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ON ERROR ESTIMATION FOR APPROXIMATION METHODS INVOLVING DOMAIN DISCRETIZATION VII: DETERMINISTIC PROBLEMS VI. FULLY DISCRETE FINITE DIFFERENCE SCHEMES FOR LINEAR PDES I: ESTIMATES IN TERMS OF PROPERTIES OF THE SOLUTION

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Abstract: This is the seventh of a sequence of 12 papers, preceded by [17, 18, 19, 20, 21, 22] and followed by [23, 24, 25, 26, 27] (in this order), dedicated to the study of error estimates for approximation problems based on discretization of the domain of the approximated functions. Within this sequence, in [18] and [19], for a model example of a Cauchy problem for a linear differential equation with variable coefficients, we applied an indirect method for error estimation based on results known in advance for the respective semi-discrete approximating problems.

In a follow-up subsequence of six of these papers, of which this is the fourth one, preceded by [20, 21, 22] and followed by [23, 24] (in this order) we develop a direct discrete method for error estimation based on an *extended Lax principle*, essentially proposed first in [6] but explicitly formulated for the first time in [20, Section 2]. In [20, 21, 22] we applied the proposed method to obtain sharp error estimates for a model boundary problem for a linear ordinary differential equation of second order with variable coefficients and right-hand side. This study is being continued in the present paper and [23, 24] of the subsequence

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by an analogous application of the proposed method to obtain sharp error estimates for a model initial-boundary problem for a linear parabolic partial differential equation of second order.

In the present paper we discuss the first 4 stages of the extended Lax principle for the model problem in consideration. These stages essentially correspond to the classical Lax principle, but classical error estimates obtained via this principle are rather coarse [8]. They were essentially sharpened in [29, 6] using more advanced tools for error estimation, such as integral and averaged moduli of smoothness. Here we provide a systematic exposition of the results of [6], sharpening, generalizing, and upgrading the results of [29], and complementing them with some additional ones, in order to prepare the error estimates at Stage 4 for use in the derivation of appropriate *a priori* estimates at Stage 5 and the final results of the new method at Stage 6 of the extended Lax principle, related to partial differential equations, in the subsequent papers [23, 24].

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1. Introduction

This is the seventh of a sequence of 12 papers, preceded by [17, 18, 19, 20, 21, 22] and followed by [23, 24, 25, 26, 27] (in this order), dedicated to the study of error estimates for approximation problems based on *discretization of the domain* of the approximated functions, and (in the concluding paper [27]) comparison of the similarities and differences with the error estimates derived by alternative approximation methods (typically, of projection type) based on finite-dimensional subspaces of functions having the same *continual domain* as the target function.

Within this sequence, in [18] and [19], for model examples of Cauchy problems, we applied an indirect method for error estimation based on results known in advance for the respective semi-discrete approximating problems.

In a sequence of six papers, of which this is the fourth one, preceded by [20, 21, 22] and followed by [23, 24] (in this order) we develop a direct discrete method for error estimation based on an *extended Lax principle*, essentially proposed first in [6] but explicitly formulated for the first time in [20, Section 2]. In this sequence of papers we apply the proposed method to obtain sharp error estimates for the following two model initial-boundary problems (one for an ordinary differential equation (ODE), and one for a partial differential equation (PDE)):

1. for a model linear ODE with variable coefficients and right-hand side (RHS) (see [20, 21, 22]);

2. for a model linear parabolic PDE with constant coefficients and a variable RHS (see the present paper and [23, 24]).

2. A Fully Discrete Method for Error Estimation

The *fully discrete method for error estimation directly in terms of properties of the problem's data*, proposed first in [6, Chapter 3], is comprised of a sequence of stages, called (see [20, Section 2]) *an* ***extended*** *Lax principle*. For self-consistency of the paper, we repeat here the main stages of this principle.

1. Derivation of estimates for the *local approximation error of the residual*.

2. Using the linearity, derivation of a *discrete approximating problem for the error*).

3. Derivation of *a priori estimates* for the solution of the discrete approximating problem for the error.

4. Combining the results obtained at Stages 1 and 3 to derive *error estimates in terms of properties of the solution.*

5. Derivation of ***consistent*** *a priori* estimates for the solution of the exact target problem.

6. Combining the results obtained at Stages 4 and 5 to derive *error estimates directly in terms of properties of the data functions* of the exact target problem.

The sub-sequence of Stages 1–4 constitutes *the classical Lax principle: local convergence of the residual* (to zero) (Stage 1) *and stability* (of the discrete approximating problem for the error) (Stages 2 and 3) *imply convergence of the error* (to zero) (Stage 4).

This classical qualitative formulation of the Lax principle was upgraded to quantitative relatively very early (see, e.g., a collection of classical references in [8]) but the quantitative error estimates derived at Stage 4 in this early period were rather coarse, in the sense that they required too high order of smoothness of the exact solution to ensure a given rate of approximation (see, e.g., [8]), thus unnecessarily limiting the range of applicability of the error estimates at Stage 4. This was due mainly to the rather limited choice (at that time) of available functional characteristics measuring regularity.

With the appearance of newer functional characteristics, notably, functional moduli such as the integral and the averaged moduli of smoothness, and the intensive development of their theory, in particular, by relating them to Peetre K-functionals (see, e.g., [15, 16] and the relevant references therein) it became possible to derive sharp error estimates at Stage 4 under minimal, or very close to minimal, assumptions about regularity of the solution. Such early results were obtained, notably, in [1] for ODEs, and in [29, 6] for PDEs. In [6] we proposed the upgrading and extending of the 'Lax principle paradigm' with Stages 5 and 6 (although the term 'extended Lax principle' was used for the first time in [20]).

The purpose of the present paper is to provide a systematic exposition of the results of [6], also sharpening, generalizing, and upgrading the results of [29], and complementing them with some additional ones, in order to prepare the error estimates at Stage 4 for use in the derivation of consistent *a priori* estimates at Stage 5 and the final results of the new method at Stage 6 related to PDEs, see the subsequent papers [23, 24].

This is a repetition of the same organization as for ODEs, where [20] was dedicated to sharp error estimates at Stage 4, in preparation for the derivation of the results at Stages 5 and 6 in the subsequent papers [21, 22].

2.1. Discrete Approximating Problems for PDEs

Consider the Dirichlet boundary-value (Cauchy–Dirichlet initial-boundary) problem for the heat equation:

$$\begin{aligned}
&\tfrac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t), \quad x \in [0,1],\ t \in (0,T],\ T > 0,\\
&u(x,0) = g(x), \quad x \in [0,1],\\
&u(j,t) = g_j(t), \quad t \in [0,T], \quad j = 0,1,\\
&g(\cdot) \in BM[0,1], g_j(\cdot) \in BM[0,T], \quad j = 0,1,\\
&f - \text{defined and bounded everywhere on}\\
&\Omega = [0,1] \times [0,T] \text{ and measurable.}
\end{aligned} \tag{1}$$

(Measurability is with respect to the customary one- and two-dimensional Lebesgue measures for $[0,1], [0,T]$ and Ω, respectively.) For the definition of the space BM, see section 3, item 1.

We assume that a "conjugation condition" is also fulfilled: u – continuous on the interior of Ω (cf. [8, p. 71]).

Approximate (1) by the following homogeneous conservative finite difference scheme (see [8, pp. 110-119, 185-192]):

$$\begin{aligned}
&\tfrac{v_{i,j+1}-v_{ij}}{d} - \left(\sigma \tfrac{v_{i+1,j+1}-2v_{i,j+1}+v_{i-1,j+1}}{h^2} + (1-\sigma)\tfrac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{h^2}\right) = \varphi_{ij},\\
&\qquad x \in \mathring{\Sigma}_h = \{\xi_i : \xi_i = ih,\ i = 1,\ldots,N-1;\ Nh = 1\},\\
&\qquad t \in \mathring{\Sigma}_d = \{\eta_j : \eta_j = jd,\ j = 0,1,\ldots,M-1;\ Md = T\},\\
&v_{i0} = g_i,\ x_i \in \{ih : i = 0,1,\ldots,N\} = \Sigma_h,\\
&v_{0j} = (g_0)_j,\ v_{1j} = (g_1)_j,\\
&j = 0,1,\ldots,M,\ t_j \in \{jd : j = 0,1,\ldots,M\} = \Sigma_d,
\end{aligned} \tag{2}$$

The error in quest $z_{ij} = v_{ij} - u_{ij}$ in (1, 2) is uniquely determined as the solution of the following finite difference problem (cf. [8, p. 189]):

$$\begin{aligned}
&\tfrac{z_{i,j+1}-z_{ij}}{d} - \left(\sigma \tfrac{z_{i+1,j+1}-2z_{i,j+1}+z_{i-1,j+1}}{h^2} + (1-\sigma)\tfrac{z_{i+1,j}-2z_{i,j}+z_{i-1,j}}{h^2}\right) = \psi_{ij},\\
&\qquad i = 1,2,\ldots,N-1;\ j = 0,1,\ldots,M-1\\
&z_{i0} = 0,\ i = 0,1,\ldots,N\\
&z_{0j} = z_{1j} = 0,\ j = 0,1,\ldots,M.
\end{aligned} \tag{3}$$

(For simplicity of exposition, the approximation on the boundary is assumed to be exact.)

Estimation of the RHS $\psi = \{\psi_{ij}\}$ in (3) yields the convergence rate of the local approximation of the residual and the conditions for this rate to be attained; estimation of the solution $z = \{z_{ij}\}$ of (3) yields the rate and conditions for convergence to zero of the error of the considered finite difference method. The function ψ is defined only for $i = 1, 2, \ldots, N-1$. We extend its definition to $i = 0, N$, as follows: $\psi_0 = \psi_N = 0$.

The function φ_{ij} in (2) is an approximation of f_{ij}. More precisely, in view of the homogeneity of the finite difference scheme (see [8, pp. 116-119, 187])

$$\varphi_{ij} = \mathrm{F}[f(x+sh, t_{j+\frac{1}{2}})], \tag{4}$$

where the template functional F is the same as in [20, 21, 22]: it is defined for $\bar{f}(s) \in BM\left[-\frac{1}{2}, \frac{1}{2}\right]$ and has the following properties

1. $\mathrm{F}[1] = 1$;
2. F is linear over $BM\left[-\frac{1}{2}, \frac{1}{2}\right]$;
3. $\mathrm{F}[\bar{f}(s)] \geq 0$ for $\bar{f}(s) \geq 0,\ s \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

For conciseness of the presentation, we introduce the following additional notation:

- all functions $a(x,t)$ are considered defined in $[0,1] \times [0,T]$,
  - (i) $a(x)$ – in $[0,1]$,
  - (ii) $a(t)$ – in $[0,T]$;
- $\Sigma_{hd}^{j} = \Sigma_h \times \{\mu d : \mu = 0, 1, \ldots, j\}, \quad \Sigma_{hd}^{M} = \Sigma_{hd}$;
- $a_{ij} = a(x_i, t_j), \quad (x_i, t_j) \in \Sigma_{hd}$;
- $a_{\bar{x},ij} = \frac{a_{ij} - a_{i-1j}}{h}, \quad a_{x,ij} = \frac{a_{i+1j} - a_{ij}}{h}$;
- for mesh functions φ and ψ, $\varphi = \psi$ denotes $\varphi_{ij} = \psi_{ij}, \forall (x_i, t_j) \in \Sigma_{hd}$;
- $[\varphi_j, \psi_j] = \sum\limits_{i=0}^{N} h\varphi_{ij}\psi_{ij}$;
- $[[\varphi, \psi]]_{hd} = \sum\limits_{i=0}^{N}\sum\limits_{j=0}^{M} hd\varphi_{ij}\psi_{ij}$;
- $\Lambda_h = \{\varphi : \mathrm{Dom}\varphi = \Sigma_h, \mathrm{Cod}\varphi \subset \mathbb{R}\}$;

- $\mathring{\Lambda}_h = \{\varphi : \varphi \in \Lambda_h, \varphi(0) = \varphi(1) = 0\}$;
- $\omega_\mu(x_\nu)f = \omega_k(f(\cdot, t_\mu + \beta d), x_\nu + \alpha h; ch)$,

  $\omega_\nu(t_\mu)f = \omega_k(f(x + \alpha h, \cdot), t_\mu + \beta d, cd)$,

  where f is the function in consideration, $k \in \mathbb{N}$, $c > 0$, $\alpha, \beta \in \mathbb{R}$. Obviously, $\omega_\mu(x_\nu)$, $\omega_\nu(t_\mu)$ are semi-norms.

3. Preliminaries: Functional Moduli and Function Spaces

Here we present the notation for the functional characteristics needed in the sequel. As a general reference about the functional characteristics introduced in this section concerning the notation, definitions, meaning and range of parameters, relevant properties and interrelations, as well as further details, we cite [15] together with [16], and the references therein. Another concise reference source on the material in this section is [17, Section 2] and the references therein.

1. The function space $C(\Omega) \subset BM(\Omega)$ of all (real-valued) continuous functions on Ω, where Ω and $BM(\Omega)$ were defined in [22, section 2.1.1]. The restriction of the norm $\|\cdot\|_{BM}$ on C will be denoted, as usual, by $\|\cdot\|_C$; recall that with respect to this norm C is a Banach space which is a closed subspace of the Banach space BM (see, e.g., [4]). The meaning of Ω is the same in all subsequent items of this list.

2. For a function $f \in BM(\Omega)$,
$$S(t, f; x) = sup\,\{f(y) : y \in [x - t, x + t] \cap \Omega\}$$
is the upper Baire's function of f at $x \in \Omega$ with step $t > 0$ (see [15] and the references therein).

3. The local modulus of smoothness of order $k \in \mathbb{N}$ at $x \in \Omega$ with $t > 0$
$$\omega_k(f, x; t) = \sup\left\{\|\Delta_h^k f(y)\| : y, y + kh \in \left[x - \frac{kt}{2}, x + \frac{kt}{2}\right] \cap \Omega\right\}$$
(see [15] and the references therein); it is natural to define also $\omega_0(f, x; t) := S(t, f; x)$.

4. The sequence space $l^p(\Sigma_h)$ defined over the mesh Σ_h (see [30, 11, 12, 13, 6, 15]).

5. The integral modulus of smoothness (ω-modulus) $\omega_m(f;h)_{L_p}$ (short: $\omega_m(f;h)_p$) (see [30, 11, 12, 13, 10, 6, 15], cf. also [7, 2] where the notation is modified).

6. The averaged modulus of smoothness (τ-modulus) $\tau_m(f;h)_{L_p}$ (short: $\tau_m(f;h)_p$) (see [10, 30, 11, 12, 13, 14, 6, 15]).

7. The Steklov-means $f_{k,t}$ (see [3, 9, 10, 15]).

8. The Wiener amalgam space $A_{p,h}(\Omega)$, with norm $\|.\|_{A_{p,h}} = \tau_0(f;h)_{L_p}$ (see [13, 14, 6, 15, 16]).

[17, Section 2.3] contains an important comparison between the properties of the integral and averaged moduli of smoothness.

Following our practice in [6] and [15], in order to distinguish between previously known results and the new ones obtained here, we shall add the additional marker '**(K.)**' (abbreviated from '(K)nown') to the enumeration of every statement in the sequel which has been previously known (with respective reference to available relevant literature, whenever possible).

The following known lemma will be used frequently.

Lemma 1. **(K.)** *(Cf. [1, 10, 5]). Let*

1. $\Upsilon = \sum_{\nu=1}^{N-1} \sum_{\mu=0}^{M-1} hda_{\nu\mu}$,

2. $|a_{\nu\mu}| \le c\omega_\mu(x_\nu)f + c\omega_\nu(t_\mu)f + |\Theta_{\nu\mu}|$, $\forall(\nu,\mu)$.

Then,

$$|\Upsilon| \le c\|\tau_{k_1}(f(\cdot,t);h)_{L_1[0,1]}\|_{A_{1,d}[0,T]} + c\|\tau_{k_2}(f(x,\cdot);d)_{L_1[0,T]}\|_{A_{1,h}[0,1]} + \Theta,$$

where $\Theta = \sum_{\nu=1}^{N-1} \sum_{\mu=0}^{M-1} hd|\Theta_{\nu\mu}|$, *and* $k_1,\ k_2$ *are the orders of the local moduli of smoothness* $\omega_\mu(x_\nu),\ \omega_\nu(t_\mu)$*, respectively.*

(For the proof of Lemma 1, see [1, 10, 5].)

Remark 1. Because $[0,1],[0,T]$ are bounded, in Lemma 1 the spaces $L_1,\ A_{1,h},\ A_{1,d}$ can be replaced with $L_p,\ A_{q,h},\ A_{r,d},\ p,q,r \ge 1$, respectively.

4. Heat Conductivity PDE with Constant Coefficients and Variable Right-Hand Side

4.1. Estimates of the Local Approximation Error of the Residual

Now we move on to the study of the local approximation of the residual.

Lemma 2. **(K.)** *(See [29].) For the error of the local approximation of the residual in (3) it holds that*

$$|\psi_{ij}| \leq \; c_1\omega_2(\tfrac{\partial u}{\partial t}(x,\cdot),t_j;d) + c_2\omega_2(\tfrac{\partial^2 u}{\partial x^2}(\cdot,t_{j+1}),x_i;h) + c_3\omega_2(\tfrac{\partial^2 u}{\partial x^2}(\cdot,t_j),x_i;h) + c_4|\sigma - \tfrac{1}{2}|\omega_1(\tfrac{\partial u}{\partial t}(x,\cdot),t_j;d),$$

where $c_j,\ j=1,2,3,4,$ are absolute constants.

Making use of the properties of the local moduli of smoothness, it is easy to note that in the cases, considered in Lemma 2, the local approximation of the residual has convergence rate $O(h^2+d),\ \sigma \neq \frac{1}{2}$ and $O(h^2+d^2),\ \sigma = \frac{1}{2}$, for sufficiently smooth functions.

There is one known case (see [8, pp. 74–75]), in which this rate is $O(h^4+d^2)$. Our first new result concerns this case.

Lemma 3. *Assume that in (1)*

1. $\frac{\partial f}{\partial t}, \frac{\partial^2 f}{\partial x^2}$ *exist everywhere on* $[0,1]\times[0,T]$*;*
2. $\frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial x^2}$ *is continuous;*
3. $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^4 u}{\partial x^4}, \frac{\partial}{\partial t}(\frac{\partial^2 u}{\partial x^2}), \frac{\partial^2}{\partial x^2}(\frac{\partial u}{\partial t})$ *exist and are continuous everywhere on* $[0,1]\times[0,T]$*;*
4. *in (2)* $d=\frac{h^2}{6},\ \varphi_{ij} = \frac{h^2}{12}\left(\frac{\partial f}{\partial t}(x_i,t_j) + \frac{\partial^2 f}{\partial x^2}(x_i,t_j)\right),\ \sigma = 0.$

Then, $\forall (x_i,t_j) \in \Sigma^{\mu}_{hd} \Rightarrow$

$$|\psi_{ij}| \leq \frac{h^2}{12}\left(\omega_1\left(\frac{\partial^2 u}{\partial t^2}(x_i,\cdot),t_j;d\right) + \omega_2\left(\frac{\partial^4 u}{\partial x^4}(\cdot,t_j),x_i;h\right)\right).$$

Proof. Under the lemma's assumptions, we can apply the formula for Taylor expansion with integral remainder, which allows to represent the local approximation of the residual ψ in the following way:

$$\begin{aligned}
\psi_{ij} =&\varphi_{ij} - \frac{u_{ij+1}-u_{ij}}{d} + \frac{u_{i+1j}-2u_{ij}+u_{i-1j}}{h^2} \\
=&\varphi_{ij} - \frac{1}{d}\left(u_{ij} + d\frac{\partial u}{\partial t}(x_i,t_j) + \int_0^d (d-\eta)\frac{\partial^2 u}{\partial t^2}(x_i,t_j+\eta)d\eta - u_{ij}\right) \\
&+ \frac{1}{h^2}[u_{ij} + h\frac{\partial u}{\partial x}(x_i,t_j) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_i,t_j) + \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x_i,t_j) \\
&+ \frac{1}{6}\int_0^d (h-\zeta)^3\frac{\partial^4 u}{\partial x^4}(x_i+\zeta,t_j)d\zeta - 2u_{ij} \\
&+ u_{ij} - h\frac{\partial u}{\partial x}(x_i,t_j) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_i,t_j) - \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x_i,t_j) \\
&+ \frac{1}{6}\int_0^d (h-\zeta)^3\frac{\partial^4 u}{\partial x^4}(x_i+\zeta,t_j)d\zeta] = \varphi_{ij} - \left(\frac{\partial u}{\partial t}(x_i,t_j) - \frac{\partial^2 u}{\partial x^2}(x_i,t_j)\right) \\
&- \frac{1}{d}\int_0^d (d-\eta)\left(\frac{\partial^2 u}{\partial t^2}(x_i,t_j+\eta) - \frac{\partial^2 u}{\partial t^2}(x_i,t_j)\right)d\eta \\
&- \frac{1}{6h^2}\int_0^h (h-\zeta)^3\left(\frac{\partial^4 u}{\partial x^4}(x_i+\zeta,t_j) - 2\frac{\partial^4 u}{\partial x^4}(x_i,t_j) + \frac{\partial^4 u}{\partial x^4}(x_i-\zeta,t_j)\right)d\zeta \\
&- \left(\frac{1}{d}\int_0^d (d-\eta)d\eta\right)\frac{\partial^2 u}{\partial t^2}(x_i,t_j) + \left(\frac{2}{6h^2}\int_0^h (h-\zeta)^3 d\zeta\right)\frac{\partial^4 u}{\partial x^4}(x_i,t_j) \\
=&f_{ij} + \frac{h^2}{12}\left(\frac{\partial f}{\partial t}(x_i,t_j) - \frac{\partial^2 f}{\partial x^2}(x_i,t_j)\right) - \left(\frac{\partial u}{\partial t}(x_i,t_j) - \frac{\partial^2 u}{\partial x^2}(x_i,t_j)\right)
\end{aligned}$$

$$- \frac{1}{d}\int_0^d (d-\eta)\left(\frac{\partial^2 u}{\partial t^2}(x_i, t_j+\eta) - \frac{\partial^2 u}{\partial t^2}(x_i, t_j)\right) d\eta$$

$$+ \frac{1}{6h^2}\int_0^h (h-\zeta)^3\left(\frac{\partial^4 u}{\partial x^4}(x_i+\zeta, t_j) - 2\frac{\partial^4 u}{\partial x^4}(x_i, t_j) + \frac{\partial^4 u}{\partial x^4}(x_i-\zeta, t_j)\right) d\zeta$$

$$- \frac{1}{2}\left(d\frac{\partial^2 u}{\partial t^2}(x_i, t_j) - \frac{h^6}{6}\frac{\partial^4 u}{\partial x^4}(x_i, t_j)\right).$$

Clearly, from the PDE,

$$f_{ij} = \frac{\partial u}{\partial t}(x_i, t_j) - \frac{\partial^2 u}{\partial x^2}(x_i, t_j).$$

Besides, it is easy to see that

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2}.$$

For $d = \frac{h^2}{6}$ we get

$$\psi_{ij} = -\frac{1}{d}\int_0^d (d-\eta)\left(\frac{\partial^2 u}{\partial t^2}(x_i, t_j+\eta) - \frac{\partial^2 u}{\partial t^2}(x_i, t_j)\right) d\eta$$

$$+ \frac{1}{6h^2}\int_0^h (h-\zeta)^3\left(\frac{\partial^4 u}{\partial x^4}(x_i+\zeta, t_j) - 2\frac{\partial^4 u}{\partial x^4}(x_i, t_j) + \frac{\partial^4 u}{\partial x^4}(x_i-\zeta, t_j)\right) d\zeta.$$

Hence,

$$|\psi_{ij}| \le \frac{1}{d}\int_0^d (d-\eta)\omega_1\left(\frac{\partial^2 u}{\partial t^2}(x_i, \cdot), t_j; d\right) d\eta +$$

$$\frac{1}{6h^2}\int_0^h (h-\zeta)^3\omega_2\left(\frac{\partial^4 u}{\partial x^4}(\cdot, t_j), x_i; h\right) d\zeta,$$

from where the lemma's statement follows immediately, because the ω-integrands do not depend on the respective integration variables. □

Lemma 4. *The RHS of (2), given via (4), is bounded by*

$$|\varphi_{ij}| \leq S\left(\frac{h}{2}, |f(\cdot, t_{j+\frac{1}{2}})|; x_i\right).$$

Proof. Let

$$g \in BM\left([-\frac{h}{2} + x_i, x_i + \frac{h}{2}] \cap [0, 1]\right).$$

We have

$$|\mathrm{F}[g(x_i + sh)]| \leq \mathrm{F}\left[|g(x_i + sh)|\right].$$

Indeed, the linearity (homogeneity) and the positivity of F imply

$$\mathrm{F}[g(x_i + sh)] \leq \mathrm{F}[|g(x_i + sh)|],$$

$$-\mathrm{F}[g(x_i + sh)] = \mathrm{F}[-g(x_i + sh)] \leq \mathrm{F}[|g(x_i + sh)|].$$

From the homogeneity, positivity and normalization of F follows

$$\begin{aligned}|\mathrm{F}[g(x_i + sh)]| &\leq \mathrm{F}[|g(x_i + sh)|] \leq \\ &\leq \mathrm{F}[\|g\|_{BM([-\frac{h}{2}+x_i, x_i+\frac{h}{2}]\cap[0,1])} = \\ &= \|g\|_{BM}\mathrm{F}[1] = \\ &= \|g\|_{BM} = \\ &= S\left(\frac{h}{2}, |g|; x_i\right).\end{aligned}$$

This easily implies

$$|\varphi_{ij}| = |\mathrm{F}[f(x_i + sh, t_{j+\frac{1}{2}})]| \leq S\left(\frac{h}{2}, |f(\cdot, t_{j+\frac{1}{2}})|; x_i\right). \qquad \square$$

4.2. A Priori Estimates for the Solution of the Discrete Approximating Problem

In order to apply Lemma 1 to problem (3), for the latter problem we shall construct a discrete Green's function $G_{ij}^{\nu\mu};\ i, \nu = 0, \ldots, N;\ j, \mu = 0, \ldots, M$, which allows the representation of the solution of (3) in the "integral form"

$$z_{ij} = [[G_{ij}, \psi]]_{hd} = \sum_{\nu=0}^{N}\sum_{\mu=0}^{M} hdG_{ij}^{\nu\mu}\psi_{\nu\mu} = \sum_{\nu=1}^{N-1}\sum_{\mu=0}^{M} hdG_{ij}^{\nu\mu}\psi_{\nu\mu}.$$

The last equality holds because of $\psi_j \in \mathring{\Lambda}_h,\ \forall t \in \Sigma_d$, by definition.

Lemma 5. **(K.)** *(See [5, 6, 28].) Problem (3) has the following Green's function:*

$$G_{ij}^{\nu\mu} = \begin{cases} 2\sum\limits_{n=1}^{N-1} \frac{1}{1+\lambda_n d\sigma} \left(\frac{1-\lambda_n d(1-\sigma)}{1+\lambda_n d\sigma}\right)^{j-\mu-1} \sin \pi n x_\nu \sin \pi n x_i, & 0 \le \mu < j, \\ 0, & j \le \mu \le M, \end{cases}$$

$i, \nu = 1, 2, \ldots, N-1;\ \lambda_n = (\frac{4}{h^2}) \sin^2(\pi n \frac{h}{2}),\ n = 1, 2, \ldots, N-1.$

This lemma has been formulated and proven in [28] but, for self-consistency of this presentation, we present here the original proof in [5, 6] (which was later published in bigger generality in [28]). This will allow us to introduce in due context the associated eigenvalues and eigenfunctions which will be used in the derivation of the *a priori* estimate in Lemma 6 below.

Proof. We shall prove the lemma using the analogy with the continual case (i.e., (1) with homogeneous initial and boundary conditions).

We search for a solution to (3) with zero right-hand side (RHS) by the method of separation of variables (Fourier's method) $z_{ij} = X_i T_j$, from where

$$T_{t,j} X - (\sigma T_{j+1} + (1-\sigma) T_j)\, X_{\bar{x}x},$$

i.e.,

$$\frac{T_{t,j}}{\sigma T_{j+1} + (1-\sigma) T_j} = \frac{X_{\bar{x}x}}{X}.$$

Since the left-hand side (LHS) does not depend on x, while the RHS does not depend on t, they must be both constant, denoted here by $-\lambda$, and, hence, we obtain the following discrete Sturm–Liouville problem:

$$\begin{cases} X_{\bar{x}x} + \lambda X = 0, & x \in \mathring{\Sigma}_h \\ X \in \mathring{\Lambda}_h \end{cases}$$

In, e.g., [8] it is shown that the non-trivial eigenvalues of this problem are $\lambda_n = (\frac{4}{h^2}) \sin^2(\pi n \frac{h}{2})$, while $X_n(x) = \sqrt{2} \sin \pi n x$ are (normalized) eigensolutions corresponding to $\lambda_n, n = 1, \ldots, N-1$, and $\{X_n, n = 1, \ldots, N-1\}$ is an orthonormal basis in $\mathring{\Lambda}_h$. Since $z, \psi \in \mathring{\Lambda}_h,\ \forall j$, these two functions have the following representation in this basis:

$$z(x,t) = \sum_{n=1}^{N-1} z^{(n)}(t) X_n(x),$$

$$\psi(x,t) = \sum_{n=1}^{N-1} \psi^{(n)}(t) X_n(x),$$

where $(x,t) \in \Sigma_h \times \mathring{\Sigma}_d$, $\psi^{(n)}(t) = [f(\cdot, t), X_n]_h = \sum\limits_{\nu=1}^{N-1} h f(x_\nu, t) X_n(x)$, because of the orthonormality.

We substitute the basis representations for $z(x,t), \psi(x,t)$ into the equation of problem (3) and obtain for $z^{(n)}(t)$:

$$\begin{cases} z_{t,j}^{(n)} + \lambda_n(\sigma z_{j+1}^{(n)} + (1-\sigma)z_j^{(n)}) = \psi_j^{(n)}; \quad (x,t) \in \mathring{\Sigma}_h \times \mathring{\Sigma}_d, \\ z_0^{(n)} = c_n. \end{cases}$$

Since (3) has zero initial condition, it follows that $c_n = 0$, wherefrom, by induction,

$$z_j^{(n)} = \frac{d}{1+\lambda_n d\sigma} \sum_{\mu=0}^{j-1} \psi_\mu^{(n)} \left(\frac{1-\lambda_n d(1-\sigma)}{1+\lambda_n d\sigma} \right)^{j-\mu-1}.$$

Taking in consideration the basis representation for $\psi^{(n)}t$, and commuting the summation order, we get:

$$z_{ij} = \sum_{\mu=0}^{j-1} \sum_{\nu=1}^{N-1} hd \left(\sum_{n=1}^{N-1} \frac{1}{1+\lambda_n d\sigma} \left(\frac{1-\lambda_n d(1-\sigma)}{1+\lambda_n d\sigma} \right)^{j-\mu-1} X_n(x_\nu) X_n(x_i) \right) \psi_{\nu\mu}.$$

Equalizing the latter expression to the assumed "integral representation" $z_{ij} = \sum_{\mu=0}^{j-1} \sum_{\nu=1}^{N-1} hdG_{ij}^{\nu\mu} \psi_{\nu\mu}$ for any choice of ψ, we obtain, according the principle of undetermined coefficients, the lemma's statement. □

Remark 2. Note that (see [28] for the details) if in the conditions of Lemma 5 we had a non-zero initial condition $z(x,0) = \varphi(x)$, we would have had $\varphi \in \mathring{\Lambda}_h$, because of the need for compatibility with the zero boundary conditions, i.e.,

$$\varphi(x) = \sum_{n=1}^{N-1} hc_n X_n(x),$$

where

$$c_n = \sum_{\nu=1}^{N-1} h\varphi_\nu X_n(x_\nu).$$

From Lemma 5 we immediately obtain

$$|z_{ij}| \leq \sum_{\mu=0}^{j-1} \sum_{\nu=1}^{N-1} hd|G_{ij}^{\nu\mu}||\psi_{\nu\mu}| \leq \sum_{\mu=0}^{M-1} \sum_{\nu=1}^{N-1} hd|G_{ij}^{\nu\mu}||\psi_{\nu\mu}|$$

The latter result will yield the *a priori estimate* which will be used in the sequel. To this end, we have to estimate $|G_{ij}^{\nu\mu}|$. The following *a priori estimate* and its proof are published here for the first time.

Lemma 6. *(See [5, 6].) Uniformly in i and ν, it is fulfilled that*

$$|G_{ij}^{\nu\mu}| \leq \sum_{n=1}^{N-1} \left(1 - \lambda_n \varepsilon \frac{h^2}{4}\right)^{j-\mu-1}, \quad \varepsilon \in (0,1],$$

under the condition:

$$\max\left\{0, \frac{1}{2-\varepsilon} - \frac{h^2}{4d}\right\} \leq \sigma \leq \min\left\{\frac{1}{\varepsilon} - \frac{h^2}{4d}, 1\right\}.$$

Proof. The expression for $G_{ij}^{\nu\mu}$ implies that it suffices to prove that under the afore-mentioned conditions the following holds true:

$$\left|\frac{1 - \lambda_n d(1-\sigma)}{1 + \lambda_n d\sigma}\right| \leq 1 - \lambda_n \varepsilon \frac{h^2}{4}, \quad n = 1, 2, \ldots, N-1.$$

Let first

$$\frac{1 - \lambda_n d(1-\sigma)}{1 + \lambda_n d\sigma} \leq 1 - \lambda_n \varepsilon \frac{h^2}{4}.$$

From here it follows that

$$\frac{\varepsilon h^2}{4\left(1 - \lambda_n \varepsilon \frac{h^2 \sigma}{4}\right)} \leq d.$$

The LHS in the latter inequality increases with the increase of n, thereforefore, in view of $\lambda_{n-1} \leq \frac{4}{h^2}$, the condition

$$\frac{\varepsilon h^2}{4(1-\varepsilon\sigma)} \leq d$$

is sufficient for the correctness of the initial inequality, uniformly in $n = 1, \ldots, N$ -1.

Let now

$$-1 + \lambda_n \varepsilon \frac{h^2}{4} \leq \frac{1 - \lambda_n d(1-\sigma)}{1 + \lambda_n d\sigma}.$$

This implies

$$\lambda_n d\left(1 - 2\sigma + \lambda_n \varepsilon \sigma \frac{h^2}{4}\right) \leq 2 - \lambda_n \varepsilon \frac{h^2}{4}.$$

For any $n = 1, 2, \ldots, N-1$, two cases are possible:

1. $1 - 2\sigma + \lambda_n \varepsilon \sigma \frac{h^2}{4} \leq 0$, which is fulfilled when $\sigma \geq \frac{1}{2 - \lambda_n \varepsilon \frac{h^2}{4}}$.

2. $1 - 2\sigma + \lambda_n \varepsilon \sigma \frac{h^2}{4} > 0$.

We consider the following alternative:

A. $\forall n : n = 1, \ldots, N-1$, case 1 holds true. A sufficient condition for this is $\sigma \geq \frac{1}{2-\varepsilon}$, again because of $0 < \lambda_n \leq \lambda_{n+1},\ \lambda_{N-1} < \frac{4}{h^2}$.

B. $\exists n$, for which case 2 holds true. Let n_0 be the smallest with this property. Clearly, $\forall n : n = n_0, n_0+1, \ldots, N-1$, case 2 holds true, i.e., it is necessary that

$$d \leq \frac{2 - \lambda_n \varepsilon \frac{h^2}{4}}{\lambda_n \left(1 - 2\sigma + \lambda_n \varepsilon \sigma \frac{h^2}{4}\right)}$$

for $n_0 \leq n \leq N-1$. The RHS decreases with the increase of n, i.e., the necessary condition

$$d \leq \frac{2 - \lambda_{N-1} \varepsilon \frac{h^2}{4}}{\lambda_n \left(1 - 2\sigma + \lambda_{N-1} \varepsilon \sigma \frac{h^2}{4}\right)}$$

is also sufficient for the correctness of the previous inequality. In this case, the stronger

$$d \leq \frac{2-\varepsilon}{4} \frac{h^2}{1 - (2-\varepsilon)\sigma}$$

is also sufficient. It is possible to directly verify that the inequality

$$\frac{\varepsilon h^2}{4(1 - \varepsilon\sigma)} \leq \frac{2-\varepsilon}{4} \frac{h^2}{1 - (2-\varepsilon)\sigma}$$

holds true $\forall \varepsilon \in (0,1],\ \forall \sigma : 0 \leq \sigma < \frac{1}{2-\epsilon}$.

It remains to notice that the so-obtained sufficient conditions can be written in the equivalent and more concise form indicated in the formulation of the lemma.

□

Lemma 6 yields one of the *a priori* estimates which we shall use in the sequel. Besides it, we shall be using also the following known *a priori* estimate.

Lemma 7. (K.) *(See [8, p. 83-86]) For the problem (3), under the condition*

$$\max\left\{0, \frac{1}{2} - (1-\varepsilon)\frac{h^2}{4d}\right\} \leq \sigma \leq 1,\ \varepsilon \in (0,1],$$

it holds true that

$$\|z_{j+1}\|_{l^\infty(\Sigma_h)} \leq \frac{1}{2\sqrt{2\varepsilon}} \left(\sum_{\mu=0}^{j} d \|\psi_\mu\|^2_{l^2(\Sigma_h)}\right)^{\frac{1}{2}}.$$

4.3. Error Estimates in Terms of Properties of the Solution

In this section we shall obtain the final estimate of the error z in (3) in norm

$$\|\cdot\|_{l^\infty(\Sigma^j_{hd})} : \|z\|_{l^\infty(\Sigma^j_{hd})} = \max\{|z_{\nu\mu}| : \nu = 0, \dots, N;\ \mu = 0, \dots, j\}$$

in terms of properties of derivatives of the solution.

Theorem 1. *Let $j \in \mathbb{N},\ 0 \le j \le M,\ t = jd,\ \varepsilon \in (0, \frac{1}{2}]$.*

Then,

$$\|z\|_{l^\infty(\Sigma^j_{hd})} \le c_1 \left(\frac{d}{h^2}\right)^{\frac{1}{q}} \|\tau_2\left(\frac{\partial^2 u}{\partial x^2}; h\right)_{L_1[0,1]}\|_{A_{p,d}[0,t]}$$

$$+ c_2 \left(\|\tau_2\left(\frac{\partial u}{\partial t}; d\right)_{L_2[0,t]}\|_{A_{2,h}[0,1]} + |\sigma - \frac{1}{2}|\|\tau_1\left(\frac{\partial u}{\partial t}; d\right)_{L_2[0,t]}\|_{A_{2,h}[0,1]}\right),$$

where:

(i) $p \in (2, \infty]$,

(ii) $\frac{1}{p} + \frac{1}{q} = 1$,

(iii) $c_2 = \frac{c}{2\sqrt{2\varepsilon}}$,

(iv) $c_1 = \frac{c2^{1-\frac{1}{q}}}{\varepsilon^{\frac{1}{q}} q^{\frac{1}{q}}} \zeta(\frac{2}{q})$,

(v) c – absolute constant,

(vi) $\zeta(s)$ – Riemann's ζ-function;

(vii) (it is possible to include also the case $p = 2$, but then $c_1 = O(\ln(\frac{1}{h}))$);

(viii) it holds true that

$$\max\{0, \frac{1}{2} - (1-\varepsilon)\frac{h^2}{4d}, \frac{1}{2-\varepsilon} - \frac{h^2}{4d}\} \le \sigma \le \min\{1, \frac{1}{\varepsilon} - \frac{h^2}{4d}\}.$$

Proof. Making use of the definition of $\omega_\nu(t_\mu)$ $\omega_\nu(x_\mu)$ and Lemma 2, after direct verification we find that

$$|\psi_{\nu\mu}| \le c\left(\omega_\nu(t_\mu)\frac{\partial u}{\partial t} + \omega_\nu(x_\mu)\frac{\partial^2 u}{\partial x^2}\right).$$

Because of the linearity and positivity, in the following estimates it will suffice to limit the considerations to only $\omega_\nu(t_\mu)$ and $\omega_\nu(x_\mu)$ instead of their linear combinations. Let us decompose (3) into the superposition of two problems (3') (3") of the same type, but with right-hand sides (RHSs) ψ' and ψ'', respectively, where $\psi' + \psi'' = \psi$ (the RHS of (3)). The linearity of the resolving operator of (3) and the homogeneity of the initial and boundary conditions imply the following connection between the solutions of the above-said two auxiliary problems:

$$z = z' + z''.$$

Making use of the proof of Lemma 2 (see [29]), we select ψ'' and ψ', as follows:

$$\psi''_{ij} = \sigma \tfrac{\partial u}{\partial t}(x_i, t_{j+1}) + (1-\sigma)\tfrac{\partial u}{\partial t}(x_i, t_j) - \tfrac{(u_{ij+1} - u_{ij})}{d},$$

$$\psi'_{ij} = \psi_{ij} - \psi''_{ij}$$

So defined, the RHSs of (3') and (3") are being majorized in absolute value by the local moduli of smoothness only in x, respectively only in t. To the first problem we apply Lemma 6, while for the second problem we invoke Lemma 7; together with Lemma 1 and $z = z' + z''$, these estimates will yield the theorem's statement.

Let

$$y'_{ij} = \sum_{\nu=1}^{N-1} \sum_{\mu=0}^{j-1} hdG_{ij}^{\nu\mu}\omega_\mu(x_\nu)\tilde{f}.$$

By Lemma 6,

$$|y'_{ij}| \le \sum_{\nu=1}^{N-1} \sum_{\mu=0}^{j-1} hd \left(\sum_{n=1}^{N-1} (1 - \lambda_n \varepsilon \frac{h^2}{4})^{j-\mu-1} \right) \omega_\mu(x_\nu)\tilde{f}$$

$$= \sum_{\mu=0}^{j-1} d \sum_{n=1}^{N-1} (1 - \lambda_n \varepsilon \frac{h^2}{4})^{j-\mu-1} \sum_{\nu=1}^{N-1} h\omega_\mu(x_\nu)\tilde{f}$$

$$\le \sum_{\mu=0}^{j-1} d\tau_k(f(\cdot, t_\mu); ch)_{L_1[0,1]} \sum_{n=1}^{N-1} (1 - \lambda_n \varepsilon \frac{h^2}{4})^{j-\mu-1}$$

$$\le \left(\sum_{\mu=0}^{j-1} d\tau_k(\tilde{f}(\cdot, t_\mu); ch)^p_{L_1[0,1]} \right)^{\frac{1}{p}} \left(\sum_{\mu=0}^{j-1} d \left(\sum_{n=1}^{N-1} (1 - \lambda_n \varepsilon \frac{h^2}{4})^{j-\mu-1} \right)^q \right)^{\frac{1}{q}}$$

$$\leq \|\tau_k(\tilde{f}; ch)_{L_1[0,1]}\|_{A_{p,d}[0,t]} \sum_{n=1}^{N-1} \left(\sum_{\mu=0}^{j-1} d(1-\lambda_n \varepsilon \frac{h^2}{4})^{(j-\mu-1)q} \right)^{\frac{1}{q}}$$

$$= \|\tau_k(\tilde{f}; ch)_{L_1[0,1]}\|_{A_{p,d}[0,t]} \sum_{n=1}^{N-1} \left(\sum_{\mu=0}^{j-1} d(1-\lambda_n \varepsilon \frac{h^2}{4})^{\mu q} \right)^{\frac{1}{q}}$$

$$= \|\tau_k(\tilde{f}; ch)_{L_1[0,1]}\|_{A_{p,d}[0,t]} \sum_{n=1}^{N-1} \left(d \frac{1-(1-\lambda_n \varepsilon \frac{h^2}{4})^{jq}}{1-(1-\lambda_n \varepsilon \frac{h^2}{4})^{q}} \right)^{\frac{1}{q}}.$$

We have

$$\lambda_n \varepsilon \frac{h^2}{4} = \varepsilon \sin^2(\pi n \frac{h}{2}) \in (0, \varepsilon) \subset (0,1), \ n = 1, \ldots, N-1.$$

Therefore,

$$1-(1-\lambda_n \varepsilon \frac{h^2}{4})^{jq} \in (0,1).$$

For the estimate of $(1-(1-\lambda_n \varepsilon \frac{h^2}{4})^{q})^{-1}$ from below we shall use the inequality

$$(1-x)^q \leq 1 - \frac{q}{2^{q-1}} x, \ 0 < x \leq \frac{1}{2}, \ 1 \leq q \leq 2.$$

(Let us prove this inequality. Let

$$\varphi(x) = (1-x)^q - (1 - \frac{qx}{2^{q-1}}).$$

Obviously, $\varphi(0) = 0$. For $x \in [0, \frac{1}{2}]$ we have

$$\varphi'(x) = -q(1-x)^{q-1} + \frac{q}{2^{q-1}} = q((\frac{1}{2})^{q-1} - (1-x)^{q-1}) \leq 0.)$$

For $\varepsilon \in (0, \frac{1}{2}]$, clearly,

$$\lambda_n \varepsilon \frac{h^2}{4} \in (0, \frac{1}{2}), \ n = 1, \ldots, N-1.$$

Therefore,

$$1-(1-\lambda_n \varepsilon \frac{h^2}{4})^{q} \geq 1 - (1 - q2^{1-q} \lambda_n \varepsilon \frac{h^2}{4}) = q2^{1-q} \lambda_n \varepsilon \frac{h^2}{4} = q2^{1-q} \varepsilon \sin^2(\pi n \frac{h}{2}).$$

Clearly, $\frac{\pi n h}{2} \in (0, \frac{\pi}{2})$, therefore, $\sin(\pi n \frac{h}{2}) \geq \frac{\pi}{2} \frac{\pi n h}{2} = nh$. Hence,

$$\sum_{n=1}^{N-1} \left(d \frac{1-(1-\lambda_n \varepsilon \frac{h^2}{4})^{jq}}{1-(1-\lambda_n \varepsilon \frac{h^2}{4})^{q}} \right)^{\frac{1}{q}} \leq \sum_{n=1}^{N-1} \left(d \frac{1}{\varepsilon q 2^{1-q} n^2 h^2} \right)^{\frac{1}{q}}$$

$$= \frac{2^{1-\frac{1}{q}}}{\varepsilon^{\frac{1}{q}} q^{\frac{1}{q}}} \left(\frac{d}{h^2}\right)^{\frac{1}{q}} \sum_{n=1}^{N-1} n^{-\frac{2}{q}}.$$

It remains to note that

$$\sum_{n=1}^{N-1} n^{-\frac{2}{q}} \begin{cases} \leq \zeta(\frac{2}{q}) < \infty, & q \in [1,2) \\ = O(\ln N) = O\left(\ln(\frac{1}{h})\right), & q = p = 2. \end{cases}$$

In conclusion,

$$|y'_{ij}| \leq \frac{2^{1-\frac{1}{q}}}{\varepsilon^{\frac{1}{q}} q^{\frac{1}{q}}} \left(\frac{d}{h^2}\right)^{\frac{1}{q}} \|\tau_k(\tilde{f}; ch)_{L_1[0,1]}\|_{A_{p,d}[0,t]}.$$

Let y''_{ij} be solution to (3") with RHS which, upon applying the triangle inequality, is being majorized by $\omega_\nu(t_\mu)\tilde{f}$. By Lemma 7,

$$|y''_{ij}| \leq |y''_{ij}|_{l^\infty(\Sigma_h)} \leq \frac{1}{2\sqrt{2\varepsilon}} \left(\sum_{\mu=0}^{j-1} d\|\omega_\nu(t_\mu)\tilde{f}\|_{l^2(\Sigma_h)}\right)^{\frac{1}{2}}, \ \varepsilon \in (0,1],$$

under the condition

$$\max\{0, \frac{1}{2} - (1-\varepsilon)\frac{h^2}{4d}\} \leq \sigma \leq 1.$$

Hence,

$$|y''_{ij}| \leq \frac{1}{2\sqrt{2\varepsilon}} \left(\sum_{\mu=0}^{j-1} d \sum_{\nu=1}^{N-1} h\omega_\nu(t_\mu)\tilde{f}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{2\varepsilon}} \left(\sum_{\nu=1}^{N-1} h \sum_{\mu=0}^{j-1} d\omega_\nu(t_\mu)\tilde{f}\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2\sqrt{2\varepsilon}} \left(\sum_{\nu=1}^{N-1} h\tau_k(\tilde{f}(x_\nu,\cdot); cd)^2_{L_2[0,t]}\right)^{\frac{1}{2}}$$

$$\leq \frac{c}{2\sqrt{2\varepsilon}} \left(\sum_{\nu=1}^{N-1} h\tau_k(\tilde{f}(x_\nu,\cdot); d)^2_{L_2[0,t]}\right)^{\frac{1}{2}}$$

$$\leq \frac{c}{2\sqrt{2\varepsilon}} \|\tau_k(\tilde{f}; d)^2_{L_2[0,t]}\|_{A_{2,h}[0,1]}.$$

The estimates for y'_{ij} y''_{ij}, together with the linearity, finally imply the theorem's statement. □

From Lemma 3 we get, analogously to Theorem 1,

Theorem 2. *Assume that:*

1. *the conditions of Theorem 1 hold;*
2. $\sigma = 0,\ d = \frac{h^2}{6}$;
3. $\frac{\partial f}{\partial t},\ \frac{\partial^2 f}{\partial x^2},\ \frac{\partial^2 u}{\partial x^2},\ \frac{\partial^4 u}{\partial x^4},\ \frac{\partial}{\partial t}(\frac{\partial^2 u}{\partial x^2}),\ \frac{\partial^2}{\partial x^2}(\frac{\partial u}{\partial t})$ *exist everywhere on* $[0,1]\times[0,t]$ *and are measurable;*
4. $\frac{\partial^2}{\partial x^2}(\frac{\partial u}{\partial t}),\ \frac{\partial}{\partial t}(\frac{\partial^2 u}{\partial x^2}),\ \frac{\partial f}{\partial t}+\frac{\partial^2 f}{\partial x^2}$ *are continuous on* $[0,1]\times[0,t]$;
5. *in (2)* $\varphi_{ij} = f_{ij} + \frac{h^2}{12}\left(\frac{\partial f}{\partial t}(x_i,t_j) + \frac{\partial^2 f}{\partial x^2}(x_i,t_j)\right)$ *holds.*

Then,

$$\|z\|_{l^\infty(\Sigma^j_{hd})} \le c_1 h^2 \|\tau_2(\frac{\partial^4 u}{\partial x^4};h)_{L_1[0,1]}\|_{A_{p,h^2}[0,t]} + c_2 h^2 \|\tau_1(\frac{\partial^2 u}{\partial t^2};h^2)_{L_2[0,t]}\|_{A_{2,h}[0,1]},$$

where $c_1,\ c_2$ *are being determined as in Theorem 1, up to a multiplicative factor which is an absolute constant.*

4.4. Some Implications

Making use of the properties of the τ-moduli and those of the spaces $A_{p,t}$, we immediately obtain the following sequence of corollaries. First, we consider some implications of Theorem 1.

Corollary 1. *Assume that:*

1. *the conditions of Theorem 1 hold;*
2. $\bigvee_0^1 \frac{\partial^2 u}{\partial x^2}(\cdot,t) \in BM[0,t]$;
3. $\bigvee_0^t{}_2 \frac{\partial u}{\partial t}(x,\cdot) \in BM[0,1]$.

Then, for $d \le Ah^2,\ A>0,\ \exists c>0$:

$$\|z\|_{l^\infty(\Sigma^M_{hd})} \le ch.$$

Corollary 2. *Assume that:*

1. *the conditions of Theorem 1 hold;*

2. $\bigvee_0^1 \frac{\partial^3 u}{\partial x^3}(\cdot, t) \in BM[0,t]$*;*

3. $\|\frac{\partial^2 u}{\partial t^2}(x,\cdot)\|_{L_2[0,t]} \in BM[0,1]$*.*

Then, for $d \le Ah^2,\ A > 0,\ \exists c > 0$*:*

$$\|z\|_{l^\infty(\Sigma_{hd}^M)} \le ch^2.$$

In Section [23] we shall make use of a result which follows from the estimate

$$\begin{aligned} \|z\|_{l^\infty(\Sigma_{hd}^j)} \le \frac{c}{2\sqrt{2\varepsilon}} \Big(& \|\tau_2(\frac{\partial^2 u}{\partial x^2}; h)_{L_2[0,1]}\|_{A_{2,d}[0,t]} \\ & + \|\tau_2(\frac{\partial u}{\partial t}; d)_{L_2[0,t]}\|_{A_{2,h}[0,1]} \\ & + |\frac{1}{2} - \sigma|.\|\tau_1(\frac{\partial u}{\partial t}; d)_{L_2[0,t]}\|_{A_{2,h}[0,1]} \Big), \end{aligned} \tag{5}$$

which is obtained in the same way as Theorem 1, but only with invocation of Lemma 7.

Next, we consider some implications of Theorem 2.

Corollary 3. *Assume that:*

1. *the conditions of Theorem 2 hold;*

2. $\bigvee_0^1 \frac{\partial^4 u}{\partial x^4}(\cdot, t) \in BM[0,t]$*;*

3. $\bigvee_0^t {}_2\, \frac{\partial^2 u}{\partial t^2}(x,\cdot) \in BM[0,1]$*.*

Then, $\exists c > 0$*:*

$$\|z\|_{l^\infty(\Sigma_{hd}^M)} \le ch^3.$$

Corollary 4. *Assume that:*

1. *the conditions of Theorem 2 hold;*

2. $\bigvee_0^1 \frac{\partial^5 u}{\partial x^5}(\cdot, t) \in BM[0,t]$;

3. $\|\frac{\partial^3 u}{\partial t^3}(x,\cdot)\|_{L_2[0,t]} \in BM[0,1]$.

Then, $\exists c > 0$*:*

$$\|z\|_{l^\infty(\Sigma_{hd}^M)} \leq ch^4.$$

5. Concluding Remarks

Remark 3. (Comparison with previous results.)

- In [8, pp. 198-199] the following results are obtained:
  - convergence $O(h)$ is proved in the weaker norm $l^2(\Sigma_h)$ under the additional conditions $\frac{\partial^2 u}{\partial t^2}, \frac{\partial^4 u}{\partial t^4}$ – continuous;
  - convergence $O(h^2)$ is proved in norm $\|\cdot\|_{l^\infty(\Sigma_{hd}^j)}$ under the same additional conditions as in the previous item, but only for $\sigma = \frac{1}{2}$.
- In [5],
  - a bound $\|z\|_{l^\infty(\Sigma_{hd}^j)} \leq ch$ is proved under conditions imposed on the second-order derivatives of the solution;
  - a bound $\|z\|_{l^\infty(\Sigma_{hd}^j)} \leq ch^2$ is proved under conditions on the third-order derivatives.
- In [29] an even more precise estimate is obtained, sharpening the conditions imposed in [5].
- The estimates obtained in the present section essentially sharpen the ones from [29].
- The estimate in (5) is also a sharpening of the ones from [29].

Remark 4. Corollaries 1–4 are only a few of the new implications of Theorems 1 and 2. For example, it is possible

- to obtain fractional error rates in terms of A-spaces and Besov spaces;

- to improve the estimates of the constant factors to the rates, by using norms in the Wiener-amalgam spaces $A_{p,h}$.

The starting point for all of these estimates is Lemma 1. However, we skip here the consideration of these results because our error estimates here are in terms of properties of the (unknown) solution, hence, all estimates considered here are O-estimates without exact knowledge of the constant factors to the rates. In [23, 24], when we address error estimates in terms of the problem's data, we shall study fractional rates and efficient bounds on the constant factors to these rates in considerable detail. (We adopted a similar approach in [20, 21, 22] in the case of ODEs.)

Remark 5. The method developed and the results obtained in this paper (and the next papers [23, 24] on this topic) are an application of the general theory developed in [15, 16].

Remark 6. The material in this paper (and the next papers [23, 24] on this topic) covers the part of the previously unpublished results in [6, Chapter 3] related to PDEs and complementary to the unpublished results in [6, Chapter 3] about ODEs which appear in [20, 21, 22].

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