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# ON ERROR ESTIMATION FOR APPROXIMATION METHODS INVOLVING DOMAIN DISCRETIZATION VI: DETERMINISTIC PROBLEMS V. FULLY DISCRETE FINITE DIFFERENCE SCHEMES FOR LINEAR ODES III: ESTIMATES IN TERMS OF PROPERTIES OF THE PROBLEM'S DATA FUNCTIONS II: INHOMOGENEOUS BOUNDARY CONDITIONS

Lubomir T. Dechevsky
Priority R&D Group for Mathematical Modelling
Numerical Simulation and Computer Visualization
Narvik University College
2, Lodve Lange's St., P.O. Box 385, N-8505 Narvik, NORWAY
e-mail: ltd@hin.no
url: http://ansatte.hin.no/ltd/

**Abstract:** This is the sixth of a sequence of 12 papers, preceded by [16, 17, 18, 19, 20] and followed by [21, 22, 23, 24, 25, 26] (in this order), dedicated to the study of error estimates for approximation problems based on discretization of the domain of the approximated functions. Within this sequence, in [17] and [18], for a model example of a Cauchy problem for a linear differential equation with variable coefficients, we applied an indirect method for error estimation based on results known in advance for the respective semi-discrete approximating problems.

In a follow-up subsequence of six of these papers, of which this is the third one, preceded by [19, 20] and followed by [21, 22, 23] (in this order) we develop a direct discrete method for error estimation based on an *extended Lax principle*, essentially proposed first in [5] but explicitly formulated for the first time in [19, section 2].

In [19, 20] and here we apply the proposed method to obtain sharp error estimates for a model boundary problem for a linear ordinary differential equation of second order with variable coefficients and right-hand side. This study is being continued in the remaining three papers [21, 22, 23] of the subsequence by an

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analogous application of the proposed method to obtain sharp error estimates for a model initial-boundary problem for a linear parabolic partial differential equation of second order.

In [19] we discussed the first 4 stages of the extended Lax principle for the model problem in consideration. These stages essentially correspond to the classical Lax principle, but the rather coarse classical error estimates obtained via this principle have been essentially sharpened in [1] using more advanced tools for error estimation, such as integral and averaged moduli of smoothness. In [19] we provided a systematic exposition of the results of [1], sharpened, generalized, and upgraded these results, and complemented them with some additional ones, in order to prepare the error estimates at Stage 4 for use in the derivation of appropriate *a priori* estimates at Stage 5 and the final results of the new method at Stage 6 of the extended Lax principle, related to ordinary differential equations, in [20] and the present paper.

In [20] we addressed two major topics (corresponding to Stages 5 and 6 of the extended Lax principle), as follows.

- Stage 5: We developed *a priori* estimates for the continual problem (1) which are *consistent* with the error estimates in terms of properties of the solution from [19].
- Stage 6: Based on the results obtained on Stage 4 in [19] and the results obtained on Stage 5 in [20], we derived sharp error estimates directly in terms of the data (variable coefficients and variable right-hand side) of the continual boundary-value problem (1) for the case of *homogeneous boundary conditions*. These new error estimates implied a diversity of corollaries providing certain approximation rates under minimal assumptions about regularity of the data.

The considerations made in [20] were made only for the special case of homogeneous boundary conditions. The present paper is dedicated to extending the results obtained in [20] for homogeneous boundary conditions to *the general case of inhomogeneous boundary conditions*.

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norm, bound, boundary, differential equation, ordinary, stationary, template, functional, linear, non-linear, positive, univariate

## 1. Introduction

This is the sixth of a sequence of 12 papers, preceded by [16, 17, 18, 19, 20] and followed by [21, 22, 23, 24, 25, 26] (in this order), dedicated to the study of error estimates for approximation problems based on *discretization of the domain* of the approximated functions, and (in the concluding paper [26]) comparison of the similarities and differences with the error estimates derived by alternative approximation methods (typically, of projection type) based on finite-dimensional subspaces of functions having the same *continual domain* as the target function.

Within this sequence, in [17] and [18], for model examples of Cauchy problems, we applied an indirect method for error estimation based on results known in advance for the respective semi-discrete approximating problems.

In a sequence of six papers, of which this is the third one, preceded by [19, 20] and followed by [21, 22, 23] (in this order), we develop a direct discrete method for error estimation based on an *extended Lax principle*, essentially proposed first in [5] but explicitly formulated for the first time in [19, Section 2].

In [19, 20], here and in the next papers [21, 22, 23] we apply the proposed method to obtain sharp error estimates for the following two model boundary problems (a boundary problem for an ordinary differential equation (ODE), and an initial-boundary problem for a partial differential equation (PDE)):

1. for a model linear ODE with variable coefficients and right-hand side (RHS) (see [19, 20] and the present paper);

2. for a model linear parabolic PDE with constant coefficients and a variable RHS (see [21, 22, 23]).

In [19] we discussed the first 4 stages of the extended Lax principle for the model problem in consideration. These stages essentially correspond to the classical Lax principle, but the rather coarse classical error estimates obtained via this principle have been essentially sharpened in [1] using more advanced tools for error estimation, such as integral and averaged moduli of smoothness. In [19] we provided a systematic exposition of the results of [1], sharpened, generalized, and upgraded these results, and complemented them with some additional ones, in order to prepare the error estimates at Stage 4 for use in the derivation of appropriate *a priori* estimates at Stage 5 and the final results of the new method at Stage 6 of the extended Lax principle, related to ordinary differential equations, in [20] and the present paper.

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In [20] we addressed only the special case of homogeneous boundary conditions. In the present paper we extend the results obtained in [20] for homogeneous boundary conditions to *the general case of inhomogeneous boundary conditions*.

## 2. Preliminaries

### 2.1. A Fully Discrete Method for Error Estimation

The *fully discrete method for error estimation directly in terms of properties of the problem's data*, proposed in [5, Chapter 3], is comprised of a sequence of stages, the entirety of which we chose in [19] to term as *an* ***extended*** *Lax principle*. In brief, the main stages of the proposed method are, as follows.

A. Classical Lax principle

- Stage 1. Derivation of estimates for the *local approximation error of the residual*.

- Stage 2. Derivation of a discrete approximating problem whose solution is *the error at the nodes of the mesh* (termed *discrete approximating problem for the error*).
- Stage 3. Derivation of *a priori estimates* for the solution of the discrete approximating problem for the error.
- Stage 4. Combining the results obtained at Stages 1 and 3 to derive *error estimates in terms of properties of the solution* (of *the exact target problem*).

B. Extension of the Lax principle

- Stage 5. Derivation of ***consistent*** *a priori* estimates for the solution of the exact target problem.
- Stage 6. Combining the results obtained at Stages 4 and 5 to derive *error estimates directly in terms of properties of the data functions and/or the data scalar parameters* (of the exact target problem).

For more details, see [19, Section 2].

Within the above framework, the organization of the exposition in the current sequence of relevant papers is, as follows.

- Part A.
  - ODEs: [19]
  - PDEs: [21]
- Part B.
  - ODEs:
    - Homogeneous boundary conditions: [20]
    - Inhomogeneous boundary conditions: the present paper (see also the concluding Section 4 for future work on the topic).
  - PDEs:
    - Homogeneous boundary conditions: [22]
    - Inhomogeneous boundary conditions: [23]

### 2.1.1. Discrete Approximating Problems for ODEs

Consider as a model exact target problem the following boundary-value problem for the *stationary ODE of heat conductivity and diffusion* (see, e.g., [7]).

$$\begin{aligned} &(k(x)u'(x))' - q(x)u(x) = -f(x), \quad 0 < x < 1, \\ &u(0) = \alpha, \quad u(1) = \beta, \end{aligned} \tag{1}$$

where:

- $\alpha,\ \beta \in \mathbb{R}$;
- $k(x) \geq c_0 > 0,\ \ q(x) \geq 0,\ \ \forall x \in [0,1]$;
- $k,\ q,\ f \in BM[0,1]$;
- $k,\ q,\ f$ have (eventually) only discontinuities of the first kind, forming a set with zero measure;
- *"conjugation conditions"* are fulfilled, as follows: $u$ – continuous, $ku'$ – continuous, $\forall x \in [0,1]$ (including the (eventual) points of discontinuity of $k,\ q,\ f$).

Solve (1) numerically via the following *homogeneous conservative finite difference scheme* (see [7]) on the $(N+1)$-node, $N \in \mathbb{N}$, uniform mesh $\Sigma_h = \{x_i = i/N,\ i = 0, \ldots, N\},\ h = 1/N$:

$$\begin{aligned} &(ay_{\bar{x}})_x - dy = -\varphi, \quad x \in \mathring{\Sigma}_h, \\ &y_0 = \alpha, \quad y_N = \beta, \end{aligned} \tag{2}$$

where, as customary (see, e.g., [7]), $y_x = y_{x,i}$ and $y_{\bar{x}} = y_{\bar{x},i}$ are the *forward* and *backward divided difference operators*, respectively:

$$y_{x,i} = \frac{y_{i+1} - y_i}{h}, \quad i = 0, \ldots, N-1; \qquad y_{\bar{x},i} = \frac{y_i - y_{i-1}}{h}, \quad i = 1, \ldots, N.$$

Thanks to the linearity of the continual and discrete approximating problems, for the error $z_i,\ i = 0, 1, \ldots, N$ we get the uniquely defined linear discrete problem

$$\begin{aligned} &(az_{\bar{x}})_x - dz = -\psi, \\ &z_0 = z_N = 0. \end{aligned} \tag{3}$$

Here $\psi$ is the *local approximation error of the residual*:

$$\psi = (au_{\bar{x}})_x - du + \varphi$$

Our purpose is to estimate $z$.

In view of the homogeneity, $a_i, d_i, \varphi_i$ are determined via *template functionals*, as follows:

$$\begin{aligned} a_i &= \mathrm{A}[k(x_i + sh], \\ d_i &= \mathrm{F}[q(x_i + sh)], \\ \varphi_i &= \mathrm{F}[q(x_i + sh)]. \end{aligned} \tag{4}$$

Here F is a linear template functional defined for $\bar{f}(s) \in BM\left[-\frac{1}{2}, \frac{1}{2}\right]$, where

$$\begin{aligned} BM(\Omega) = \{ \quad & f : \mathrm{Dom} f = \Omega, \mathrm{Cod} f \in \mathbb{R}, \\ & f - \text{measurable and bounded everywhere on } \Omega, \\ & \|f\|_{BM(\Omega)} = \sup_{x\in\Omega} |f(x)| < \infty \} \end{aligned}$$

($\Omega$ – an open, closed or semi-open interval in $\mathbb{R}$; Dom $g$ and Cod $g$ – the domain and codomain of a function $g$, respectively).

The functionals F and A have the following properties:

- the template functional F:
  - is linear over $BM\left[-\frac{1}{2}, \frac{1}{2}\right]$;
  - is exact over the constants: $\mathrm{F}[1] = 1$;
  - is positive: $\mathrm{F}[\bar{f}(s)] \geq 0$ for $\bar{f}(s) \geq 0, \ s \in \left[-\frac{1}{2}, \frac{1}{2}\right]$;
- for the (possibly, *non-linear*) template functional A, see [7, p. 116].

Additional conditions are being imposed on the template functionals A and F, in order to ensure rate $O(h^2)$ of the local approximation error of the residual:

- in the case of F: $\mathrm{F}[s] = 0$;
- in the case of A: see [7, p. 118].

In particular, when

- $\mathrm{F}[\bar{f}(s)] = \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} f(\cdot + sh) ds,$
- $\mathrm{A}[\bar{K}(s)] = \left( \int\limits_{-1}^{0} \frac{ds}{\bar{K}(\cdot + sh)} \right)^{-1},$

- $a$, $d$, $\varphi$ are obtained via (4),

the respective scheme is termed *"the best"* (cf. [7], [1]).

In our argumentation in [19, 20] and the sequel of the present paper we are assuming that $k$, $q$, $f$ have the minimum of properties needed in the course of the exposition.

In particular, all arguments in the exposition are valid, if $k$, $q$, $f$ are piecewise continuous in $[0, 1]$ (not necessarily having bounded variation).

In some of the remarks in [20, Section 4.3.3] and in the concluding remarks in the present paper we show that our results continue to hold true also under assumptions on $k$, $q$, $f$ which are much more general than piecewise continuity, and are close to those in the formulation of (1).

### 2.2. Functional Characteristics

Here we present the notation for the functional characteristics needed in the sequel. As a general reference about the functional characteristics introduced in this section concerning the notation, definitions, meaning and range of parameters, relevant properties and interrelations, as well as further details, we cite [14] together with [15], and the references therein. Another concise reference source on the material in this section is [16, Section 2] and the references therein.

1. The function space $C(\Omega) \subset BM(\Omega)$ of all (real-valued) continuous functions on $\Omega$, where $\Omega$ and $BM(\Omega)$ were defined in Section 2.1.1. The restriction of the norm $\|\cdot\|_{BM}$ on $C$ will be denoted, as usual, by $\|\cdot\|_C$; recall that with respect to this norm $C$ is a Banach space which is a closed subspace of the Banach space $BM$ (see, e.g., [4]).

2. For a function $f \in BM(\Omega)$,
$$S(t, f; x) = sup\,\{f(y) : y \in [x - t, x + t] \cap \Omega\}$$
is the upper Baire's function of $f$ at $x \in \Omega$ with step $t > 0$ (see [14] and the references therein).

3. The local modulus of smoothness of order $k \in \mathbb{N}$ at $x \in \Omega$ ($\Omega$ as in the definition of the upper Baire's function) with $t > 0$
$$\omega_k(f, x; t) = \sup\left\{\|\Delta_h^k f(y)\| : y, y + kh \in \left[x - \frac{kt}{2}, x + \frac{kt}{2}\right] \cap \Omega\right\}$$

(see [14] and the references therein); it is natural to define also $\omega_0(f, x; t) := S(t, f; x)$.

4. The sequence space $l^p(\Sigma_h)$ defined over the mesh $\Sigma_h$ (see [29, 10, 11, 12, 5, 14]).

5. The inhomogeneous Sobolev space $W_p^m(\Omega)$, with $W_p^0 = L_p$ (see, e.g., [2, 6, 14]); $\Omega \subset \mathbb{R}$ is as in section 2.1.1.

6. The integral modulus of smoothness ($\omega$-modulus) $\omega_m(f; h)_{L_p}$ (short: $\omega_m(f; h)_p$) (see [29, 10, 11, 12, 9, 5, 14], cf. also [6, 2] where the notation is modified).

7. The averaged modulus of smoothness ($\tau$-modulus) $\tau_m(f; h)_{L_p}$ (short: $\tau_m(f; h)_p$) (see [9, 29, 10, 11, 12, 13, 5, 14]).

8. The Steklov-means $f_{k,t}$ (see [3, 8, 9, 14]).

9. The Wiener amalgam space $A_{p,h}(\Omega)$, with norm $\|.\|_{A_{p,h}} = \tau_0(f; h)_{L_p}$ (see [12, 13, 5, 14, 15]); $\Omega \subset \mathbb{R}$ is as in section 2.1.1.

10. The inhomogeneous Besov space $B_{pq}^s(\Omega)$ (see [2, 29, 10, 11, 12, 13, 5, 14]).

11. The inhomogeneous $A$-space $A_{pq}^s(\Omega)$, an analogue of the respective Besov space where the $\omega$-modulus in the definition of the norm in $B_{pq}^s$ is replaced by the respective $\tau$-modulus (with the same parameters) in the definition of the norm in $A_{pq}^s$ (see [29, 10, 11, 12, 13, 5, 14]).

12. The Wiener–Young $p$-variation $\bigvee_{\Omega} {}_p\, g$ of $g$, the case $p = 1$ corresponding to the customary Jordan variation (see [30, 31]), with $\bigvee_{\Omega} {}_p = \bigvee_a^b$ for $\Omega = [a, b]$.

13. The space $AC(\Omega)$ of all absolutely continuous functions on $\Omega \subset \mathbb{R}$, $\Omega$ as in section 2.1.1.

[16, Section 2.3] contains an important comparison between the properties of the integral and averaged moduli of smoothness.

Following our practice in [5] and [14], in order to distinguish between previously known results and the new ones obtained here, we shall add the additional marker '**(K.)**' (abbreviated from '(K)nown') to the enumeration of every statement in the sequel which has been previously known (with respective reference to available relevant literature).

### 2.3. The Results in [19, 20]

Proceeding through Stages 1–6 of the extended Lax principle, in [19, 20] we arrived at the following results (see also the relevant comments in Section 1).

1. In [19, Section 4.1] we obtained *estimates of the local approximation error of the residual* (Stage 1).
2. In [19, Section 4.2] a respective *a priori estimate* was given for *the solution of the discrete approximating problem* (Stages 2 and 3; the linearity of the exact differential operator in problem (1) and the approximating finite-difference operator in problem (2) was essentially used in the derivation of the discrete problem (3) for the error, which was with homogeneous boundary conditions, since the approximation on the boundary was assumed to be exact.
3. In [19, Section 4.3] we combined the results from the previous two items to obtain *error estimates in terms of properties of the solution* of the exact target problem (1), with which Stage 4 was completed.
4. In [20, Section 4.2] we derived *a priori estimates for the solution of the continual problem* (1) with homogeneous boundary conditions, thus completing Stage 5 for the case of such particular boundary conditions. A big challenge in this very technically involved part of the realization of the overall program of the extended Lax principle for problems (1–3) was to derive at Stage 5 such *a priori* estimates that the following would simultaneously hold true:

   (a) these *a priori* estimates would be *consistent* with the *error estimates in terms of properties of the solution* of (1) obtained at Stage 4 (i.e., the left-hand sides (LHS) of the *a priori* estimates at Stage 5 had to be in terms of some of the functional characteristics appearing in the RHS of the error estimates at Stage 4 (e.g., average moduli of smoothness in the RHS of some of the error estimates in [19, Section 4.3.1], or some of the functional characteristics, e.g., $A$-space norms, Besov space norms, or Wiener-Young $p$-variation, in the RHS of some of the error estimates in [19, Section 4.3.2]);

   (b) the resulting error estimates at the final Stage 6 should be *sharp*.

   In [5], where all of the present results were first derived, the task of combining items (a) and (b) above proved to require very fine tuning of the results obtained at each of Stages 4, 5 and 6, in order to fit them all together. In [5] this task was resolved iteratively, after several consecutive steps of improving the error estimates at Step 4 and refining the *a priori* estimates at Stage 5, until indications appeared that the resulting error estimates at the final Stage 6 are sharp. At the initial step of this iterative process we started with the error estimates in terms of properties of the solution as they were formulated in [1] (see [19, Theorems 1, 2]); at the final step of this iterative process we ended up with the new Stage-4 error estimates obtained in [19, Theorems 3, 4, Corollary 1]; in parallel to this, the *a priori* estimates at Stage 5 had also to be refined, and the results in [20, Section 4.2] are the final product of this refinement.

5. Based on the refined *a priori* estimates derived at Stage 5 in [20, Section 4.2] (see the previous item in the present list), in [20, Sections 4.3] we obtained *sharp error estimates directly in terms of properties of the coefficients and RHS of the ODE in problem (1), in the special case of homogeneous boundary conditions.*

Our current objective in the present paper is *to extend the results of [20, Section 4.3] to* ***the general case of inhomogeneous boundary conditions in (1)***.

## 3. Error Estimates Directly in Terms of Properties of the Coefficients and Right-Hand Side: the General Case of Inhomogeneous Boundary Conditions

Now we shall show that it is possible to get rid of the limitation about homogeneous boundary conditions $\alpha = \beta = 0$ in (1), under which we proved [20, Theorems 4 and 5]. In the general case of inhomogeneous boundary conditions in (1) and (2) (and the general assumptions about $k$, $q$, $f$ from [20, Remark 4]), the error $z$ can be estimated in one of the following two ways.

A. Via detailed repetition of the proof of [20, Theorems 4 and 5], taking in consideration also [20, Remark 3]. In other words, it is necessary to take the following additional steps:

   (a) to construct the Green's function for problem (1) in the general case of inhomogeneous boundary conditions;

(b) to use the construction in the previous item to derive sharp *a priori* estimates for the solution of (1) in the general case of inhomogeneous boundary conditions; that is, to upgrade [20, Lemmata 2 and 3] and the arguments in [20, Remark 2] for the case of inhomogeneous boundary conditions.

These two additional steps eventually lead to sharp error estimates at the final Stage 6 of the extended Lax principle, but require additional technical elaboration and very precise work in order not to coarsen the *a priori* estimates at Stage 5, hence also the final error estimates at Stage 6. The level of sophistication of this technical elaboration increases considerably in the case of multidimensional boundary-value and initial-boundary problems for PDEs; for a model example, see [23].

B. Via reduction of the general case to the particular case of homogeneous boundary conditions (essentially using the linearity of the problem). To this end, consider the function

$$\begin{aligned} &\tilde{u}(x) = u(x) - w(x),\ \ x \in [0,1], \\ &w(x) = w_{\alpha,\beta}(x) = w_{\alpha,\beta}(k;x) = (1-\Theta(x))\alpha + \Theta(x)\beta, \\ &\Theta(x) = \Theta(k;x) = \frac{\int\limits_0^x \frac{dt}{k(t)}}{\int\limits_0^1 \frac{dt}{k(t)}}. \end{aligned} \tag{5}$$

Clearly,

$$0 \le \Theta(x) \le 1, \quad x \in [0,1]. \tag{6}$$

It is easy to see that $\tilde{u}$ is solution to problem (1) with homogeneous boundary conditions and new RHS, equal to

$$\begin{aligned} \tilde{f}(x) = \tilde{f}_{\alpha,\beta}(x) = \tilde{f}_{\alpha,\beta}(k,q;x) &= f(x) + (k(x)w'(x))' - q(x)w(x) \\ &= f(x) - q(x)w(x). \end{aligned} \tag{7}$$

Using (5–7), it is possible to extend the results in [20] to the general case of inhomogeneous boundary conditions, by reducing the general case to the special case of homogeneous boundary conditions, and then applying the results in [20] valid for this special case.

In the sequel in this paper we shall study in detail alternative B.; for a further discussion of alternatives A. and B., see Remarks 11 and 12 in Section 4.

### 3.1. Main Results

The generalization of [20, Theorem 4] for the case of inhomogeneous boundary conditions is, as follows.

**Theorem 1.** *Let us solve problem (1) approximately via the discrete problem (2), under the assumptions that*

1. *$k$, $q$ and $f$ are piecewise continuous;*

2. *$\tilde{f}$ is defined via $f$, $q$, $k$, $\alpha$ and $\beta$ in (5–7);*

3. *$\tilde{f}$ is continuous.*

*Then,*

*for the error $z$ satisfying (3), the following estimate holds true:*

$$
\begin{aligned}
\|z\|_{l^\infty(\Sigma_h)} \le \frac{2}{c_0}[ & c_1(k,q,\tilde{f})\left(\frac{3}{2c_0}\tau_2(k;h)_{L_1} + 2\|k\|_{A_{p',4h}}\tau_2(\frac{1}{k};2h)_{L_p}\right) \\
& + c_2(k,\tilde{f})\tau_2(q;h)_{L_1} + 2\tau_2(f;h)_{L_1} \\
& + c_3(k,q,\tilde{f};p,p_1)h\tau_1(\frac{1}{k};2h)_{L_{p_1}} \\
& + c_4(k,q,\tilde{f};r)h\omega_1(\frac{1}{k};h)_{L_r} \\
& + c_5(k,\tilde{f};p)h\omega_1(q;2h)_{L_p} \\
& + c_6(k,q;p)h\omega_1(\tilde{f};2h)_{L_p} \\
& + c_7(k,q,\tilde{f};p,p_2,p_3,r)h^2 \\
& + \frac{|\beta-\alpha|}{\|\frac{1}{k}\|_{L_1}}\left(\frac{1}{c_0}\tau_2(k;h)_{L_1} + \|k\|_{A_{p',h}}\tau_2(\frac{1}{k};h)_{L_p} + \frac{c}{2}\|q\|_{A_{r',h}}h\omega_1(\frac{1}{k};h)_{L_r}\right) \\
& + \max\{|\alpha|,|\beta|\}\tau_2(q;h)_{L_1}],
\end{aligned} \tag{8}
$$

*where* $\frac{1}{p}+\frac{1}{p'}=\frac{1}{r}+\frac{1}{r'}=1$, $\frac{1}{p_j}+\frac{1}{p_j'}=\frac{1}{p}$, $j=1,2,3$, $c: 0<c<16$ *is the* *constant appearing in the error estimate in [19, Theorem 4],*

$$
\begin{aligned}
c_1(k,q,\tilde f) \leq& (1+\frac{\|q\|_{L_1}}{c_0})\|\tilde f\|_{L_1},\\
c_2(k,\tilde f) \leq& \frac{\|\tilde f\|_{L_1}}{c_0},\\
c_3(k,q,\tilde f;p,p_1) \leq& 16\|k\|_{A_{p',4h}}\left[(1+\frac{\|q\|_{L_1}}{c_0})\|\tilde f\|_{L_{p_1'}}+\|q\|_{L_{p_1'}}\frac{\|\tilde f\|_{L_1}}{c_0}\right],\\
c_4(k,q,\tilde f;r) \leq& 16\|q\|_{A_{r',h}}(1+\frac{\|q\|_{L_1}}{c_0})\|\tilde f\|_{L_1},\\
c_5(k,\tilde f;p) \leq& \frac{64}{c_0}\|k\|_{A_{p',4h}}\|\tilde f\|_{L_1},\\
c_6(k,q;p) \leq& \frac{64}{c_0}\|k\|_{A_{p',4h}}(1+\frac{\|q\|_{L_1}}{c_0}),\\
c_7(k,q,\tilde f;p,p_2,p_3,r) \leq& \frac{128}{c_0^2}\|k\|_{A_{p',4h}}\left\{\|q\|_{L_{p_2'}}\left[\|\tilde f\|_{L_{p_2}}+(1+\frac{\|q\|_{L_1}}{c_0})\|\tilde f\|_{L_1}\right]\right.\\
&\left.+\|\tilde f\|_{L_{p_3'}}\left[\frac{\|q\|_{L_{p_3}}}{c_0}+\frac{1}{\|\frac{1}{k}\|_{L_1}}+(1+\frac{(1+\|q\|_{L_1})}{c_0})\|q\|_{L_1}\right]\right\}\\
&+\frac{12}{c_0}\|q\|_{A_{r',h}}\left[(1+\frac{\|q\|_{L_1}}{c_0})\|\tilde f\|_{L_r}+\|q\|_{L_r}\frac{\|\tilde f\|_{L_1}}{c_0}\right].
\end{aligned}
\tag{9}
$$

**Remark 1.** Note that the second summand on the second line of (8) contains $f$ and not $\tilde f$. This is the same term which appears in the error estimate in [19, Theorem 4] and is independent of $u$, $u'$, so $\tilde f$ does not get involved in its estimation.

The generalization of [20, Theorem 5] for the "best" scheme in the case of inhomogeneous boundary conditions is, as follows.

**Theorem 2.** *Assume that*

1. *the conditions of Theorem 1 hold (including the definition of $\tilde f$);*
2. *in problem (2) the "best" scheme has been selected.*

*Then,*

*for the error $z$, the following estimate holds true:*

$$\begin{aligned} \|z\|_{l^\infty(\Sigma_h)} \leq \frac{2h}{c_0}[ \quad & c_1(k,q,f,\tilde{f};r)\tau_1(k;h)_{L_r} \\ & +c_2(k,q,\tilde{f};p)\omega_1(\frac{1}{k};h)_{L_p} \\ & +c_3(k,q,\tilde{f})\tau_1(q;h)_{L_1} \\ & +c_4(k,\tilde{f})\omega_1(q;h)_{L_1} \\ & +\omega_1(\tilde{f};h)_{L_1} \\ & +c_5(k,q,\tilde{f};p)h \\ & +\frac{|\beta-\alpha|}{\|\frac{1}{k}\|_{L_1}}\tau_1(q;h)_{L_1} \\ & +\max\{|\alpha|,|\beta|\}\|q\|_{A_{r',h}}\tau_1(k;h)_{L_r}], \end{aligned} \tag{10}$$

*where* $\frac{1}{p}+\frac{1}{p'}=\frac{1}{r}+\frac{1}{r'}=1$,

$$\begin{aligned} c_1(k,q,f,\tilde{f};r) &\leq \frac{1}{2c_0}(\|f\|_{A_{r',h}}+\|q\|_{A_{r',h}}\frac{\|\tilde{f}\|_{L_1}}{c_0}), \\ c_2(k,q,\tilde{f};r) &\leq 16\|q\|_{A_{r',h}}\|\tilde{f}\|_{L_1}(1+\frac{\|q\|_{L_1}}{c_0}), \\ c_3(k,q,\tilde{f}) &\leq \frac{1}{2c_0}\|\tilde{f}\|_{L_1}(1+\frac{\|q\|_{L_1}}{c_0}), \\ c_4(k,\tilde{f}) &\leq \frac{\|\tilde{f}\|_{L_1}}{c_0}, \\ c_5(k,q,\tilde{f};p) &\leq \frac{16}{c_0}\left(\|q\|_{L_1}\|\tilde{f}\|_{L_1}(1+\frac{\|q\|_{L_1}}{c_0})\right. \\ &\quad \left.+\|q\|_{A_{p',h}}\left((1+\frac{\|q\|_{L_1}}{c_0})\|\tilde{f}\|_{L_p}+\|q\|_{L_p}\|\tilde{f}\|_{L_1}\right)\right). \end{aligned} \tag{11}$$

**Remark 2.** Note that $c_1$ in (11) depends both on $f$ and $\tilde{f}$. This is because the first summand in the brackets in the expression

$$\frac{h}{2c_0}(\|f\|_{A_{r',h}}+\|q\|_{A_{r',h}}\|u\|_c)\tau_1(k;h)_{L_r}$$

appearing in the last line of the error estimate in [19, Corollary 1] depends on $f$ and is independent of $u$, $u'$, so $\tilde{f}$ does not get involved in its estimation.

*Proof. (Outline of proofs of Theorems 1 and 2.)*

Of course, [19, Theorem 4 and Corollary 1] hold true for problems (1–3) with inhomogeneous boundary conditions. (This is true provided that we continue assuming that the approximation on the boundary of the discrete approximating problem (2) is exact, so that problem (3) about the error continues to be with homogeneous boundary conditions also in the case of inhomogeneous boundary conditions in (1, 2). In this case the *a priori* estimate about the solution of (3) from [19, Lemma 6] continues to hold.) In order to derive the generalization of [20, Theorems 4 and 5] in the present Theorems 1 and 2, respectively, it therefore suffices to substitute the solution $u$ with $\tilde{u}+w$ in the RHSs of the error estimates in [19, Theorem 4 and Corollary 1]. The next steps of the substantiation are the same as in the proof of [20, Theorems 4 and 5]. Let us begin the estimation, for example, with $\tau_2(u';h)_{L_\varrho},\ 1\le\varrho\le\infty$.

$$\tau_2(u';h)_{L_\varrho}=\tau_2(\tilde{u}'+w';h)_{L_\varrho}\le\tau_2(\tilde{u}';h)_{L_\varrho}+\tau_2(w';h)_{L_\varrho}. \tag{12}$$

For the estimation of the second summand in the RHS of (12), from the explicit form of $w$ it follows that

$$\tau_2(w';h)_{L_\varrho}=\frac{|\beta-\alpha|}{\|\frac{1}{k}\|_{L_1}}\tau_2(\frac{1}{k};h)_{L_\varrho}. \tag{13}$$

For the estimation of the first summand in the RHS of (12), we can apply [20, inequality (34)] to $\tau_2(\tilde{u}';h)_{L_\varrho}$, with $f$ replaced by $\tilde{f}=f-qw$, according to (7).

In the same manner we treat, one by one, all the summands in the RHSs of the error estimates in [19, Theorem 4 and Corollary 1] containing $u$ or its respective derivatives, while the remaining summands remain unchanged. Thus, we get the following (with $\varrho:1\le\varrho\le\infty$, $\varrho':\frac{1}{\varrho}+\frac{1}{\varrho'}=1$, $h>0$).

$$\|u'\|_{A_{\varrho',h}}\le\|u'\|_{BM}\le\|\tilde{u}'\|_{BM}+\frac{|\beta-\alpha|}{c_0\|\frac{1}{k}\|_{L_1}}, \tag{14}$$

$$\|u\|_{A_{\varrho',h}}\le\|u\|_C\le\|\tilde{u}\|_C+\max\{|\alpha|,|\beta|\}, \tag{15}$$

$$\begin{aligned}\omega_1(u';h)_{L_\varrho}\ &\le\omega_1(\tilde{u}';h)_{L_\varrho}+\omega_1(w';h)_{L_\varrho}=\\ &=\omega_1(\tilde{u}';h)_{L_\varrho}+\tfrac{|\beta-\alpha|}{\|\frac{1}{k}\|_{L_1}}\omega_1(\tfrac{1}{k};h)_{L_\varrho},\end{aligned} \tag{16}$$

$$\begin{aligned}\tau_2(ku';h)_{L_\varrho}\ &\le\tau_2(k\tilde{u}';h)_{L_\varrho}+\tau_2(kw';h)_{L_\varrho}=\\ &=\tau_2(k\tilde{u}';h)_{L_\varrho}.\end{aligned} \tag{17}$$

Analogously to the first summand in the RHS of (12), for every summand in the RHSs of (14–17) which contains $\tilde{u}$ or $\tilde{u}'$ there is a respective sharp upper bound in the proof of [20, Theorem 4 or 5]. More precisely, a sharp estimate of the first summand in the RHS of (14, 15, 16, 17) is provided by [20, inequalities (13, 12, 18, 35)], respectively, with $f$ being replaced by $\tilde{f}$ throughout these bounds.

Similarly to the proofs of [20, Theorem 4 and 5], we combine all of the above results together and, after multiplication with the respective factors to the LHSs of (12, 14, 15, 16, 17) appearing in the RHSs of the error estimates in [19, Theorem 4 and Corollary 1] and subsequent reduction of the similar terms, keeping the direct (not via $\tilde{f}$) contribution of the inhomogeneous boundary conditions separate, we obtain the statements of Theorems 1 and 2. □

### 3.2. Some Implications

Theorem 1 and the properties of the $\omega$- and $\tau$-moduli imply a variety of corollaries which generalize [20, Corollaries 1–4] for the case of inhomogeneous boundary conditions.

**Corollary 1.** *(Generalization of [20, Corollary 1] for the case of inhomogeneous boundary conditions.) Assume that*

1. *the conditions of Theorem 1 hold;*
2. $\bigvee_0^1{}_p\, k,\ \bigvee_0^1{}_p\, q,\ \bigvee_0^1{}_p\, f \ < \infty$;
3. $1 \le p < \infty$.

*Then,*

$$\|z\|_{l^\infty(\Sigma_h)} \le h^{\frac{1}{p}} \left( c_{11} \bigvee_0^1{}_p\, k \ + c_{12} \bigvee_0^1{}_p\, q \ + c_{13} \bigvee_0^1{}_p\, f \right) + c_{14} h, \tag{18}$$

*where* $c_{1j} = c_{1j}(k, q, f; |\beta - \alpha|, \max\{|\alpha|, |\beta|\}) \in (0, \infty)$, $j = 1, 2, 3, 4$.

**Corollary 2.** *(Generalization of [20, Corollary 2] for the case of inhomogeneous boundary conditions.) Assume that*

1. *the conditions of Theorem 1 hold;*
2. $k,\ q,\ f \in AC[0, 1]$;

3. $\bigvee_0^1{}_p k',\ \bigvee_0^1{}_p q',\ \bigvee_0^1{}_p f' < \infty$;

4. $1 \le p < \infty$.

*Then,*

$$\|z\|_{l^\infty(\Sigma_h)} \le h^{1+\frac{1}{p}} \left( c_{21} \bigvee_0^1{}_p k' + c_{22} \bigvee_0^1{}_p q' + c_{23} \bigvee_0^1{}_p f' \right) + c_{24} h^2. \tag{19}$$

*where* $c_{2j} = c_{2j}(k, q, f; |\beta - \alpha|, \max\{|\alpha|, |\beta|\}) \in (0, \infty)$, $j = 1, 2, 3, 4$.

**Corollary 3.** *(Generalization of [20, Corollary 3] for the case of inhomogeneous boundary conditions.) Assume that*

1. *the conditions of Theorem 1 hold;*

2. (i) *either* $k,\ q,\ f \in A^s_{1\infty}$, $s \in (0, 1) \cap (1, 2)$,
   (ii) *or* $k, q, f \in W^s_1$, $s = 1, 2$.

*Then,*

$$\|z\|_{l^\infty(\Sigma_h)} \le h^s \begin{cases} \left(c_{301}\|k\|_{A^s_{1\infty}} + c_{302}\|q\|_{A^s_{1\infty}} + c_{303}\|f\|_{A^s_{1\infty}}\right) \\ \qquad + c_{304} h^{1+[s]}, \quad s \in (0,1) \cap (1,2) \\ \left(c_{311}\|k\|_{W^s_1} + c_{312}\|q\|_{W^s_1} + c_{313}\|f\|_{W^s_1}\right), \quad s = 1, 2. \end{cases} \tag{20}$$

*where* $c_{3ij} = c_{3ij}(k, q, f; |\beta - \alpha|, \max\{|\alpha|, |\beta|\}) \in (0, \infty)$, *and* $c_{3ij}$ *can be chosen independent of* $s$, $i = 0, 1$, $j = 1, 2, 3$; *the same holds true also for* $c_{304}$: $c_{304} = c_{304}(k, q, f; |\beta - \alpha|, \max\{|\alpha|, |\beta|\}) \in (0, \infty)$, *and* $c_{304}$ *can be chosen independent of* $s$.

**Corollary 4.** *(Generalization of [20, Corollary 4] for the case of inhomogeneous boundary conditions.) Assume that*

1. *the conditions of Theorem 1 hold;*

2. $k,\ q,\ f \in B^s_{p(s),r(s)}$, *where*

$$p(s) = \begin{cases} s^{-1}, & s \in (0,1), \\ 1, & s \in (1,2), \end{cases} \tag{21}$$

$$r(s) = \begin{cases} 1, & s \in (0,1), \\ \infty, & s \in (1,2), \end{cases} \tag{22}$$

$s \in (0,1) \cap (1,2)$ .

*Then,*

$$\|z\|_{l^\infty(\Sigma_h)} \leq h^s \left( c_{301}\|k\|_{B^s_{p(s),r(s)}} + c_{302}\|q\|_{B^s_{p(s),r(s)}} + c_{303}\|f\|_{B^s_{p(s),r(s)}} \right) + \\ + c_{304} h^{1+[s]}, \tag{23}$$

*where* $[s]$ *is the integer part of* $s$*, and the constants* $c_{30j}$*,* $j = 1,2,3,4$*, are the same as the respective constants in the case* $i = 0$ *of the error estimate (20) in Corollary 3.*

**Remark 3.** The dependence of the constants $c_{kj}$ in Corollaries 1, 2 and the constants $c_{kij}$ in Corollaries 3, 4 on the functional parameters $k$, $q$, $f$ and the scalar parameters $|\beta - \alpha|$, $\max\{|\alpha|, |\beta|\}$ can be estimated via Theorem 1, the bounds in its proof, and [20, Lemmata 5, 6]. (See also Remark 5 below.)

In [20, Section 4.3.2], for the case of homogeneous boundary conditions in problem (2), we noted that in the particular case when "the best" scheme has been selected for the solution of the discrete approximating problem (2), [20, Theorem 5] implies a variety of corollaries exhibiting improved convergence rates compared to the general case (i.e., either improved rates for the same assumptions on the data functions of the continual problem (1), or the same rates for relaxed assumptions on these data functions). Now we shall show that this continues to hold true also in the general case of inhomogeneous boundary conditions in (2). Namely, the following Corollaries 5–7 generalize [20, Corollaries 5–7] for the case of inhomogeneous boundary conditions.

**Corollary 5.** *(Generalization of [20, Corollary 5] for the case of inhomogeneous boundary conditions.) Assume that*

1. *the conditions of Theorem 2 hold;*
2. $\bigvee_0^1{}_p\, k,\ \bigvee_0^1{}_p\, q,\ \bigvee_0^1{}_p\, f \ < \infty;$
3. $1 \leq p < \infty$.

*Then,*

$$\|z\|_{l^\infty(\Sigma_h)} \le h^{1+\frac{1}{p}} \left( c_{51} \bigvee_0^1{}_p\, k \; + c_{52} \bigvee_0^1{}_p\, q \; + c_{53} \bigvee_0^1{}_p\, f \right) + c_{54} h^2, \tag{24}$$

*where* $c_{5j} = c_{5j}(k, q, f; |\beta - \alpha|, \max\{|\alpha|, |\beta|\}) \in (0, \infty),\ j = 1, 2, 3, 4.$

**Corollary 6.** *(Generalization of [20, Corollary 6] for the case of inhomogeneous boundary conditions.) Assume that*

1. *the conditions of Theorem 2 hold;*
2. *(i) either* $k,\, q \in A^s_{1\infty},\ \ f \in B^s_{1\infty},\ \ s \in (0, 1),$
   *(ii) or* $k,\, q,\, f \in W^1_1,\ \ s = 1.$

*Then,*

$$\|z\|_{l^\infty(\Sigma_h)} \le \begin{cases} h^{1+s} \left( c_{601} \|k\|_{A^s_{1\infty}} + c_{602} \|q\|_{A^s_{1\infty}} + c_{603} \|f\|_{B^s_{1\infty}} \right) \\ \qquad\qquad +c_{604} h^2, \quad s \in (0, 1), \\ h \left( c_{611} \|k\|_{BM} + c_{612} \|q\|_{BM} + c_{613} \|f\|_{L_1} \right) \\ \qquad\qquad +c_{614} h^2, \quad s = 0, \\ h^2 \left( c_{621} \|k\|_{W^1_1} + c_{622} \|q\|_{W^1_1} + c_{623} \|f\|_{W^1_1} \right), \quad s = 1, \end{cases} \tag{25}$$

*where* $c_{6ij} = c_{6ij}(k, q, f; |\beta - \alpha|, \max\{|\alpha|, |\beta|\}) \in (0, \infty)$*, and* $c_{6ij}$ *can be chosen independent of* $s$*, for all* $(i, j) :\ i = 0, 1,\ j = 1, 2, 3, 4,$ *and for all* $(i, j) :\ i = 2,\ j = 1, 2, 3.$

**Corollary 7.** *(Generalization of [20, Corollary 7] for the case of inhomogeneous boundary conditions.) Assume that*

1. *the conditions of Theorem 2 hold;*
2. $k,\, q \in B^s_{\frac{1}{s},1},\ f \in B^s_{1\infty},\ s \in (0, 1).$

*Then,*

$$\|z\|_{l^\infty(\Sigma_h)} \le h^{1+s} \left( c_{71} \|k\|_{B^s_{\frac{1}{s},1}} + c_{72} \|q\|_{B^s_{\frac{1}{s},1}} + c_{73} \|f\|_{B^s_{1\infty}} \right) + c_{74} h^2, \tag{26}$$

*where* $c_{7j} = c_{7j}(k, q, f; |\beta - \alpha|, \max\{|\alpha|, |\beta|\}) \in (0, \infty),\ j = 1, 2, 3, 4.$

**Remark 4.** The constants $c_{kj}$ in Corollaries 5, 7 and $c_{kij}$ in Corollary 6 can be estimated via Theorem 2, the bounds in its proof, and [20, Lemmata 5, 6]. (See also Remark 5 below.)

**Remark 5.** In order to obtain the error estimates in Corollaries 1–7 above, the functional characteristics of $\tilde{f}$ appearing in the error estimates in Theorems 1, 2 are being bounded from above by respective functional characteristics of $1/k$, $q$ and $f$ using the triangle inequality and [20, Lemma 5]. Furthermore, the appearing functional characteristics of $1/k$ can be bounded from above by respective functional characteristics of $k$ using the uniform lower bound $c_0 > 0$ for $k$ and [20, Lemma 6]. All elements of this type of estimation argument have already appeared in other context in the proofs of [20, Theorems 4, 5], and will therefore not be explicitly repeated here.

## 4. Concluding Remarks

**Remark 6.** It is possible to eliminate the presence of $\tilde{f}$ from the RHSs of the error estimates (8,10) in Theorems 1, 2 by estimating from above every modulus of smoothness and functional norm of $\tilde{f}$ present in (8,10) with respective functional characteristics of $1/k$, $q$, $f$ and, eventually, further bounds from above of the involved moduli and norms of $1/k$ by respective moduli and norms of $k$. The technique for achieving this is the same as described in Remark 5. In (8,10) we gave preference to RHSs involving $\tilde{f}$, because the elimination of the functional characteristics of $\tilde{f}$ is achieved via a sequence of triangle and Hölder inequalities, the result of which preserves the sharp rates but coarsens the multiplicative constant factors to these rates.

**Remark 7.** There are two alternative strategies for the realization of the approach (outlined in Remark 5) for derivation of Corollaries 1–7 from Theorems 1, 2, as follows.

1. Strategy 1. Perform the following two steps:
   (a) eliminate the dependence of the RHSs of (8,10) on $\tilde{f}$ by the approach outlined in Remark 6; the resulting new RHSs of (8,10) contain only functional norms, $\tau$-moduli and $\omega$-moduli of $k$, $q$, $f$;
   (b) for the new RHSs of (8,10) obtained at the previous step, apply the properties of the $\tau$-moduli and $\omega$-moduli (for references to these properties, see Section 2.2).

2. Strategy 2. Perform the steps in reverse order, i.e.:

   (a) for the original RHSs of (8,10) containing $\tilde{f}$, apply the properties of the $\tau$-moduli and $\omega$-moduli; the resulting new RHSs of (8,10) contain only functional norms and, eventually, quasi-norms of (eventually, derivatives of) $k$, $q$, $f$ and $\tilde{f}$;

   (b) on the functional norms/quasi-norms of (eventually, derivatives of) $\tilde{f}$ obtained at the previous step, apply respective triangle/quasi-triangle inequality followed by Hölder inequality (for the product $qw$ or its eventual derivatives).

Each of these two strategies produces the results in Corollaries 1–7 with the same rates, but there are some minor variations in the upper bounds that can be obtained for the constant factors $c_{kj}$, $c_{kij}$ to these rates (see also Remarks 3, 4). A more flexible strategy within this approach is to consider the use of Strategies 1, 2 individually and independently for each summand in the RHSs in (8,10) and find which of the two strategies gives the smaller upper bounds of the constants related to this particular summand. Note, however, that while the rates in Corollaries 1–7 are, in a certain sense, sharp, even the best of the above combined strategies produce upper estimates for the constant factors $c_{kj}$, $c_{kij}$ in Corollaries 1–7 which are efficient but not sharp.

**Remark 8.** Note that the knowledge of the boundary values $\alpha$ and $\beta$ implies the knowledge of $|\beta-\alpha|$ and $\max\{|\alpha|,|\beta|\}$ but, in the opposite direction, the knowledge of $|\beta-\alpha|$ and $\max\{|\alpha|,|\beta|\}$ does not determine uniquely the boundary values $\alpha$ and $\beta$. This means that the assumptions about knowledge of $|\beta-\alpha|$ and $\max\{|\alpha|,|\beta|\}$ in Theorems 1, 2 and Corollaries 1–7 are less stringent than the assumptions of knowledge of the exact $\alpha$ and $\beta$. It also means that besides the case $(\alpha,\beta)=(0,0)$ of homogeneous boundary conditions, there is also one more general particular case of inhomogeneous boundary conditions for which the RHSs of (8,10) simplify: namely, this is the case $\alpha=\beta$.

**Remark 9.** Comparison between the rates in Theorems 1 and 2, Corollary 1 and Corollary 5, Corollary 3 and Corollary 6, Corollary 4 and Corollary 7, respectively, shows that, indeed, the "best" scheme performs best, to the extent that, under the same assumptions on the problem's data, the rate of approximation by the "best" scheme is one order higher than the approximation rate achieved via an arbitrary admissible scheme. This is true not only for homogeneous boundary conditions but for the general case of inhomogeneous boundary conditions, as well.

**Remark 10.** In [20, Section 4.3.3], in the context of homogeneous boundary conditions considered in [20], we observed that our results continue to hold true also under assumptions on $k$, $q$, $f$ which are much more general than piecewise continuity (see the formulation of Theorems 1 and 2), and are close to those in the formulation of (1). This observation continues to hold true also in the general case of inhomogeneous boundary conditions.

**Remark 11.** Throughout [19, 20] and the present paper we assumed that in the discrete approximating problem (2) the approximation at the boundary is exact. This ensured that problem (3) about the error is with homogeneous boundary conditions. For such discrete problems, the *a priori* estimate in [19, Lemma 6] holds true, and it is a key tool for deriving all error estimates in terms of properties of the solution in [1, 19], hence, also for all main results in [20] and the present paper. In order to handle the more general case of inexact approximation at the boundary in (2), we can invoke a discrete analogue of alternatives A. and B. for the continual case (see Section 3):

A. by directly proving a generalization of the *a priori* estimate for the error in (3) for the case of arbitrary inhomogeneous boundary conditions in (3) (which means that problems (2) and (3) now coincide); this objective requires:
   (a) constructing the discrete Green's function for problems (2,3) in the general case of inhomogeneous boundary conditions;
   (b) using the construction in the previous item to derive the above-said *a priori* estimate for the solution of (2,3) in the general case of inhomogeneous boundary conditions;

B. via reduction of the general case of inexact approximation at the boundary to the case of exact approximation at the boundary, essentially transferring the approximation error at the boundary of the domain to the local approximation error of the residual in the interior of the domain; this results in the reduction of the general case of inhomogeneous boundary conditions in (3) to the particular case of homogeneous boundary conditions for which [19, Lemma 6] suffices.

**Remark 12.** All techniques developed in [19, 20] and the present paper for the fulfillment of Stages 1–6 of the extended Lax principle for ODEs and applied to the study of the approximation error for the model ODE problem (1,2) have their analogues in the case of PDEs on Cartesian-product domains but, as mentioned in Section 3, in the PDE case the level of technical sophistication considerably increases. For example, in [23] we shall show that reduction of the general case of inhomogeneous boundary conditions on the boundary of a (hyper-)rectangular domain to the particular case of homogeneous boundary conditions can be achieved by Boolean-sum interpolation (see, e.g., [28, 27]), in combination with the use of an intermediate approximation like, e.g., the Steklov-mean, with special tuning of its parameters.

**Remark 13.** The method developed and the results obtained in this paper (and the previous papers [19, 20] on this topic) are an application of the general theory developed in [14], [15].

**Remark 14.** The material in this paper (and the previous papers [19, 20] on this topic) covers the part of the previously unpublished results in [5, Chapter 3] related to ODEs and complementary to the unpublished results in [5, Chapter 3] about PDEs which are scheduled to appear in [21, 22, 23].

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