicit representation: $A(x),\ B(x)$ in **P3** are respectively determined by

$$\begin{cases} (kA')' = 0,\ A(0) = 0,\ k(0+0)A'(0+0) = 1, \\ (kB')' = 0,\ B(1) = 0,\ k(1-0)B'(1-0) = -1, \\ A, B, kA', kB' - \text{continuous}, \end{cases}$$

wherefrom

$$A(x) = \int_0^x \frac{dt}{k(t)},$$

$$B(x) = \int_x^1 \frac{dt}{k(t)},$$

and the lemma's statement about G_0 follows easily.

Let us consider now the general case $q \geq 0$ a.e. in $[0,1]$. We set $\rho(x,\xi) = G_0(x.\xi) - G(x,\xi),\ x,\xi \in [0,1]$. Since G_0 and G are continuous in x and ξ, it follows that also ρ has the same property. Because **P2** implies that $\frac{\partial}{\partial x}(k\frac{\partial}{\partial x}G_0)$ and $\frac{\partial}{\partial x}(k\frac{\partial}{\partial x}G)$ are summable in x in $[0,\xi)$ and $(\xi,1]$, then, $k\frac{\partial}{\partial x}G_0$ and $k\frac{\partial}{\partial x}G$ are absolutely continuous in $[0,\xi)$ and $(\xi,1]$, hence, also $k\frac{\partial\rho}{\partial x}$ has the same property. Moreover, since $k\frac{\partial}{\partial x}G_0$ and $k\frac{\partial}{\partial x}G$ have the same jump at $x=\xi$, it follows that $k\frac{\partial\rho}{\partial x}$ is continuous in x for $x=\xi$, hence, absolutely continuous in x in $[0,1]$. From here and from **P2**, we obtain $\forall\xi \in [0,1]$

$$\begin{cases} \dfrac{\partial}{\partial x}\left(k(x)\dfrac{\partial\rho}{\partial x}(x,\xi)\right) - q(x)\rho(x,\xi) = -q(x)G_0(x,\xi), \quad x \in [0,1], \\ \rho(0,\xi) = \rho(1,\xi) = 0 \end{cases}$$

Obviously, $-q(x)G_0(x,\xi) \leq 0$, a.e. $x \in [0,1]$. Lemma 1 and $\rho(0,\xi) = \rho(1,\xi) = 0$ yield $\rho(x,\xi) \geq 0$. $x,\xi \in [0,1]$, i.e., $G(x,\xi) \leq G_0(x,\xi)$. From here and from $G(x,\xi) > 0$ we get $0 < G(x,\xi) \leq \frac{1}{c_0},\ x,\xi \in [0,1]$.

Furthermore, since for $0 \leq x < \xi\ \ \xi < x \leq 1$ $k\frac{\partial}{\partial x}G$ is absolutely continuous in x, we have (for $0 \leq x < \xi$)

$$\begin{aligned}
|\frac{\partial}{\partial x}G(x,\xi)| &= \frac{1}{k(x)}|k(x)\frac{\partial}{\partial x}G(x,\xi)| \\
&= \frac{1}{k(x)}|k(0)\frac{\partial}{\partial x}G(0,\xi) + \int_0^x q(t)G(t,\xi)dt| \\
&= \frac{1}{k(x)}|k(0)A'(0)\frac{B(\xi)}{A(1)} + \int_0^x q(t)G(t,\xi)dt| \\
&\leq \frac{1}{c_0}|1.\frac{B(\xi)}{A(1)}| + |\int_0^x q(t)G(t,\xi)dt| \\
&\leq \frac{1}{c_0}1 + |\int_0^x q(t)G(t,\xi)dt| \\
&\leq \frac{1}{c_0}(1 + \frac{\|q\|_{L_1}}{c_0}).
\end{aligned}$$

Here we used **P3.1** $|G(t,\xi)| = G(t,\xi) \leq \frac{1}{c_0}$.

The same estimate is being proved in an analogous way for $\xi < x \leq 1$. (In this case, the integral representation is $-\frac{A(\xi)}{A(1)} - \int\limits_0^x q(t)G(t,\xi)dt$.)

It is easy to verify with the help of **P3** that

$$\frac{\partial}{\partial \xi}G(a,b) = \frac{\partial}{\partial x}G(b,a), \quad a,b \in [0,1],$$

wherefrom the estimate for $\frac{\partial}{\partial \xi}G$ follows immediately from the one for $\frac{\partial}{\partial x}G$. □

Lemma 3. *Under the conditions of Lemma 2 it is fulfilled that*

$$\frac{\partial}{\partial x}\left(k(\alpha)\frac{\partial}{\partial \alpha}G(\alpha,x)\mid_{\alpha=1-0}\right) \leq \frac{1}{c_0}\left(\frac{1}{\|\frac{1}{k}\|_{L_1}} + \frac{\|q\|_{L_1}}{c_0}\right),$$

$$\frac{\partial}{\partial x}\left(k(\alpha)\frac{\partial}{\partial \alpha}G(\alpha,x)\mid_{\alpha=0+0}\right) \leq \frac{1}{c_0}\left(\frac{1}{\|\frac{1}{k}\|_{L_1}} + \frac{\|q\|_{L_1}}{c_0}\right), \quad x \in [0,1].$$

Proof. We shall prove the correctness of the first formula. The correctness of the second one can be proven analogously.

Under the conditions of the lemma, it is easy to verify that

$$\begin{aligned}\frac{\partial}{\partial x}\left(k(\alpha)\frac{\partial}{\partial \alpha}G(\alpha,x)\mid_{\alpha=1-0}\right) &= \varphi(x)\\ &= k(1-0)A'(x)\frac{B'(1-0)}{A(1)}\\ &= A'(x)k(1)\frac{B'(1)}{A(1)}\\ &= \frac{A'(x)}{A(1)}.\end{aligned}$$

$$\begin{aligned}|\varphi(x)| = &= \frac{|A'(x)|}{A(1)}\\ &= \frac{|k(x)A'(x)|}{k(x)A(1)}\\ &= \tfrac{1}{k(x)A(1)}|k(0)A'(0) + \int\limits_0^x q(t)A(t)dt|\\ &= \frac{1}{k(x)A(1)}|1 + \int\limits_0^x q(t)A(t)dt| \leq\\ &\leq \tfrac{1}{c_0 A(1)} + \tfrac{\|q\|_{L_1}A(1)}{c_0 A(1)}\\ &= \frac{1}{c_0 A(1)} + \frac{\|q\|_{L_1}}{c_0}\end{aligned}$$

We shall estimate $A(1)$ from below with a positive quantity.

$$\begin{aligned} A(1) &= A(0) + \int_0^1 A'(t)dt \\ &= \int_0^1 A'(t)dt \\ &= \int_0^1 \frac{k(t)}{k(t)} A'(t)dt \\ &= \int_0^1 \frac{1}{k(t)} \left(k(0)A'(0) + \int_0^t (k(\xi)A'(\xi))' d\xi \right) dt \\ &= \int_0^1 \frac{1}{k(t)} \left(1 + \int_0^t q(\xi)A'(\xi) d\xi \right) dt \\ &\geq \int_0^1 \frac{dt}{k(t)}. \end{aligned}$$

This immediately yields the estimate in quest. □

In the sequel we shall be making use of the following sufficient condition for differentiation in a parameter under the sign of the integral (i.e., commutation of differentiation with respect to a parameter, and integration).

Theorem 3. **(K.)** *(See [12, p. 748].) Assume that*

1. *$\varphi = \varphi(x,t), x \in [a,b], t \in [c,d]$ is summable with respect to t in $[c,d]$ for any $x \in [a,b]$;*
2. *$\frac{\partial}{\partial x}\varphi(x,t)$ exists for every $x \in [a,b],\ t \in [c,d]$;*
3. *$\frac{\partial}{\partial x}\varphi(x,t)$ is also summable with respect to $t \in [c,d]$ for any $x \in [a,b]$;*
4. *$\frac{\partial}{\partial x}\varphi(x,t)$ is bounded uniformly in x,t: $|\frac{\partial}{\partial x}\varphi(x,t)| \leq c < \infty$, for every $x \in [a,b],\ t \in [c,d]$.*

Then,

$$\left(\int_c^d \varphi(x,t)dt \right)' = \int_c^d \frac{\partial}{\partial x}\varphi(x,t)dt, \quad \forall x \in [a,b].$$

We shall be using more often the following

Lemma 4. *Assume that*

1. *the conditions of Theorem 3 hold;*
2. $a = c,\ b = d$*;*
3. $\forall x \in [a, b]$ *the point* $t = x$ *is a Lebesgue point for* $\varphi(x, \cdot)$ *considered as a function of* $t \in [a, b]$.

Then,

$\forall x \in [a, b]$ *it holds true that*

$$\left(\int_a^x \varphi(x,t)dt\right)' = \varphi(x,x) + \int_a^x \frac{\partial}{\partial x}\varphi(x,t)dt. \tag{7}$$

Proof. Choose $\xi : x + \xi \in [a, b], \xi \neq 0$. Evidently,

$$\frac{1}{\xi}\left(\int_x^{x+\xi} \varphi(x+\xi,t)dt - \int_a^x \varphi(x,t)dt\right)$$
$$= \frac{1}{\xi}\int_x^{x+\xi} \varphi(x+\xi,t)dt + \frac{1}{\xi}\int_a^x (\varphi(x+\xi,t) - \varphi(x,t))dt$$

$$\frac{1}{\xi}\int_a^x (\varphi(x+\xi,t) - \varphi(x,t))dt \to \int_a^x \frac{\partial}{\partial x}\varphi(x,t)dt, \quad \xi \to 0, x \in [a,b],$$

follows from a theorem by Arzelà (see [12, p. 745]), analogously to the proof of Theorem 3.

It remains to show that

$$\frac{1}{\xi}\int_x^{x+\xi} \varphi(x+\xi,t)dt \to \varphi(x,x), \quad \xi \to 0,\ x \in [a,b].$$

Obviously,

$$\frac{1}{\xi}\int_x^{x+\xi} \varphi(x+\xi,t)dt = \frac{1}{\xi}\int_x^{x+\xi} (\varphi(x+\xi,t) - \varphi(x,t))dt + \frac{1}{\xi}\int_x^{x+\xi} \varphi(x,t)dt.$$

Under the sign of the integral in the first summand we apply the formula for Taylor expansion with integral remainder:

$$\begin{aligned}|\frac{1}{\xi}\int\limits_{x}^{x+\xi}(\varphi(x+\xi,t)-\varphi(x,t))dt| &\le \frac{1}{|\xi|}\int\limits_{\min\{x,x+\xi\}}^{\max\{x,x+\xi\}}|\varphi(x+\xi,t)-\varphi(x,t)|dt\\ &\le \frac{1}{|\xi|}\int\limits_{\min\{x,x+\xi\}}^{\max\{x,x+\xi\}}\int\limits_{\min\{x,x+\xi\}}^{\max\{x,x+\xi\}}|\frac{\partial}{\partial\eta}\varphi(\eta,t)|d\eta dt\\ &\le c|\xi|\to+0,\quad \xi\to 0\end{aligned}$$

Therefore, it suffices to show that

$$\frac{1}{\xi}\int\limits_{x}^{x+\xi}\varphi(x,t)dt\to\varphi(x,x),\quad \forall x\in[a,b].$$

This immediately follows from the assumption that $t=x$ is a Lebesgue point for $\varphi(x,\cdot)$. ☐

4.2.2. Remarks

Remark 1. In some cases in the sequel, the existence is needed of only one-sided (left-hand or right-hand) derivative in the parameter x under the sign of the integral (i.e., commutation between one-sided differentiation in the parameter x, and integration). In this case, Lemma 4 continues to hold, with the following modifications (with no loss of generality, consider the case of left-hand onesided derivative). The condition that $t=x$ be a Lebesgue point for $\varphi(x.\cdot)$ (cf. [7, p. 303]) is being replaced by $\varphi(x,x-0)\ne\pm\infty$, $\lim\limits_{\xi\to+0}\frac{1}{\xi}|\varphi(x,t)-\varphi(x,x-0)|dt=0$.

The conclusion of (7) takes the form

$$\frac{\partial}{\partial\alpha}\int\limits_{a}^{\alpha}\varphi(\alpha,t)dt\,|_{\alpha=x-0}=\varphi(x,x-0)+\int\limits_{a}^{x}\frac{\partial}{\partial\alpha}\varphi(\alpha,t)\,|_{\alpha=x-0}\,dt.$$

Remark 2. In the presence of "conjugation conditions" (u,ku' – continuous) the properties **P1–4** continue to hold true also under the more general assumptions for $k,\ q,\ f$ under which problem (1) was considered in section 2.1, under the condition that the problems (5) and (6) have a solution. In this more general case the proof is analogous to the classical case plus the following specific modifications:

(a) uniqueness in **P1** follows from Lemma 1;

(b) existence can be proved via a direct verification that $\int\limits_0^1 G(x,\xi)f(\xi)d\xi$ is a solution to the problem (1) (cf. [6, p. 299]). Here the validity of the differentiation with respect to the parameter x in the integral $\int\limits_0^1 G(x,\xi)f(\xi)d\xi$ follows from Theorem 3, while in the integrals

$$\int\limits_0^x k(x)\frac{\partial}{\partial x}G(x,\xi)f(\xi)d\xi, \qquad \int\limits_x^1 k(x)\frac{\partial}{\partial x}G(x,\xi)f(\xi)d\xi$$

it follows from Lemma 4, by taking in consideration also Remark 1.

(c) $\int\limits_0^1 G(x,\xi)f(\xi)d\xi$ satisfies the equation in (1) only in the points of continuity of f (cf. [6, p. 299]).

Remark 3. The argumentation in this section, with relatively minor technical modifications, can be carried through also in the general case of inhomogeneous boundary conditions. For the construction of the Green's function in this more general case, cf., e.g., [6, p. 300].

4.3. Error Estimates Directly in Terms of Properties of the Coefficients and Right-Hand Side

4.3.1. Main Results

For the proof of the main results we shall need the following lemma.

Lemma 5. *Let Ω be an arbitrary interval,* $1 \le p \le \infty,\ n \in \mathbb{N},\ h > 0$. *Then,*

$$\omega_n(fg;h)_{L_p(\Omega)} \le \sum_{\nu=0}^{n} \binom{n}{\nu} \omega_\nu(f;h)_{L_{p_\nu}(\Omega)} \omega_{n-\nu}(g;h)_{L_{p'_\nu}(\Omega)},$$

$$\tau_n(fg;h)_{L_p(\Omega)} \le \sum_{\nu=0}^{n} \binom{n}{\nu} \tau_\nu(f;\frac{2nh}{r_\nu})_{L_{p_\nu}(\Omega)} \tau_{n-\nu}(g;2h)_{L_{p'_\nu}(\Omega)},$$

$$r_0 = r_1,\ r_\nu = \nu,\ \nu \in \mathbb{N}.$$

for all f, g, for which the integral (averaged) moduli in the above inequalities make sense, setting also, by definition,

$$\omega_0(\varphi;h)_{L_r(\Omega)} := \|\varphi\|_{L_r(\Omega)},$$

$$\tau_0(\varphi;h)_{L_r(\Omega)} := \|\varphi\|_{A_{r,h}},$$

Here $p_\nu:\ p_\nu \geq p,\ p'_\nu \geq p,\ \frac{1}{p_\nu}+\frac{1}{p'_\nu}=\frac{1}{p},\ \nu=0,1,\dots,n.$

Proof. Essentially, the formulae in the lemma are analogues of the Leibniz' formula, with analogous proof. For $t, t+nd \in \Omega$ the formula

$$(\Delta_\delta^n fg)(x) \leq \sum_{\nu=0}^{n} \binom{n}{\nu} (\Delta_\delta^n f)(x+(n-\nu)\delta)(\Delta_\delta^n g)(x)$$

is being proved by induction in the same way as the Leibniz' formula.

This easily implies

$$\omega_n(f,x;h) \leq \sum_{\nu=0}^{n} \binom{n}{\nu} \omega_\nu(f,x;\frac{2nh}{r_\nu})\omega_{n-\nu}(g,x;2h).$$

Raising the left-hand and right-hand sides in the latter inequality to power p, integrating in x from 0 to 1, raising in power $\frac{1}{p}$, and applying consecutively the inequalities of Minkowski and Hölder, we obtain the second formula in the lemma. The proof of the first formula in the lemma is analogous, but simpler. □

For simplicity, we shall prove the main results under the assumptions of the classical case ($k,\ q$ – piecewise continuous, f – continuous), and for homogeneous boundary conditions $\alpha=\beta=0$ (1, 2). In Remarks 4, 6 below and in [23] we shall show that our arguments remain valid also under much more general assumptions about $k,\ q,\ f$, close to those in the formulation of problem (1) in section 2.1, and for the general case of inhomogeneous boundary conditions.

Theorem 4. *Let us solve problem (1) approximately via the discrete problem (2), under the assumptions that*

1. *k, q are piecewise continuous;*

2. *f is continuous.*

Then, for the error z satisfying (3), the following estimates holds true:

$$
\begin{aligned}
\|z\|_{l^\infty(\Sigma_h)} \le \frac{2}{c_0}\Big[\quad & c_1(k,q,f)\left(\tfrac{3}{2c_0}\tau_2(k;h)_{L_1} + 2\|k\|_{A_{p',4h}}\tau_2(\tfrac{1}{k};2h)_{L_p}\right)\\
&+c_2(k,f)\tau_2(q;h)_{L_1} + 2\tau_2(f;h)_{L_1}\\
&+c_3(k,q,f;p,p_1)h\tau_1(\frac{1}{k};2h)_{L_{p_1}}\\
&+c_4(k,q,f;r)h\omega_1(\frac{1}{k};h)_{L_r}\\
&+c_5(k,f;p)h\omega_1(q;2h)_{L_p}\\
&+c_6(k,q;p)h\omega_1(f;2h)_{L_p}\\
&+c_7(k,q,f;p,p_2,p_3,r)h^2\Big],
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{p'}=\frac{1}{r}+\frac{1}{r'}=1,\ \frac{1}{p_j}+\frac{1}{p_j'}=\frac{1}{p},\ j=1,2,3,$

$$
\begin{aligned}
c_1(k,q,f) &\le (1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1},\\
c_2(k,f) &\le \frac{\|f\|_{L_1}}{c_0},\\
c_3(k,q,f;p,p_1) &\le 16\|k\|_{A_{p',4h}}\left[(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_{p_1'}} + \|q\|_{L_{p_1'}}\frac{\|f\|_{L_1}}{c_0}\right],\\
c_4(k,q,f;r) &\le 16\|q\|_{A_{r',h}}(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1},\\
c_5(k,f;p) &\le \frac{64}{c_0}\|k\|_{A_{p',4h}}\|f\|_{L_1},\\
c_6(k,q;p) &\le \frac{64}{c_0}\|k\|_{A_{p',4h}}(1+\frac{\|q\|_{L_1}}{c_0}),\\
c_7(k,q,f;p,p_2,p_3,r) &\le \frac{128}{c_0^2}\|k\|_{A_{p',4h}}\left\{\|q\|_{L_{p_2'}}\left[\|f\|_{L_{p_2}} + (1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}\right]\right.\\
&\left. + \|f\|_{L_{p_3'}}\left[\frac{\|q\|_{L_{p_3}}}{c_0} + \frac{1}{\|\frac{1}{k}\|_{L_1}} + (1+\frac{(1+\|q\|_{L_1})}{c_0})\|q\|_{L_1}\right]\right\}\\
&+\frac{12}{c_0}\|q\|_{A_{r',h}}\left[(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_r} + \|q\|_{L_r}\frac{\|f\|_{L_1}}{c_0}\right].
\end{aligned}
$$

Proof. It suffices to estimate $\|u'\|_{A_{r',h}},\ \tau_2(u';h)_{L_{r_2}},\ \|u\|_{A_{r_3',h}},\ \omega_1(u';h)_{L_{r_4}}$ in Theorem 1.

We shall use repeatedly that $k(x)\frac{\partial}{\partial x}G(x,\xi)$ is absolutely continuous in $x \in$

$[0,\xi)\cup(\xi,0]$, wherefrom,

$$k(x)\frac{\partial}{\partial x}G(x,\xi)=k(0)\frac{\partial}{\partial x}G(0,\xi)+\int\limits_0^x q(\zeta)G(\zeta,\xi)d\zeta,\quad x\in[0,\xi),\tag{8}$$

$$k(x)\frac{\partial}{\partial x}G(x,\xi)=k(1)\frac{\partial}{\partial x}G(1,\xi)-\int\limits_x^1 q(\zeta)G(\zeta,\xi)d\zeta,\quad x\in(\xi,1],\tag{9}$$

where, moreover, it is fulfilled that

$$|k(0)\frac{\partial}{\partial x}G(0,\xi)|\le 1,\ |k(1)\frac{\partial}{\partial x}G(1,\xi)|\le 1,\quad \xi\in[0,1].\tag{10}$$

(10) follows from P3 in section 4.2. Indeed,

$$|k(0)\frac{\partial}{\partial x}G(0,\xi)|=\left|k(0)A'(0)\frac{B(\xi)}{A(1)}\right|=\frac{B(\xi)}{A(1)}\le 1,$$

and analogously for $k(1)\frac{\partial}{\partial x}G(1,\xi)$.

We shall also be using repeatedly

$$\|\frac{1}{k}\|_{BM}=\frac{1}{c_0}\text{ and }\|\cdot\|_{BM}=\|\cdot\|_{A_{\infty.h}}.\tag{11}$$

Now we proceed to the estimation of the above-said quantities in Theorem 1.

By Lemma 2,

$$\|u\|_{A_{r_3',h}}\le\|u\|_c\le\frac{\|f\|_{L_1}}{c_0}\tag{12}$$

By Theorem 3 and Lemma 2,

$$\|u'\|_{A_{r_3',h}}\le\|u'\|_{BM}\le\frac{1}{c_0}\left(1+\frac{\|q\|_{L_1}}{c_0}\right)\|f\|_{L_1}.\tag{13}$$

Let us estimate $\omega_1(u';h)_{L_r},\ 1\le r\le\infty$.

$$\omega_1(u';h)_{L_r}=\omega_1\left(\int\limits_0^{\cdot}\frac{\partial}{\partial x}G(\cdot,t)f(t)dt+\int\limits_{\cdot}^1\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_2}\le$$

$$\le\omega_1\left(\int\limits_0^{\cdot}\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_2}+\omega_1\left(\int\limits_{\cdot}^1\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_2}.\tag{14}$$

The estimates of the two summands are analogous. Let us estimate the first one of them.

$$\begin{aligned}
\omega_1\left(\int_0^{\cdot}\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_2} &= \omega_1\left(\frac{1}{k(\cdot)}\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_2}\\
&\le \frac{1}{c_0}\omega_1\left(\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_2}\\
&+\|\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt\|_{BM}\omega_1(\frac{1}{k};h)_{L_r}.
\end{aligned} \tag{15}$$

Here we used (11) and Lemma 5.

We can estimate $\|\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt\|_{BM}$ in two ways, as follows.

On the one side,

$$\|\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt\|_{BM}\le\|kf\|_{L_1}.\frac{1}{c_0}(1+\frac{\|q\|_{L_1}}{c_0}),$$

according to Lemma 2.

On the other side,

$$\begin{aligned}
\|\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt\|_{BM} &= \|\int_0^{\cdot}\left(k(1)G(1,t)-\int_{\cdot}^{1}q(\xi)G(\xi,t)d\xi\right)f(t)dt\|_{BM}\\
&\le\|\int_0^{\cdot}\left(1+\int_{\cdot}^{1}|q(\xi)|G(\xi,t)d\xi\right)|f(t)|dt\|_{BM}\\
&\le(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1},
\end{aligned}$$

according to (9, 10), Lemma 2.

It is easy to see that the second estimate is sharper than the first one. Indeed, $\|f\|_{L_1}\le\frac{\|kf\|_{L_1}}{c_0}$, because $\forall x\in[0,1]$ $|f(x)|=\frac{k(x)}{k(x)}|f(x)|\le\frac{|k(x)f(x)|}{c_0}$ and the norm in L_1 is monotone. In conclusion,

$$\|\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt\|_{BM}\le\left(1+\frac{\|q\|_{L_1}}{c_0}\right)\|f\|_{L_1}. \tag{16}$$

Let us estimate $\omega_1\left(\int\limits_0^{\cdot} k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_r}$. On the one side,

$$\omega_1\left(\int\limits_0^{\cdot} k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_r}$$

$$\leq h\left\|k(\cdot)\frac{\partial}{\partial x}G(\cdot,\cdot-0)f(\cdot)+\int\limits_0^{\cdot}\frac{\partial}{\partial x}(k(\cdot)\frac{\partial}{\partial x}G(\cdot,t))f(t)dt\right\|_{L_r}$$

$$\leq h\left\|k(\cdot)\frac{\partial}{\partial x}G(\cdot,\cdot-0)f(\cdot)+q(\cdot)\int\limits_0^{\cdot}G(\cdot,t))f(t)dt\right\|_{L_r}$$

$$\leq h\|kf\|_{L_r}\frac{1}{c_0}(1+\frac{\|q\|_{L_1}}{c_0})+h\left\|q(\cdot)\int\limits_0^{\cdot}G(\cdot,t))f(t)dt\right\|_{L_r}$$

$$\leq h\|kf\|_{L_r}\frac{1}{c_0}(1+\frac{\|q\|_{L_1}}{c_0})+\frac{h}{c_0}\|q\|_{L_r}\|f\|_{L_1}$$

by Lemma 4, Remark 1 and Lemma 2. On the other side,

$$\omega_1\left(\int\limits_0^{\cdot} k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_r}$$

$$=\omega_1\left(\int\limits_0^{\cdot} k(\cdot)\left(k(1)G(1,t)-\int\limits_{\cdot}^{1}q(\xi)G(\xi,t)d\xi\right)f(t)dt;h\right)_{L_r}$$

$$\leq h\left\|\left(k(1)\frac{\partial}{\partial x}G(1,\cdot)-\int\limits_{\cdot}^{1}q(\xi)G(\xi,\cdot)d\xi\right)f(\cdot)+q(\cdot)\int\limits_0^{\cdot}G(\cdot,t)f(t)dt\right\|_{L_r}$$

$$\leq h\left\|\left(1+\int\limits_0^{\cdot}|q(\xi)|G(\xi,\cdot)d\xi\right)f(\cdot)\right\|_{L_r}+h\left\|q(\cdot)\int\limits_0^{\cdot}G(\cdot,t)f(t)dt\right\|_{L_r}$$

$$\leq h(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_r}+\frac{h}{c_0}\|q\|_{L_r}\|f\|_{L_1},$$

according to (9), Lemma 4, (10), Lemma 2.

In the same way as with (16), it is seen that the second estimate is better than the first one. In conclusion,

$$\omega_1\left(\int_0^{\cdot} k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_r} \le h(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_r} + \frac{h}{c_0}\|q\|_{L_r}\|f\|_{L_1}. \quad (17)$$

(15 – 17) yield

$$\omega_1\left(\int_0^{\cdot} \frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_2} = \frac{h}{c_0}\left((1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_r} + \|q\|_{L_r}\frac{\|f\|_{L_1}}{c_0}\right) + (1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}\omega_1(\frac{1}{k};h)_{L_r}.$$

In the same way, and by the same quantity, also $\omega_1\left(\int_x^1 \frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_2}$ is being estimated. Therefore, in view of (14),

$$\omega_1(u';h)_{L_r} = \frac{2h}{c_0}\left((1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_r} + \|q\|_{L_r}\frac{\|f\|_{L_1}}{c_0}\right) + 2(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}\omega_1(\frac{1}{k};h)_{L_r}. \quad (18)$$

Let us estimate $\tau_2(u';h)_{L_p},\ 1 \le p \le \infty$.

$$\tau_2(u';h)_{L_p} = \tau_2\left(\int_0^1 \frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_p} \le$$

$$\le \tau_2\left(\int_0^{\cdot} \frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_p} + \tau_2\left(\int_{\cdot}^1 \frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_p}. \quad (19)$$

The estimates of the two summands are analogous. Let us estimate the first one of them. By Lemma 5 and (11)

$$\tau_2\left(\int_0^{\cdot}\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_p}=\tau_2\left(\frac{1}{k(\cdot)}\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_p}$$

$$\le\|\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt\|_{BM}\tau_2(\frac{1}{k};2h)_{L_p}+$$

$$+2\tau_1\left(\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;4h\right)_{L_{p_1'}}\tau_1(\frac{1}{k};2h)_{L_p} \tag{20}$$

$$+\frac{1}{c_0}\tau_2\left(\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;2h\right)_{L_p}.$$

In view of (16), it is enough to estimate $\tau_1\left(\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;4h\right)_{L_{p_1'}}$, $\tau_2\left(\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;2h\right)_{L_p}$.

The estimate of $\tau_1\left(\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;4h\right)_{L_{p_1'}}$ is derived analogously to (17). The differences are also in the bounds for the constants:

$$\tau_1\left(\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;4h\right)_{L_{p_1'}}$$
$$\le 4h(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_r}+4h\|q\|_{L_r}\frac{\|f\|_{L_1}}{c_0}. \tag{21}$$

It remains to estimate $\tau_2\left(\int_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;2h\right)_{L_p}$. From (9), the properties of the τ-moduli, Lemma 4 and Remark 1 it follows that

$$\tau_2\left(\int\limits_0^{\cdot} k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;2h\right)_{L_p}$$

$$=\tau_2\left(\int\limits_0^{\cdot}\left(k(1)\frac{\partial}{\partial x}G(1,t)-\int\limits_{\cdot}^{1}q(\xi)G(\xi,t)d\xi\right)f(t)dt;2h\right)_{L_p}$$

$$\leq 32h\omega_1\left(\left(k(1)\frac{\partial}{\partial x}G(1,\cdot)-\int\limits_{\cdot}^{1}q(\xi)G(\xi,\cdot)d\xi\right)f(\cdot)+q(\cdot)\int\limits_0^{\cdot}G(\cdot,t)f(t)dt;2h\right)_{L_p}$$

$$\leq 32h\omega_1\left(\left(k(1)\frac{\partial}{\partial x}G(1,\cdot)-\int\limits_{\cdot}^{1}q(\xi)G(\xi,\cdot)d\xi\right)f(\cdot);2h\right)_{L_p}$$

$$+32h\omega_1\left(q(\cdot)\int\limits_0^{\cdot}G(\cdot,t)f(t)dt;2h\right)_{L_p}. \tag{22}$$

Let us estimate the second summand. By Lemma 5, the properties of the ω-moduli, Lemma 4, Lemma 2 and (11)

$$\omega_1\left(q(\cdot)\int\limits_0^{\cdot}G(\cdot,t)f(t)dt;2h\right)_{L_p}$$

$$\leq \|q\|_{L'_{p_2}}\omega_1\left(\int\limits_0^{\cdot}G(\cdot,t)f(t)dt;2h\right)_{L_p}+\omega_1(q;2h)_{L_p}\|\int\limits_0^{\cdot}G(\cdot,t)f(t)dt\|_{BM}$$

$$\leq \|q\|_{L'_{p_2}}.2h\|G(\cdot,\cdot)f(\cdot)+\int\limits_0^{\cdot}\frac{\partial}{\partial x}G(\cdot,t)f(t)dt\|_{L_{p_2}}+\frac{\|f\|_{L_1}}{c_0}\omega_1(q;2h)_{L_p}.$$

By Lemma 2

$$\|G(\cdot,\cdot)f(\cdot)+\int\limits_0^{\cdot}\frac{\partial}{\partial x}G(\cdot,t)f(t)dt\|_{L_{p_2}}\leq\|G(\cdot,\cdot)f(\cdot)\|_{L_{p_2}}+\|\int\limits_0^{\cdot}\frac{\partial}{\partial x}G(\cdot,t)f(t)dt\|_{L_{p_2}}$$

$$\leq \frac{\|f\|_{L_{p_2}}}{c_0} + (1 + \frac{\|q\|_{L_1}}{c_0})\frac{\|f\|_{L_1}}{c_0}.$$

Therefore,

$$\omega_1\left(q(\cdot)\int_0^{\cdot} G(\cdot,t)f(t)dt; 2h\right)_{L_p}$$
$$\leq \frac{\|q\|_{L'_{p_2}}}{c_0}2h\left(\|f\|_{L_{p_2}} + (1 + \frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}\right) + \frac{\|f\|_{L_1}}{c_0}\omega_1(q; 2h)_{L_p}. \quad (23)$$

Let us now estimate the first summand at the right-hand side in (22). By Lemma 5 and (11),

$$\omega_1\left(\left(k(1)\frac{\partial}{\partial x}G(1,\cdot) - \int_{\cdot}^1 q(\xi)G(\xi,\cdot)d\xi\right) f(\cdot); 2h\right)_{L_p}$$
$$\leq \|k(1)\frac{\partial}{\partial x}G(1,\cdot) - \int_{\cdot}^1 q(\xi)G(\xi,\cdot)d\xi\|_{BM}\omega_1(f; 2h)_{L_p}$$
$$+ \|f\|_{L_{p'_3}}\omega_1(k(1)\frac{\partial}{\partial x}G(1,\cdot) - \int_{\cdot}^1 q(\xi)G(\xi,\cdot)d\xi; 2h)_{L_{p_3}}. \quad (24)$$

From (10) and Lemma 2 it follows that (cf. (16))

$$\|k(1)\frac{\partial}{\partial x}G(1,\cdot) - \int_{\cdot}^1 q(\xi)G(\xi,\cdot)d\xi\|_{BM} \leq \|1 + \int_{\cdot}^1 |q(\xi)|G(\xi,\cdot)d\xi\|_{BM}$$
$$\leq 1 + \frac{\|q\|_{L_1}}{c_0}. \quad (25)$$

The "conjugation conditions" in (1), Lemma 4, Remark 1 and the properties of the ω-moduli together imply

$$\omega_1(k(1)\frac{\partial}{\partial x}G(1,\cdot) - \int_{\cdot}^1 q(\xi)G(\xi,\cdot)d\xi; 2h)_{L_{p_3}}$$
$$= \omega_1(k(1-0)\frac{\partial}{\partial x}G(1-0,\cdot) - \int_{\cdot}^1 q(\xi)G(\xi,\cdot)d\xi; 2h)_{L_{p_3}}$$

$$\leq 2h\|\frac{\partial}{\partial x}\left(\frac{\partial}{\partial \alpha}G(\alpha,\cdot)\mid_{\alpha=1-0}\right)+q(\cdot+0)G(\cdot,\cdot)-\int\limits_{\cdot}^{1}q(\xi)\frac{\partial}{\partial \xi}G(\xi,\cdot)d\xi\|_{L_{p_3}}$$

$$\leq 2h\left(\|\frac{\partial}{\partial x}\left(\frac{\partial}{\partial \alpha}G(\alpha,\cdot)\mid_{\alpha=1-0}\right)\|_{L_{p_3}}+\|q(\cdot+0)G(\cdot,\cdot)\|_{L_{p_3}}\right.$$
$$\left.+\|\int\limits_{\cdot}^{1}q(\xi)\frac{\partial}{\partial \xi}G(\xi,\cdot)d\xi\|_{L_{p_3}}\right). \quad (26)$$

By Lemma 3,

$$\begin{aligned}\|\tfrac{\partial}{\partial x}\left(\tfrac{\partial}{\partial \alpha}G(\alpha,\cdot)\mid_{\alpha=1-0}\right)\|_{L_{p_3}} &\leq \|\tfrac{\partial}{\partial x}\left(\tfrac{\partial}{\partial \alpha}G(\alpha,\cdot)\mid_{\alpha=1-0}\right)\|_{BM}\leq \\ &\leq \tfrac{1}{c_0}\left(\tfrac{1}{\|\frac{1}{k}\|_{L_1}}+\tfrac{\|q\|_{L_1}}{c_0}\right).\end{aligned} \quad (27)$$

By Lemma 2,

$$\|q(\cdot+0)G(\cdot,\cdot)\|_{L_{p_3}}\leq \frac{\|q\|_{L_{p_3}}}{c_0}. \quad (28)$$

(Here we essentially used that the set of discontinuity points of q has zero measure.)

By Lemma 2,

$$\|\int\limits_{\cdot}^{1}q(\xi)\frac{\partial}{\partial \xi}G(\xi,\cdot)d\xi\|_{L_{p_3}}\leq \|\int\limits_{\cdot}^{1}q(\xi)\frac{\partial}{\partial \xi}G(\xi,\cdot)d\xi\|_{BM}\leq$$
$$\leq \frac{1}{c_0}\left(1+\frac{\|q\|_{L_1}}{c_0}\right)\|q\|_{L_1}. \quad (29)$$

(26 – 28) yield

$$\omega_1(k(1)\frac{\partial}{\partial x}G(1,\cdot)-\int\limits_{\cdot}^{1}q(\xi)G(\xi,\cdot)d\xi;2h)_{L_{p_3}}$$
$$\leq \frac{2h}{c_0}\left(\frac{1}{\|\frac{1}{k}\|_{L_1}}+\frac{\|q\|_{L_1}}{c_0}+\frac{\|q\|_{L_{p_3}}}{c_0}+(1+\frac{\|q\|_{L_1}}{c_0})\|q\|_{L_1}\right)$$
$$=\frac{2h}{c_0}\left(\frac{1}{\|\frac{1}{k}\|_{L_1}}+\frac{\|q\|_{L_{p_3}}}{c_0}+(1+\frac{(1+\|q\|_{L_1})}{c_0})\|q\|_{L_1}\right). \quad (30)$$

From (24, 25, 30) it follows

$$\omega_1\left(\left(k(1)\frac{\partial}{\partial x}G(1,\cdot)-\int\limits_{\cdot}^{1}q(\xi)G(\xi,\cdot)d\xi\right)f(\cdot);2h\right)_{L_p}$$
$$\leq (1+\frac{\|q\|_{L_1}}{c_0})\omega_1(f;2h)_{L_p}+\frac{2h}{c_0}\|f\|_{L_{p_3'}}\left(\frac{1}{\|\frac{1}{k}\|_{L_1}}+\frac{\|q\|_{L_1}}{c_0}+\frac{\|q\|_{L_{p_3}}}{c_0}\right.$$
$$\left.+(1+\frac{(1+\|q\|_{L_1})}{c_0})\|q\|_{L_1}\right). \quad (31)$$

From (22, 23, 31) it follows

$$\tau_2\left(\int\limits_0^{\cdot}k(\cdot)\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;2h\right)_{L_p}$$
$$\leq\frac{64h^2}{c_0}\|q\|_{L_{p_3'}}\left(\|f\|_{L_{p_3}}+(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}\right)$$
$$+\frac{32h}{c_0}\|f\|_{L_1}\omega_1(q;2h)_{L_p}+$$
$$+32h(1+\frac{\|q\|_{L_1}}{c_0})\omega_1(f;2h)_{L_p}+ \quad (32)$$
$$+\frac{64h^2}{c_0}\|f\|_{L_{p_3'}}\left(\frac{1}{\|\frac{1}{k}\|_{L_1}}+\frac{\|q\|_{L_1}}{c_0}+\frac{\|q\|_{L_{p_3}}}{c_0}\right.$$
$$\left.+(1+\frac{(1+\|q\|_{L_1})}{c_0})\|q\|_{L_1}\right).$$

From (16, 20, 21, 32) it follows

$$\tau_2\left(\int\limits_0^{\cdot}\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_p}(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}\tau_2(\frac{1}{k};2h)_{L_p}$$
$$+8\left((1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_{p_1'}}+\|q\|_{L_{p_1'}}\frac{\|f\|_{L_1}}{c_0}\right)h\tau_1(\frac{1}{k};2h)_{L_{p_1}}$$
$$+\frac{32}{c_0}(1+\frac{\|q\|_{L_1}}{c_0})h\omega_1(f;2h)_{L_p}$$
$$+\frac{32}{c_0^2}\|f\|_{L_1}h\omega_1(q;2h)_{L_p}$$

$$+\frac{64h^2}{c_0^2}\|q\|_{L_{p_2'}}\left(\|f\|_{L_{p_2}}+(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}\|f\|_{L_1}\right)$$
$$+\|f\|_{L_{p_3'}}\left(\frac{\|q\|_{L_{p_3}}}{c_0}+\frac{1}{\|\frac{1}{k}\|_{L_1}}+\frac{\|q\|_{L_1}}{c_0}+(1+\frac{(1+\|q\|_{L_1})}{c_0})\|q\|_{L_1}\right). \qquad (33)$$

An estimate for the other summand in the right-hand side of (19) is obtained analogously, i.e.,

$$\tau_2\left(\int_{\cdot}^{1}\frac{\partial}{\partial x}G(\cdot,t)f(t)dt;h\right)_{L_p}$$

Hence, in view of (19, 33), the final estimate of $\tau_2(u';h)_{L_p}$ takes the form

$$\begin{aligned}\tau_2(u';h)_{L_p}\le&2(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}\tau_2(\frac{1}{k};2h)_{L_p}+\\&+16\left((1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_{p_1'}}+\|q\|_{L_{p_1'}}\frac{\|f\|_{L_1}}{c_0}\right)h\tau_1(\frac{1}{k};2h)_{L_{p_1}}+\\&+\frac{64}{c_0}(1+\frac{\|q\|_{L_1}}{c_0})h\omega_1(f;2h)_{L_p}+\\&+\frac{64}{c_0^2}\|f\|_{L_1}h\omega_1(q;2h)_{L_p}+\\&+\frac{128h^2}{c_0^2}\left(\|q\|_{L_{p_2'}}\left(\|f\|_{L_{p_2}}+(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}\|f\|_{L_1}\right)+\right.\\&+\|f\|_{L_{p_3'}}\left(\frac{\|q\|_{L_{p_3}}}{c_0}+\frac{1}{\|\frac{1}{k}\|_{L_1}}+\frac{\|q\|_{L_1}}{c_0}\right.\\&\left.\left.+(1+\frac{(1+\|q\|_{L_1})}{c_0})\|q\|_{L_1}\right)\right).\end{aligned} \qquad (34)$$

Theorem 1, (12, 13, 18, 34) together imply

$$\begin{aligned}\|z\|_{l^\infty(\Sigma_h)}\le&\frac{2}{c_0}(\frac{3}{2c_0}(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}\tau_2(k;h)_{L_1}+\\&+2\|k\|_{A_{p',4h}}((1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}\tau_2(\frac{1}{k};2h)_{L_p}\\&+8((1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_{p_1'}}+\|q\|_{L_{p_1'}}\frac{\|f\|_{L_1}}{c_0})h\tau_1(\frac{1}{k};2h)_{L_{p_1}}+\end{aligned}$$

$$+\frac{32}{c_0}(1+\frac{\|q\|_{L_1}}{c_0})h\omega_1(f;2h)_{L_p}+$$

$$+\frac{32}{c_0^2}\|f\|_{L_1}h\omega_1(q;2h)_{L_p}+$$

$$+\frac{64h^2}{c_0^2}(\|q\|_{L_{p_2'}}(\|f\|_{L_{p_2}}+(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}\|f\|_{L_1})+$$

$$+\|f\|_{L_{p_3'}}(\frac{\|q\|_{L_{p_3}}}{c_0}+\frac{1}{\|\frac{1}{k}\|_{L_1}}+\frac{\|q\|_{L_1}}{c_0}+(1+\frac{(1+\|q\|_{L_1})}{c_0})\|q\|_{L_1})))+$$

$$+\frac{\|f\|_{L_1}}{c_0}\tau_2(q;h)_{L_1}+$$

$$+16\|q\|_{A_{r',h}}(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}h\omega_1(\frac{1}{k};h)_{L_r}+$$

$$+\frac{16h^2}{c_0}\|q\|_{A_{r',h}}((1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_r}+\|q\|_{L_r}\frac{\|f\|_{L_1}}{c_0})+$$

$$+2\tau_2(f;h)_{L_1}.$$

Hence, after simple computations and reduction of the common terms, we obtain the statement of the theorem. □

Lemma 6. *Under the conditions of Theorem 4 it is fulfilled that*

$$\tau_1(\frac{1}{k};h)_{L_p}\le\frac{1}{c_0^2}\tau_1(k;h)_{L_p},$$
$$\tau_2(\frac{1}{k};h)_{L_p}\le\frac{1}{c_0^2}\tau_2(k;h)_{L_p}+\frac{2}{c_0^3}\tau_2(k;4h)^2_{L_{2p}}.$$

Proof. Let $t,t+\xi\in[x-\frac{nh}{2},x+\frac{nh}{2}]\cap\Omega,\ n=1,2,\ \Omega=[0,1]$ (or, more generally, an arbitrary interval). The identities

$$(\Delta_\xi^1\frac{1}{k})(t)=-\frac{1}{k(t+\xi)k(t)}(\Delta_\xi^1k)(t),$$
$$(\Delta_\xi^2\frac{1}{k})(t)=\frac{1}{k(t+2\xi)k(t)}(-\Delta_\xi^2k)(t)+\frac{1}{k(t+\xi)}(\Delta_\xi^1k)(t+\xi)(\Delta_\xi^1k)(t).$$

can be verified directly. These identities easily imply the lemma's statement. □

In section 4.3.2 we shall use Lemma 6 in combination with Theorem 4 to derive a variety of new corollaries.

Theorem 5. *Assume that*

1. *the conditions of Theorem 4 hold;*

2. *in problem (2) the "best" scheme has been selected.*

Then, for the error z, the following estimate holds true:

$$
\begin{aligned}
\|z\|_{l^\infty(\Sigma_h)} \leq & \frac{2}{c_0}(c_1(k,q,f;r)\tau_1(k;h)_{L_r}+ \\
& + c_2(k,q,f;p)\omega_1(\frac{1}{k};h)_{L_p}+ \\
& + c_3(k,q,f)\tau_1(q;h)_{L_1}+ \\
& + c_4(k,f)\omega_1(q;h)_{L_1}+ \\
& + \omega_1(f;h)_{L_1}+ \\
& + c_5(k,q,f;p)h\ ),
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{p'}=\frac{1}{r}+\frac{1}{r'}=1,\ \frac{1}{p_j}+\frac{1}{p'_j}=\frac{1}{p},\ j=1,2,3,$

$$
\begin{aligned}
c_1(k,q,f;p) &\leq \frac{1}{2c_0}(\|f\|_{A_{r',h}}+\|q\|_{A_{r',h}}\frac{\|f\|_{L_1}}{c_0}), \\
c_2(k,q,f;p) &\leq 16\|q\|_{A_{r',h}}\|f\|_{L_1}(1+\frac{\|q\|_{L_1}}{c_0}), \\
c_3(k,q,f) &\leq \frac{1}{2c_0}\|f\|_{L_1}(1+\frac{\|q\|_{L_1}}{c_0}), \\
c_4(k,f) &\leq \tfrac{\|f\|_{L_1}}{c_0}, \\
c_5(k,q,f;p) &\leq \frac{16}{c_0}\left(\|q\|_{L_1}\|f\|_{L_1}(1+\frac{\|q\|_{L_1}}{c_0})\right. \\
& \left. +\|q\|_{A_{p',h}}\left((1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_p}+\|q\|_{L_p}\|f\|_{L_1}\right)\right),
\end{aligned}
$$

$\frac{1}{p}+\frac{1}{p'}=\frac{1}{r}+\frac{1}{r'}=1.$

Proof. We shall apply Theorem 2. In view of (12, 13, 18), it suffices to estimate $\tau_2(ku';h)_{L_1}$. The properties of the τ- and ω-moduli, the equation in (1), Lemma 5 and (12, 13) yield

$$
\begin{aligned}
\tau_2(ku';h)_{L_1} &\leq 16h\omega_1((ku')';h)_{L_1} \\
&= 16h\omega_1(qu-f;h)_{L_1} \\
&\leq 16h\left(\omega_1(qu;h)_{L_1}+\omega_1(f;h)_{L_1}\right) \\
&\leq 16h\left(\|u_c\|\omega_1(q;h)_{L_1}+\|q\|_{L_1}\omega_1(u;h)_{L_\infty}+\omega_1(f;h)_{L_1}\right)
\end{aligned}
$$

$$\leq 16h\left(\|u_c\|\omega_1(q;h)_{L_1} + \|q\|_{L_1}h\|u'\|_{BM} + \omega_1(f;h)_{L_1}\right)$$
$$\leq 16h\left(\frac{\|f\|_{L_1}}{c_0}\omega_1(q;h)_{L_1} + \|q\|_{L_1}\frac{h}{c_0}(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1} + \omega_1(f;h)_{L_1}\right)$$
$$\leq 16h\left(\frac{1}{c_0}\|f\|_{L_1}\omega_1(q;h)_{L_1} + \omega_1(f;h)_{L_1}\right)$$
$$+\frac{16h^2}{c_0}\|q\|_{L_1}\|f\|_{L_1}(1+\frac{\|q\|_{L_1}}{c_0}). \tag{35}$$

From Theorem 2, (12, 13, 18, 35) we obtain

$$\|z\|_{l^\infty(\Sigma_h)} \leq \frac{2}{c_0}(16h(\frac{1}{c_0}\|f\|_{L_1}\omega_1(q;h)_{L_1} + \omega_1(f;h)_{L_1})+$$
$$+\frac{16h^2}{c_0}\|q\|_{L_1}\|f\|_{L_1}(1+\frac{\|q\|_{L_1}}{c_0}) + \frac{h}{2c_0}(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}\tau_1(q;h)_{L_1}+$$
$$+16h\|q\|_{A_{p';h}}(\frac{h}{c_0}((1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_p} + \|q\|_{L_p}\frac{\|f\|_{L_1}}{c_0})+$$
$$+(1+\frac{\|q\|_{L_1}}{c_0})\|f\|_{L_1}\omega_1(\frac{1}{k};h)_{L_p})+$$
$$+\frac{h}{2c_0}(\|f\|_{A_{r',h}} + \|q\|_{A_{r',h}}\frac{\|f\|_{L_1}}{c_0})\tau_1(k;h)_{L_r}),$$

which after simple computations and reduction of the common terms yields the theorem's statement. □

In section 4.3.2 we shall use Theorem 5 to derive a variety of new corollaries for the particular case when "the best" scheme has been selected for the solution of the discrete approximating problem (2).

4.3.2. Some Implications

Theorem 4, Lemma 6 and the properties of the ω- and τ-moduli straightforwardly imply a variety of corollaries.

Corollary 1. *Assume that*

1. *the conditions of Theorem 4 hold;*

2. $\bigvee_0^1{}_p\, k,\ \bigvee_0^1{}_p\, q,\ \bigvee_0^1{}_p\, f\ < \infty;$

3. $1 \le p < \infty$.

Then,

$$\|z\|_{l^\infty(\Sigma_h)} \le h^{\frac{1}{p}} \left(c_1^1(k,q,f) \bigvee_0^1{}_p\, k + c_2^1(k,q,f) \bigvee_0^1{}_p\, q + c_3^1(k,q,f) \bigvee_0^1{}_p\, f \right) + c_4^1(k,q,f)h.$$

Corollary 2. *Assume that*

1. *the conditions of Theorem 4 hold;*
2. $k,\ q,\ f \in AC[0,1]$;
3. $\bigvee_0^1{}_p\, k',\ \bigvee_0^1{}_p\, q',\ \bigvee_0^1{}_p\, f' < \infty$;
4. $1 \le p < \infty$.

Then,

$$\|z\|_{l^\infty(\Sigma_h)} \le h^{1+\frac{1}{p}} \left(c_1^2(k,q,f) \bigvee_0^1{}_p\, k' + c_2^2(k,q,f) \bigvee_0^1{}_p\, q' + c_3^2(k,q,f) \bigvee_0^1{}_p\, f' \right) + c_4^2(k,q,f)h^2.$$

Corollary 3. *Assume that*

1. *the conditions of Theorem 4 hold;*
2. *(i) either* $k,\ q,\ f \in A^s_{1\infty},\ s \in (0,1) \cap (1,2)$,
   (ii) or $k,q,f \in W^s_1,\ s = 1,2$.

Then,

$$\|z\|_{l^\infty(\Sigma_h)} \le h^s \begin{cases} \left(c_{10}^3(k,q,f)\|k\|_{A^s_{1\infty}} + c_{20}^3(k,q,f)\|q\|_{A^s_{1\infty}} + c_{30}^3(k,q,f)\|f\|_{A^s_{1\infty}}\right) \\ \qquad +c_{40}^3(k,q,f)h^{1+[s]}, \quad s \in (0,1) \cap (1,2) \\ \left(c_{11}^3(k,q,f)\|k\|_{W^s_1} + c_{21}^3(k,q,f)\|q\|_{W^s_1} + c_{31}^3(k,q,f)\|f\|_{W^s_1}\right), \quad s = 1,2. \end{cases}$$

Corollary 4. *Assume that*

1. *the conditions of Theorem 4 hold;*
2. $k,\ q,\ f \in B^s_{p(s),r(s)}$, *where*

$$p(s) = \begin{cases} s^{-1}, & s \in (0,1), \\ 1, & s \in (1,2), \end{cases}$$

$$r(s) = \begin{cases} 1, & s \in (0,1), \\ \infty, & s \in (1,2), \end{cases}$$

$s \in (0,1) \cap (1,2)$.

Then,

$$\|z\|_{l^\infty(\Sigma_h)} \leq h^s \left(c^3_{10}(k,q,f)\|k\|_{B^s_{p(s),r(s)}} + c^3_{20}(k,q,f)\|q\|_{B^s_{p(s),r(s)}} + c^3_{30}(k,q,f)\|f\|_{B^s_{p(s),r(s)}} \right) + c^3_{40}(k,q,f)h^{1+[s]},$$

where $[s]$ *is the integer part of* s.

The constants c^ν_μ in Corollaries 1 – 4 can be estimated via Theorem 4 and Lemma 6.

For the particular case when "the best" scheme has been selected for the solution of the discrete approximating problem (2), Theorem 5 implies a variety of corollaries exhibiting improved convergence rates compared to the general case (i.e., either improved rates for the same assumptions on the data functions of the continual problem (1), or the same rates for relaxed assumptions on these data functions).

Corollary 5. *Assume that*

1. *the conditions of Theorem 5 hold;*
2. $\bigvee_0^1{}_p\, k,\ \bigvee_0^1{}_p\, q,\ \bigvee_0^1{}_p\, f\ < \infty;$
3. $1 \leq p < \infty$.

Then,

$$\|z\|_{l^\infty(\Sigma_h)} \leq h^{1+\frac{1}{p}} \left(c^5_1(k,q,f) \bigvee_0^1{}_p\, k + c^5_2(k,q,f) \bigvee_0^1{}_p\, q + c^5_3(k,q,f) \bigvee_0^1{}_p\, f \right) + c^5_4(k,q,f)h^2.$$

Corollary 6. *Assume that*

1. *the conditions of Theorem 5 hold;*
2. *(i) either* $k, q \in A^s_{1\infty}$, $f \in B^s_{1\infty}$, $s \in (0,1)$,
   (ii) or $k, q, f \in W^1_1$, $s = 1$.

Then,

$$\|z\|_{l^\infty(\Sigma_h)} \le \begin{cases} h^{1+s}\left(c^6_{10}(k,q,f)\|k\|_{A^s_{1\infty}} + c^6_{20}(k,q,f)\|q\|_{A^s_{1\infty}} + c^6_{30}(k,q,f)\|f\|_{B^s_{1\infty}}\right) \\ \qquad\qquad +c^6_{40}(k,q,f)h^{1+[s]}, \quad s \in (0,1), \\ h\left(c^6_{11}(k,q,f)\|k\|_{BM} + c^6_{21}(k,q,f)\|q\|_{BM} + c^6_{31}(k,q,f)\|f\|_{L_1}\right) \\ \qquad\qquad +c^6_{41}(k,q,f)h, \quad s = 0, \\ h^2\left(c^6_{12}(k,q,f)\|k\|_{W^1_1} + c^6_{22}(k,q,f)\|q\|_{W^1_1} + c^6_{32}(k,q,f)\|f\|_{W^1_1}\right), \quad s = 1. \end{cases}$$

Corollary 7. *Assume that*

1. *the conditions of Theorem 5 hold;*
2. $k, q \in B^s_{\frac{1}{s},1}$, $f \in B^s_{1\infty}$, $s \in (0,1)$.

Then,

$$\|z\|_{l^\infty(\Sigma_h)} \le h^{1+s}\left(c^7_1(k,q,f)\|k\|_{B^s_{\frac{1}{s},1}} + c^7_2(k,q,f)\|q\|_{B^s_{\frac{1}{s},1}} + c^7_3(k,q,f)\|f\|_{B^s_{1\infty}}\right) + c^7_4(k,q,f)h^2.$$

The constants c^ν_μ in Corollaries 5, 7 and $c^\lambda_{\mu\nu}$ in Corollary 6 can be estimated via Theorem 5 and Lemma 6.

Note that the properties of the integral and the averaged moduli of smoothness allow obtaining estimates also in terms of Triebel-Lizorkin norms of the data functions, but in the context of the present study the afore-mentioned results measuring the regularity in Besov norms prove to be sharper.

4.3.3. Remarks

Remark 4. It can be shown that Theorems 4 and 5 remain valid also in the general case when $k, q, f \in BM[0,1]$, with k, q, f possibly having only discontinuities of the first kind, the set of all these eventual discontinuity points for each one of $k,\ q,\ f$ being of measure zero, and each of problems (5, 6) having a continuous solution. In order to show the validity of Theorems 4, 5 in this more general case, we trace in detail the proofs of the two Theorems. Upon doing so, it turns that the following moments need additional substantiation.

(a) The existence of integral representation via Green's function, for which the a priori estimates of 4.2 hold true.

(b) The validity of the substitution in [22, Corollary 1 and Theorem 4] of u and u' with $\int\limits_0^1 G(\cdot,t)f(t)dt$ and $\int\limits_0^1 \frac{\partial}{\partial x}G(\cdot,t)f(t)dt$, respectively, which needs substantiation in view of Remark 2 (b).

(c) The validity of the differentiation in a parameter under the sign of the integral (i.e., the commutation between differentiation in a parameter, and integration), taking in consideration that f may not be continuous everywhere.

The substantiation of (a) and (c) is easy: (a) follows from the observations made in Remark 2; (c) follows from the observations made in Remark 1. Let us show (b). Since f is continuous a.e. in $[0,1]$, the function $\int\limits_0^1 G(\cdot,t)f(t)dt$ is a solution to problem (1), where the right-hand side f is replaced by $\tilde{f}$, where $\tilde{f}$ can be different from f only in the eventual discontinuity points of f. From here, and from Lemma 1 it easily follows that $u(x)-\int\limits_0^1 G(\cdot,t)f(t)dt=0,\ \forall x\in[0,1]$, which completes the substantiation of (b).

Remark 5. The results in [22] and sections 4.1, 4.2, 4.3 in the present paper can be obtained via the equivalent K-functionals of the integral and the averaged moduli of smoothness (see [17]), instead of using the moduli themselves as we did in this paper, by proving an analogue of Lemma 5 for the respective K-functionals. Concerning this issue, there are several points to make, as follows.

1. The K-functional version of Lemma 5 is more technically involved, since it additionally involves the use of an intermediate approximation (based on the Steklov-means or another smoothing operator with equivalent properties), embedding inequalities about the intermediate derivatives (see, e.g., [17]) of the intermediate approximation, *etc.*, all of which results in coarsening the constant factors to the rates in the resulting error estimates.

2. In the case of boundary-value and initial-boundary problems about linear PDEs in the multidimensional case, for hyper-rectangular domains (oriented coordinate-wise), or for ones reducible to these via change of variables, the finite differences to be used in the definition of the integral and averaged moduli of smoothness are still coordinate-wise, hence, it is still possible to work directly with the moduli, without necessarily passing on to the K-functional version. This will be the approach chosen by us in [24, 25, 26].

3. For boundary-value and initial-boundary problems about linear PDEs in the multidimensional case for domains more general than a hyper-rectangle, the definition of the moduli of smoothness becomes more technically involved (see, e.g., [31]), while the definition of the respective K-functionals continues to be relatively more straightforward. This is why for the derivation of error estimates in this case we would recommend as more technically straightforward the use of the Bramble-Hilbert lemma (see [30]) coupled with the invocation of the K-functionals equivalent to the respective moduli of smoothness. This case requires considerable additional study.

5. Concluding Remarks

Remark 6. The limitation of homogeneous boundary conditions $\alpha = \beta = 0$ in (1), under which we proved Theorems 4 and 5, is also not essential. In the general case of inhomogeneous boundary conditions in (1) and (2) (and the general assumptions about k, q, f from Remark 4), the error z can be estimated in one of the following two ways.

(a) Via detailed repetition of the proof of Theorems 4, 5, taking in consideration Remark 3.

(b) Via reduction of the general case to the particular case of homogeneous boundary conditions (essentially using the linearity of the problem). To this end, consider the function

$$\tilde{u}(x) = u(x) - w(x),\ \ x \in [0,1],$$
$$w(x) = (1-\Theta(x))\alpha + \Theta(x)\beta,$$
$$\Theta(x) = \frac{\int\limits_0^x \frac{dt}{k(t)}}{\int\limits_0^1 \frac{dt}{k(t)}}.$$

Evidently, $0 \le \Theta(x) \le 1,\ \ x \in [0,1]$.

More about this will be said and done in the subsequent paper [23] which is dedicated to the extension of the results of the present paper for homogeneous boundary conditions to the general case of inhomogeneous boundary conditions.

Remark 7. The method developed and the results obtained in this paper, the previous paper [22] and the subsequent paper [23] on this topic are an application of the general theory developed in [17], [18].

Remark 8. The material in this paper, the previous paper [22] and the subsequent paper [23] covers the part of the previously unpublished results in [5, Chapter 3] related to ODEs and complementary to the unpublished results in [5, Chapter 3] about PDEs which are scheduled to appear in [24, 25, 26].

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