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ON ERROR ESTIMATION FOR APPROXIMATION METHODS INVOLVING DOMAIN DISCRETIZATION VIII: DETERMINISTIC PROBLEMS VII. FULLY DISCRETE FINITE DIFFERENCE SCHEMES FOR LINEAR PDES II. ESTIMATES IN TERMS OF PROPERTIES OF THE PROBLEM'S DATA FUNCTIONS I: HOMOGENEOUS BOUNDARY CONDITIONS

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Abstract: This is the eighth of a sequence of 12 papers, preceded by [19, 20, 21, 22, 23, 24, 25] and followed by [26, 27, 28, 29] (in this order), dedicated to the study of error estimates for approximation problems based on discretization of the domain of the approximated functions. Within this sequence, in [20] and [21], for a model example of a Cauchy problem for a linear differential equation with variable coefficients, we applied an indirect method for error estimation based on results known in advance for the respective semi-discrete approximating problems.

In a follow-up subsequence of six of these papers, of which this is the fifth one, preceded by [22, 23, 24, 25] and followed by [26] (in this order) we develop a direct discrete method for error estimation based on an *extended Lax principle*, essentially proposed first in [4] but explicitly formulated for the first time in [22, Section 2].

In [22, 23, 24] we applied the proposed method to obtain sharp error estimates for a model boundary problem for a linear ordinary differential equation

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of second order with variable coefficients and right-hand side. This study is being continued in the remaining three papers, [25], the present paper paper and [26] of the subsequence, by an analogous application of the proposed method to obtain sharp error estimates for a model initial-boundary problem for a linear parabolic partial differential equation of second order.

In [25] we discussed the first 4 stages of the extended Lax principle for the model problem in consideration. These stages essentially correspond to the classical Lax principle, but the rather coarse classical error estimates obtained via this principle have been essentially sharpened in [4, 31] using more advanced tools for error estimation, such as integral and averaged moduli of smoothness. In [25] we provided a systematic exposition of the results of [4], sharpened, generalized, and upgraded the results of [31], and complemented them with some additional ones, in order to prepare the error estimates at Stage 4 for use in the derivation of appropriate *a priori* estimates at Stage 5 and the final results of the new method at Stage 6 of the extended Lax principle, related to partial differential equations, in the present paper and the subsequent paper [26].

In the present paper we address two major topics (corresponding to Stages 5 and 6 of the extended Lax principle), as follows.

- Stage 5: We develop *a priori* estimates for the continual problem (1) with *homogeneous initial and boundary conditions* which are *consistent* with the error estimates in terms of properties of the solution from [25].

- Stage 6: Based on the results obtained on Stage 4 in [25] and the results obtained here on Stage 5, we derive sharp error estimates directly in terms of the data of the continual initial-boundary problem (1) for the case of *homogeneous boundary conditions*. (In view of the homogeneity of the initial and boundary conditions and the constance of the linear PDE's coefficients, the data-function here is only the right-hand side.) These new error estimates imply a diversity of corollaries providing certain approximation rates under minimal assumptions about regularity of the data.

In [26] we shall extend the results obtained here for homogeneous initial and boundary conditions to the general case of inhomogeneous initial and/or boundary conditions.

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1. Introduction

This is the eighth of a sequence of 12 papers, preceded by [19, 20, 21, 22, 23, 24, 25] and followed by [26, 27, 28, 29] (in this order), dedicated to the study of error estimates for approximation problems based on *discretization of the domain* of the approximated functions, and (in the concluding paper [29]) comparison of the similarities and differences with the error estimates derived by alternative approximation methods (typically, of projection type) based on finite-dimensional subspaces of functions having the same *continual domain* as the target function.

Within this sequence, in [20] and [21], for model examples of Cauchy problems, we applied an indirect method for error estimation based on results known in advance for the respective semi-discrete approximating problems.

In a sequence of six papers, of which this is the fifth one, preceded by [22, 23, 24, 25] and followed by [26] (in this order), we develop a direct discrete method for error estimation based on an *extended Lax principle*, essentially proposed first in [4] but explicitly formulated for the first time in [22, Section 2].

In [22, 23, 24, 25], here and in the next paper [26] we apply the proposed method to obtain sharp error estimates for the following two model initial-boundary problems (one for an ordinary differential equation (ODE), and one for a partial differential equation (PDE)):

1. for a model linear ODE with variable coefficients and right-hand side

(RHS) (see [22, 23, 24]);

2. for a model linear parabolic PDE with constant coefficients and a variable RHS (see [25], the present paper, and [26]).

In [25] we discussed the first 4 stages of the extended Lax principle for the model problem in consideration. These stages essentially correspond to the classical Lax principle, but the rather coarse classical error estimates obtained via this principle have been essentially sharpened in [4, 31] using more advanced tools for error estimation, such as integral and averaged moduli of smoothness. In [25] we provided a systematic exposition of the results of [4], sharpened, generalized, and upgraded the results of [31], and complemented them with some additional ones, in order to prepare the error estimates at Stage 4 for use in the derivation of appropriate *a priori* estimates at Stage 5 and the final results of the new method at Stage 6 of the extended Lax principle, related to partial differential equations, in the present paper and the subsequent paper [26].

In the present paper we address two major topics (corresponding to Stages 5 and 6 of the extended Lax principle), as follows.

- Stage 5: We develop *a priori* estimates for the continual problem (1) which are *consistent* with the error estimates in terms of properties of the solution from [25].

- Stage 6: Based on the results obtained on Stage 4 in [25] and the results obtained here on Stage 5, we derive sharp error estimates directly in terms of the data of the continual initial-boundary problem (1) for the case of *homogeneous boundary conditions.* (Because of the homogeneity of the initial and boundary conditions and the constance of the linear PDE's coefficients, the data-function here is only the right-hand side (RHS).) These new error estimates imply a diversity of corollaries providing certain approximation rates under minimal assumptions about regularity of the data.

In [26] we shall extend the results obtained here for homogeneous initial and boundary conditions to the general case of inhomogeneous initial and/or boundary conditions.

2. A Fully Discrete Method for Error Estimation

The *fully discrete method for error estimation directly in terms of properties of the problem's data*, proposed in [4, Chapter 3], is comprised of a sequence of stages, the entirety of which we chose in [22] to term as *an* ***extended*** *Lax principle*. In brief, the main stages of the proposed method are, as follows.

A. Classical Lax principle

- Stage 1. Derivation of estimates for the *local approximation error of the residual*.
- Stage 2. Derivation of a discrete approximating problem whose solution is *the error on the nodes of the mesh* (termed *discrete approximating problem for the error*).
- Stage 3. Derivation of *a priori estimates* for the solution of the discrete approximating problem for the error.
- Stage 4. Combining the results obtained at Stages 1 and 3 to derive *error estimates in terms of properties of the solution* (of *the exact target problem*).

B. Extension of the Lax principle

- Stage 5. Derivation of ***consistent*** *a priori* estimates for the solution of the exact target problem.
- Stage 6. Combining the results obtained at Stages 4 and 5 to derive *error estimates directly in terms of properties of the data functions and/or the data scalar parameters* (of the exact target problem).

For more details, see [22, Section 2].

Within the above framework, the organization of the exposition in the current sequence of relevant papers is, as follows.

- Part A.
  - ODEs: [22]
  - PDEs: [25]
- Part B.
  - ODEs:

  * Homogeneous boundary conditions: [23]
  * Inhomogeneous boundary conditions: [24]

- PDEs:

  * Homogeneous boundary conditions: the present paper
  * Inhomogeneous boundary conditions: [26]

2.1. Discrete Approximating Problems for PDEs

Consider the Dirichlet boundary-value (Cauchy–Dirichlet initial-boundary) problem for the heat equation:

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t), \quad x \in [0,1],\ t \in (0,T],\ T > 0,\\
&u(x,0) = g(x), \quad x \in [0,1],\\
&u(j,t) = g_j(t), \quad t \in [0,T], \quad j = 0,1,\\
&g(\cdot) \in BM[0,1], g_j(\cdot) \in BM[0,T], \quad j = 0,1,\\
&f - \text{defined and bounded everywhere on } \Omega = [0,1] \times [0,T] \text{ and measurable.}
\end{aligned} \tag{1}$$

(Measurability is with respect to the customary one- and two-dimensional Lebesgue measures for $[0,1]$, $[0,T]$ and Ω, respectively.) For the definition of the space BM, see Section 3, item 1.

We assume that a "conjugation condition" is also fulfilled: u – continuous on the interior of Ω (cf. [7, p. 71]).

Approximate (1) by the following homogeneous conservative finite difference scheme (see [7, pp. 110-119, 185-192]):

$$\begin{aligned}
&\frac{v_{i,j+1}-v_{ij}}{d} - \left(\sigma \frac{v_{i+1,j+1}-2v_{i,j+1}+v_{i-1,j+1}}{h^2} + (1-\sigma)\frac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{h^2}\right) = \varphi_{ij},\\
&\qquad x \in \mathring{\Sigma}_h = \{\xi_i : \xi_i = ih,\ i = 1,\dots,N-1;\ Nh = 1\},\\
&\qquad t \in \mathring{\Sigma}_d = \{\eta_j : \eta_j = jd,\ j = 0,1,\dots,M-1;\ Md = T\},\\
&v_{i0} = g_i,\ x_i \in \{ih : i = 0,1,\dots,N\} = \Sigma_h,\\
&v_{0j} = (g_0)_j,\ v_{1j} = (g_1)_j,\ j = 0,1,\dots,M,\ t_j \in \{jd : j = 0,1,\dots,M\} = \Sigma_d,
\end{aligned} \tag{2}$$

The error in quest $z_{ij} = v_{ij} - u_{ij}$ in (1, 2) is uniquely determined as the solution of the following finite difference problem (cf. [7, p. 189]):

$$\frac{z_{i,j+1}-z_{ij}}{d} - \left(\sigma \frac{z_{i+1,j+1}-2z_{i,j+1}+z_{i-1,j+1}}{h^2} + (1-\sigma)\frac{z_{i+1,j}-2z_{i,j}+z_{i-1,j}}{h^2}\right) = \psi_{ij},$$
$$i = 1, 2, \ldots, N-1;\ j = 0, 1, \ldots, M-1 \tag{3}$$
$$z_{i0} = 0,\ i = 0, 1, \ldots, N$$
$$z_{0j} = z_{1j} = 0,\ j = 0, 1, \ldots, M.$$

(For simplicity of exposition, the approximation on the boundary is assumed to be exact.)

Estimation of the RHS $\psi = \{\psi_{ij}\}$ in (3) yields the convergence rate of the local approximation of the residual and the conditions for this rate to be attained; estimation of the solution $z = \{z_{ij}\}$ of (3) yields the rate and conditions for convergence to zero of the error of the considered finite difference method. The function ψ is defined only for $i = 1, 2, \ldots, N-1$. We extend its definition to $i = 0, N$, as follows: $\psi_0 = \psi_N = 0$.

The function φ_{ij} in (2) is an approximation of f_{ij}. More precisely, in view of the homogeneity of the finite difference scheme (see [7, pp. 116-119, 187])

$$\varphi_{ij} = \mathrm{F}[f(x+sh, t_{j+\frac{1}{2}})], \tag{4}$$

where the template functional F is the same as in [22, 23, 24]: it is defined for $\bar{f}(s) \in BM\left[-\frac{1}{2}, \frac{1}{2}\right]$ and has the following properties

1. $\mathrm{F}[1] = 1$;
2. F is linear over $BM\left[-\frac{1}{2}, \frac{1}{2}\right]$;
3. $\mathrm{F}[\bar{f}(s)] \geq 0$ for $\bar{f}(s) \geq 0,\ s \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

For conciseness of the presentation, we introduce the following additional notation:

- all functions $a(x,t)$ are considered defined in $[0,1] \times [0,T]$,
  - (i) $a(x)$ – in $[0,1]$,
  - (ii) $a(t)$ – in $[0,T]$;
- $\Sigma^j_{hd} = \Sigma_h \times \{\mu d : \mu = 0, 1, \ldots, j\}, \quad \Sigma^M_{hd} = \Sigma_{hd}$;
- $a_{ij} = a(x_i, t_j), \quad (x_i, t_j) \in \Sigma_{hd}$;
- $a_{\bar{x},ij} = \frac{a_{ij}-a_{i-1j}}{h}, \quad a_{x,ij} = \frac{a_{i+1j}-a_{ij}}{h}$;

- for mesh functions φ and ψ, $\varphi = \psi$ denotes $\varphi_{ij} = \psi_{ij}, \forall (x_i, t_j) \in \Sigma_{hd}$;

- $[\varphi_j, \psi_j] = \sum_{i=0}^{N} h\varphi_{ij}\psi_{ij}$;

- $[[\varphi, \psi]]_{hd} = \sum_{i=0}^{N}\sum_{j=0}^{M} hd\varphi_{ij}\psi_{ij}$;

- $\Lambda_h = \{\varphi : \mathrm{Dom}\varphi = \Sigma_h, \mathrm{Cod}\varphi \subset \mathbb{R}\}$;

- $\mathring{\Lambda}_h = \{\varphi : \varphi \in \Lambda_h, \varphi(0) = \varphi(1) = 0\}$;

- $\omega_\mu(x_\nu)f = \omega_k(f(\cdot, t_\mu + \beta d), x_\nu + \alpha h; ch)$,

  $\omega_\nu(t_\mu)f = \omega_k(f(x + \alpha h, \cdot), t_\mu + \beta d, cd)$,

  where f is the function in consideration, $k \in \mathbb{N}$, $c > 0$, $\alpha, \beta \in \mathbb{R}$. Obviously, $\omega_\mu(x_\nu)$, $\omega_\nu(t_\mu)$ are semi-norms.

3. Preliminaries: Functional Moduli and Function Spaces

Here we present the notation for the functional characteristics needed in the sequel. As a general reference about the functional characteristics introduced in this section concerning the notation, definitions, meaning and range of parameters, relevant properties and interrelations, as well as further details, we cite [17] together with [18], and the references therein. Another concise reference source on the material in this section is [19, Section 2] and the references therein.

1. The function space $C(\Omega) \subset BM(\Omega)$ of all (real-valued) continuous functions on Ω, where the notation for Ω and $BM(\Omega)$ was introduced in Section 2.1. The restriction of the norm $\|\cdot\|_{BM}$ on C will be denoted, as usual, by $\|\cdot\|_C$; recall that with respect to this norm C is a Banach space which is a closed subspace of the Banach space BM (see, e.g., [3]).

2. For a function $f \in BM(\Omega)$,
   $$S(t, f; x) = sup\,\{f(y) : y \in [x - t, x + t] \cap \Omega\}$$
   is the upper Baire's function of f at $x \in \Omega$ with step $t > 0$ (see [17] and the references therein).

3. The local modulus of smoothness of order $k \in \mathbb{N}$ at $x \in \Omega$ (Ω as in the definition of the upper Baire's function) with $t > 0$
$$\omega_k(f,x;t) = \sup\left\{\|\Delta_h^k f(y)\| : y, y+kh \in \left[x - \frac{kt}{2}, x + \frac{kt}{2}\right] \cap \Omega\right\}$$
(see [17] and the references therein); it is natural to define also $\omega_0(f,x;t) := S(t,f;x)$.

4. The sequence space $l^p(\Sigma_h)$ defined over the mesh Σ_h (see [32, 12, 13, 14, 4, 17]).

5. The inhomogeneous Sobolev space $W_p^m(\mathbb{R})$, with $W_p^0 = L_p$ (see, e.g., [1, 5, 17]).

6. The integral modulus of smoothness (ω-modulus) $\omega_m(f;h)_{L_p}$ (short: $\omega_m(f;h)_p$) (see [32, 12, 13, 14, 9, 4, 17], cf. also [5, 1] where the notation is modified).

7. The averaged modulus of smoothness (τ-modulus) $\tau_m(f;h)_{L_p}$ (short: $\tau_m(f;h)_p$) (see [9, 32, 12, 13, 14, 15, 4, 17]).

8. The Steklov-means $f_{k,t}$ (see [2, 8, 9, 17]). The notation $f_{k,t}$ is collective for the classical one-sided Steklov-means [9] (appropriate to use on a ray or the whole real axis), the two-sided Steklov-means in the sense of Brudnyi [2] and the warped Steklov-means in the sense of Sendov [8] (both appropriate to use on a finite interval/segment). In [4, 17, 18, 12, 13, 14, 15, 32] and in the present sequence consisting of [19]–[25], this paper, and [26]–[29] we are always using either the one-sided or the two-sided version, depending on the context. For finite interval/segment we give preference to the two-sided over the warped version because, when measured in L_p-norm, the derivatives of the warped version generate estimates in terms of averaged moduli, while the same derivatives of the two-sided version generate smaller estimates in terms of the smaller respective integral moduli.

9. The Wiener amalgam space $A_{p,h}(\mathbb{R})$, with norm $\|.\|_{A_{p,h}} = \tau_0(f;h)_{L_p}$ (see [14, 15, 4, 17, 18]).

10. The inhomogeneous Besov space $B_{pq}^s(\mathbb{R})$ (see [1, 32, 12, 13, 14, 15, 4, 17]).

11. The inhomogeneous A-space $A_{pq}^s(\mathbb{R})$, an analogue of the respective Besov space where the ω-modulus in the definition of the norm in B_{pq}^s is replaced

by the respective τ-modulus (with the same parameters) in the definition of the norm in A^s_{pq} (see [32, 12, 13, 14, 15, 4, 17]).

12. The inhomogeneous Triebel–Lizorkin space $F^s_{pq}(\mathbb{R})$ (see [1, 4, 17]).

13. The Wiener–Young p-variation $\bigvee_{-\infty}^{\infty}{}_p\, g$ of g, the case $p = 1$ corresponding to the customary Jordan variation, and the case $p = 2$ corresponding to the quadratic variation in the sense of Wiener (see [34, 35]).

[19, Section 2.3] contains an important comparison between the properties of the integral and averaged moduli of smoothness.

Following our practice in [4] and [17], in order to distinguish between previously known results and the new ones obtained here, we shall add the additional marker '**(K.)**' (abbreviated from '(K)nown') to the enumeration of every statement in the sequel which has been previously known (with respective reference to available relevant literature).

4. Heat Conductivity PDE with Constant Coefficients and Variable Right-Hand Side

4.1. Error Estimates in Terms of Properties of the Solution

The previous paper in this sequence, [25], was dedicated to the derivation of sharp error estimates in terms of properties of the solution of the continual problem (1), i.e., to Stages 1–4 of the extended Lax principle. It contained the main relevant results [25, Theorem 1] for the general formulation of the discrete approximating problem (2), and [25, Theorem 2] for the special case of the explicit finite difference scheme, corresponding to $\sigma = 0$ and $d = h^2/6$ in (2), when the order of accuracy of the discrete approximating problem increases from 2 to 4.

In [25, Section 4.4.] we also mentioned that for the purposes of the upgrading the error estimates in terms of properties of the solution to error estimates in terms of properties of the problem's data (i.e., upgrading Stage 4 via Stage 5 to Stage 6), in the place of the estimate in [25, Theorem 1] we shall be using the modified estimate [25, inequality (5)]. For the reader's convenience, we present explicitly this estimate here as a modification of [25, Theorem 1], as follows.

In this section we shall obtain the final estimate of the error z in (3) in norm

$$\|\cdot\|_{l^\infty(\Sigma^j_{hd})} : \|z\|_{l^\infty(\Sigma^j_{hd})} = \max\{|z_{\nu\mu}| : \nu = 0, \ldots, N;\ \mu = 0, \ldots, j\}$$

in terms of properties of derivatives of the solution.

Theorem 1. **(K.)** *(Cf. [25, inequality (5)] and the relevant comments therein.) Let*

1. *$j \in \mathbb{N},\ 0 \le j \le M,\ t = jd$;*
2. *$\varepsilon \in (0, 1]$.*

Then,

$$\begin{aligned} \|z\|_{l^\infty(\Sigma^j_{hd})} \le \tfrac{c}{2\sqrt{2\varepsilon}} \Big(& \|\tau_2(\frac{\partial^2 u}{\partial x^2}; h)_{L_2[0,1]}\|_{A_{2,d}[0,t]} \\ & + \|\tau_2(\frac{\partial u}{\partial t}; d)_{L_2[0,t]}\|_{A_{2,h}[0,1]} \\ & + |\frac{1}{2} - \sigma|.\|\tau_1(\frac{\partial u}{\partial t}; d)_{L_2[0,t]}\|_{A_{2,h}[0,1]} \Big), \end{aligned} \tag{5}$$

where:

(i) c is an absolute constant;

(ii) it holds true that

$$\max\left\{0, \frac{1}{2} - (1 - \varepsilon)\frac{h^2}{4d}\right\} \le \sigma \le 1.$$

Proof. (Outline) In the proof of [25, Theorem 1] two distinct *a priori* estimates were used, namely, the ones in [25, Lemma 6] and [25, Lemma 7], respectively. According to what we mentioned in [25, Section 4.4.] concerning [25, inequality (5)], Theorem 1 as formulated here is a modification of [25, Theorem 1] which is being proved in the same way as [25, Theorem 1], but with invocation of [25, Lemma 7] also in those places in the proof of [25, Theorem 1] where [25, Lemma 6] has been used. □

Remark 1. (See also Remarks 3, 4, 6, 12.) In the special case of higher order of accuracy in [25, Theorem 2], it is possible to also formulate a modification where, similar to Theorem 1 here, [25, Lemma 7] is being used throughout, including the places where in the proof of [25, Theorem 2] the *a priori* estimate from [25, Lemma 6] is being used.

4.2. Multi-Dimensional Moduli. A Priori Estimates for the Continual Problem

The purpose of the present section is to derive *a priori* estimates for the continual problem (1) needed for Stage 5 of the extended Lax principle. To this end, here we shall consider the respective properties of the continual Green's function $G = G(x, \xi);\ x, \xi \in [0, 1]$ for problem (1). For conciseness of the exposition, we shall consider the case of homogeneous initial and boundary conditions in (1). In the general case of inhomogeneous boundary conditions it is possible to apply analogous arguments (cf. [23, Remarks 2 and 6] and [24] for a similar situation in the case of ODE), however, the exposition becomes considerably more spacious and technically involved. (See also Remark 12 below.)

4.2.1. Multidimensional Moduli

For the needs of the proof of the main result in Section 4.3, it is necessary to consider only one of the possible multi-dimensional extensions of the moduli of smoothness. Here we shall limit the considerations to only the necessary minimum of information about these moduli. For our purposes, it will be enough to consider only the two-dimensional case.

Let $\Omega = [0, 1] \times [0, T],\ T > 0,\ \Omega_t = [0, 1] \times [0, t],\ 0 < t \le T$. Everywhere in this section Ω can be replaced with a variable $\Omega_t,\ 0 < t \le T$.

Definition 1. Let $x, t \in \mathbb{R};\ h, d > 0;\ m, n \in \mathbb{N} \cup \{0\}$.

$$\begin{aligned}\Delta^{m,n}_{h,d} f(x,t) &= \Delta^m_h \left(\Delta^n_d f(x,t)\right) = \Delta^n_d \left(\Delta^m_h f(x,t)\right) \\ &= \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} (-1)^{m+n+\mu+\nu} \binom{m}{\mu} \binom{n}{\nu} f(x + \mu h, t + \nu d).\end{aligned}$$

Definition 2. (Integral modulus of smoothness.) Let $m,\ n,\ h,\ d$ be as in Definition 1. The integral modulus of smoothness of order (m, n), with step (h, d), in metric $L_p,\ 1 \le p \le \infty$, has the form

$$\begin{aligned}&\omega_{(m,n)}(f; (h,d))_{L_p(\Omega)} \\ &\quad = \sup \left\{ \left(\int\limits_{\Omega^{m,n}_{\eta,\xi}} |\Delta^{m,n}_{\eta,\xi} f(x,t)|^p dx dt \right)^{\frac{1}{p}}, |\eta| \le h, |\xi| \le d \right\},\ 1 \le p \le \infty.\end{aligned}$$

(for $p = \infty$ the integral is being replaced by $\sup\limits_{\Omega^{m,n}_{\eta,\xi}}$), where

$$\Omega^{m,n}_{\eta,\xi} = \left\{(x,t) \in \mathbb{R}^2 : (x,t),\ (x,t+nd),\ (x+mh,t),\ (x+mh,t+nd) \in \Omega\right\}$$
$$(\Omega - \text{convex!}).$$

We shall make use of the following intermediate approximation

$$f_{m,h;n,d}(x,t) = (f_{m,h})_{n,d}(x,t) = (f_{n,d})_{m,h}(x,t), \quad (x,t) \in \Omega.$$

The definition of the univariate Steklov mean for a finite interval (see Section 3) and the measurability, hence, the local summability of f, imply the applicability of Fubini's theorem, which in its turn implies the commutation of $(\cdot)_{m,h}$ and $(\cdot)_{n,d}$.

Lemma 1. *Let*

1. $f \in L_p(\Omega),\ 1 \le p < \infty$;
2. $f \in BM(\Omega),\ p = \infty$ *(with an obvious modification of the definition of* BM *for two-dimensional* Ω*);*
3. $m, n \in \mathbb{N} \cup \{0\};\ \mu = 0, 1, \ldots, m;\ \nu = 0, 1, \ldots, n;$
4. $0 < h \le \frac{1}{m},\ 0 < d \le \frac{T}{n}$.

Then,

$$\left\|\frac{\partial^{\mu+\nu}}{\partial x^\mu t^\nu} f_{m,h;n,d}\right\|_{L_p(\Omega_t)} \le \frac{c}{h^\mu d^\nu}\omega_{(\mu,\nu)}(f;(h,d))_{L_p(\Omega_t)}, \quad c = c(\mu,\nu,m,n).$$

Proof. The proof is analogous to the univariate case (cf. [9, pp. 47-48]). ☐

Remark 2. Let us note also that, as easily seen,

$$\omega_{(m,0)}(f;(h,d))_{L_p(\Omega_t)} \le \|\omega_m(f;h)_{L_p[0,1]}\|_{L_p[0,t]}$$
$$\omega_{(0,n)}(f;(h,d))_{L_p(\Omega_t)} \le \|\omega_n(f;h)_{L_p[0,t]}\|_{L_p[0,1]}.$$

Lemma 2.

I. *Assume that the conditions of Lemma 1 hold.*

Then,

$$\|f - f_{m,h;n,d}\|_{L_p(\Omega_t)}$$
$$\le c\left(\omega_{(m,0)}(f;(h,d))_{L_p(\Omega_t)} + \omega_{(0,n)}(f;(h,d))_{L_p(\Omega_t)}\right).$$

II. Let also $f(\cdot,t) \in BM[0,1]$, $\|f\|_{A_{p,h}[0,1]} \in BM[0,1]$.

Then, also

$$\| \, \|f - f_{m,h;n,d}\|_{A_{p,h}[0,1]} \|_{A_{p,h}[0,t]} \leq c \left(\|\tau_m(f;h)_{L_p[0,1]}\|_{A_{p,d}[0,t]} + \|\tau_n(f;d)_{L_p[0,t]}\|_{A_{p,h}[0,1]} \right),$$

holds true, with $c = c(m,n)$.

Proof. First, we prove the inequality

$$\|f - f_{m,h;n,d}\|_{L_p(\Omega_t)} \leq$$
$$\leq \|f - f_{m,h}\|_{L_p(\Omega_t)} + \|f_{m,h} - (f_{n,d})_{m,h}\|_{L_p(\Omega_t)} \leq$$
$$\leq \omega_{(m,0)}(f;(h,d))_{L_p(\Omega_t)} + \|(f - f_{n,d})_{m,h}\|_{L_p(\Omega_t)} \leq$$
$$\leq \omega_{(m,0)}(f;(h,d))_{L_p(\Omega_t)} + \|(f - f_{n,d})_{m,h} - (f - f_{n,d})\|_{L_p(\Omega_t)} + \|f - f_{n,d}\|_{L_p(\Omega_t)} \leq$$
$$\leq \omega_{(m,0)}(f;(h,d))_{L_p(\Omega_t)} + \omega_{(m,0)}(f - f_{n,d};(h,d))_{L_p(\Omega_t)} + \omega_{(0,n)}(f;(h,d))_{L_p(\Omega_t)} \leq$$
$$\leq \omega_{(m,0)}(f;(h,d))_{L_p(\Omega_t)} + c\|f - f_{n,d}\|_{L_p(\Omega_t)} + \omega_{(0,n)}(f;(h,d))_{L_p(\Omega_t)} \leq$$
$$\leq c \left(\omega_{(m,0)}(f;(h,d))_{L_p(\Omega_t)} + \omega_{(0,n)}(f;(h,d))_{L_p(\Omega_t)} \right).$$

The second inequality is considerably harder to prove. The case $p = \infty$ is analogous to, but simpler than, the case $1 \leq p < \infty$. Let us consider in detail the case $1 \leq p < \infty$.

$$\| \, \|f - f_{m,h;n,d}\|_{A_{p,h}[0,1]} \|_{A_{p,h}[0,t]} \leq$$
$$\leq \| \, \|f - f_{m,h}\|_{A_{p,h}[0,1]} \|_{A_{p,h}[0,t]} + \| \, \|f_{m,h} - (f_{m,h})_{n,d}\|_{A_{p,h}[0,1]} \|_{A_{p,h}[0,t]} \leq$$
$$\leq \|\tau_m(f;h)_{L_p[0,1]}\|_{A_{p,d}[0,t]} + \| \, \|(f - f_{n,d})_{m,h}\|_{A_{p,h}[0,1]} \|_{A_{p,h}[0,t]}.$$

Let us estimate the second summand. The properties of the upper Baire's function and Fubini's theorem easily imply

$$\| \, \|\varphi\|_{L_p[0,1]} \|_{A_{p,d}[0,t]} \leq \| \, \|\varphi\|_{A_{p,d}[0,t]} \|_{L_p[0,1]}.$$

[17, Lemma 8] and the properties of the τ-moduli imply

$$\| \, \|(f - f_{n,d})_{m,h}\|_{A_{p,h}[0,1]} \|_{A_{p,h}[0,t]}$$
$$\leq c \left(\| \, \|(f - f_{n,d})_{m,h}\|_{L_p[0,1]} \|_{A_{p,h}[0,t]} + h \| \, \|\frac{\partial}{\partial x}(f - f_{n,d})_{m,h}\|_{L_p[0,1]} \|_{A_{p,h}[0,t]} \right).$$

From the definition of the univariate Steklov mean, for the first summand we get, upon applying the generalized Minkowski inequality, for $F_{n,d} = f - f_{n,d}$,

$$\| \|(f - f_{n,d})_{m,h}\|_{L_p[0,1]}\|_{A_{p,h}[0,t]} \leq$$

$$\leq \frac{c}{h^m} \int_0^h \cdots \int_0^h \sum_{k=0}^{m-1} \binom{m}{k} \cdot \Bigg(\| \|F_{n,d}(x + \frac{m-k}{m} \sum_{\mu=1}^{m} \xi_\mu, t)\|_{L_p[0,\frac{2}{3}]}\|_{A_{p,d}[0,t]}$$

$$+ \| \|F_{n,d}(x - \frac{m-k}{m} \sum_{\mu=1}^{m} \xi_\mu, t)\|_{L_p[\frac{1}{3},1]}\|_{A_{p,d}[0,t]} \Bigg) d\xi_1 \ldots d\xi_m \leq$$

$$\leq \frac{c}{h^m} \int_0^h \cdots \int_0^h \sum_{k=0}^{m-1} \binom{m}{k} \cdot \Bigg(\| \|F_{n,d}(x + \frac{m-k}{m} \sum_{\mu=1}^{m} \xi_\mu, t)\|_{A_{p,d}[0,t]}\|_{L_p[0,\frac{2}{3}]}$$

$$+ \| \|F_{n,d}(x - \frac{m-k}{m} \sum_{\mu=1}^{m} \xi_\mu, t)\|_{A_{p,d}[0,t]}\|_{L_p[\frac{1}{3},1]} \Bigg) d\xi_1 \ldots d\xi_m \leq$$

$$\leq \frac{c}{h^m} \int_0^h \cdots \int_0^h \sum_{k=0}^{m-1} \binom{m}{k} \cdot \Bigg(\|\tau_n(f(x + \frac{m-k}{m} \sum_{\mu=1}^{m} \xi_\mu, \cdot); d)_{L_p[0,t]}\|_{L_p[0,\frac{2}{3}]}$$

$$+ \|\tau_n(f(x - \frac{m-k}{m} \sum_{\mu=1}^{m} \xi_\mu, \cdot); d)_{L_p[0,t]}\|_{L_p[\frac{1}{3},1]} \Bigg) d\xi_1 \ldots d\xi_m \leq$$

$$\leq c\|\tau_n(f; d)_{L_p[0,t]}\|_{A_{p,h}[0,1]}.$$

For the second summand we get analogously[1]

$$h\| \|\frac{\partial}{\partial x}(f - f_{n,d})_{m,h}\|_{L_p[0,1]}\|_{A_{p,h}[0,t]} \leq$$

$$\leq \frac{1}{h^{m-1}} \int_0^h \cdots \int_0^h \sum_{k=0}^{m-1} \binom{m}{k} \frac{m}{m-k}$$

[1] Here one more summand has to be added, but it is being estimated in the same way as the first summand.

$$\cdot \Bigg(\| \, \|\Delta^1_{\frac{m}{m-k}h} F_{n,d}(x + \frac{m-k}{m} \sum_{\mu=1}^{m-1} \xi_k, t)\|_{L_p[0,\frac{2}{3}]} \|_{A_{p,d}[0,t]} +$$

$$+ \| \, \|\Delta^1_{-\frac{m}{m-k}h} F_{n,d}(x - \frac{m-k}{m} \sum_{\mu=1}^{m-1} \xi_k, t)\|_{L_p[\frac{1}{3},1]} \|_{A_{p,d}[0,t]} \Bigg) \, d\xi_1 \ldots d\xi_{m-1} \leq$$

$$\leq \frac{1}{h^{m-1}} \int_0^h \cdots \int_0^h \sum_{k=0}^{m-1} \binom{m}{k} \frac{m}{m-k}$$

$$\cdot \Bigg(\| \, \|F_{n,d}(x + \frac{m-k}{m}(h + \sum_{\mu=1}^{m-1} \xi_k), t)\|_{L_p[0,\frac{2}{3}]} \|_{A_{p,d}[0,t]} +$$

$$+ \| \, \|F_{n,d}(x + \frac{m-k}{m} \sum_{\mu=1}^{m-1} \xi_k, t)\|_{L_p[0,\frac{2}{3}]} \|_{A_{p,d}[0,t]} +$$

$$+ \| \, \|F_{n,d}(x - \frac{m-k}{m}(h + \sum_{\mu=1}^{m-1} \xi_k), t)\|_{L_p[0,\frac{2}{3}]} \|_{A_{p,d}[0,t]} +$$

$$[9pt] \qquad + \| \, \|F_{n,d}(x - \frac{m-k}{m} \sum_{\mu=1}^{m-1} \xi_k, t)\|_{L_p[0,\frac{2}{3}]} \|_{A_{p,d}[0,t]} \Bigg) \, d\xi_1 \ldots d\xi_{m-1} \leq$$

$$\leq c \|\tau_n(f; d)_{L_p[0,t]} \|_{A_{p,h}[0,1]},$$

where the last inequality is being proved in the same way as with the first summand. □

From the properties of afore-defined multi-dimensional moduli we shall need only the following obvious analogues of the univariate case:

$$\omega_{(m,n)}(f; (h,d))_{L_p(\Omega)} \leq c(m,n,\mu,nu))\omega_{(\mu,\nu)}(f; (h,d))_{L_p(\Omega)},$$
$$0 \leq \mu \leq m, \ 0 \leq \nu \leq n; m,n,\mu,\nu \in \mathbb{N} \cup \{0\}. \tag{6}$$

We now turn to the question about the *a priori* estimates for the solution of (1) to be used in the proof of the main result in Section 4.3.

Definition 3. (K.) The space $C_0^\infty(\Omega)$ is defined, as follows:

$$C_0^\infty(\Omega) = \{f : \mathrm{Dom} f = \Omega, \ \mathrm{Cod} f \subset \mathbb{R}, \ f - \text{infinitely smooth},$$
$$\exists K - \text{open set} : \sup f \subset K \subset \Omega\}.$$

Lemma 3. **(K.)** *(See [30, p. 120], [11, p. 54].)*
$C_0^\infty(\tilde{\Omega})$ *is dense on* $W_p^r(\tilde{\Omega}),\ 1 \le p < \infty,\ \forall r \in \mathbb{N} \cup \{0\}$,
where

$$\begin{aligned} W_p^r(\tilde{\Omega}) = \{ \quad & f : f \in L_p(\tilde{\Omega}),\ \tfrac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta} f \in L_p(\Omega), \\ & (\alpha,\beta) \in I = \{(\mu,\nu) : \mu,\nu \in \mathbb{N} \cup \{0\},\ \mu+\nu \le r\}, \\ & \|f\|_{W_p^r(\tilde{\Omega})} = \sum_{(\alpha,\beta)\in I} \|\tfrac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\|_{L_p(\tilde{\Omega})} < \infty \ \}, \end{aligned}$$

and where $\tilde{\Omega} \subset \mathbb{R}^2$ *is a domain with Lipschitz boundary.*

Lemma 4. *Let*

1. $f(\cdot,t) \in BM[0,1],\ \forall t \in [0,T]$;
2. $\|f\|_{A_{p,h}[0,1]} \in BM[0,t],\ \forall t \in [0,T]$;
3. $1 \le p \le \infty$;
4. $r \in \mathbb{N} \cup \{0\}$;
5. $0 < h \le \frac{1}{r},\ 0 < d \le \frac{T}{r}$.

Then,

$$\forall \varepsilon > 0\ \exists \varphi = \varphi(\varepsilon, f) \in C_0^\infty(\Omega),$$

such that, simultaneously,

$$\|\frac{\partial^{\mu+\nu}\varphi}{\partial x^\mu t^\nu}\|_{L_p} \le c_{\mu\nu} h^{-\mu} d^{-\nu} \omega_{(\mu,\nu)}(f;(h,d))_{L_p} + \varepsilon, \tag{7}$$

and

$$\begin{aligned} &\|\ \|f-\varphi\|_{A_{p,h}[0,1]}\|_{A_{p,d}[0,t]} \\ &\qquad \le c(r)\left(\|\tau_r(f;h)_{L_p[0,1]}\|_{A_{p,d}[0,t]} + \|\tau_r(f;d)_{L_p[0,t]}\|_{A_{p,h}[0,1]} + \varepsilon\right), \end{aligned} \tag{8}$$

are fulfilled, where $c_{\mu\nu},\ \mu = 0,1,\dots;\ \nu = 0,1,\dots;\ \mu+\nu \le r$, *depend only on* μ,ν,r.

Proof. Let us show (7).
Remark 9 $\Rightarrow f_{r,h;r,d} \in W_p^r(\Omega)$, because, clearly,

$$f_{r,h;r,d},\ \frac{\partial^r f_{r,h;r,d}}{\partial x^r},\ \frac{\partial^r f_{r,h;r,d}}{\partial t^r} \in L_p(\Omega).$$

Lemma 3 $\Rightarrow \forall\varepsilon>0\ \exists\varphi=\varphi_\varepsilon\in C_0^\infty(\Omega)$:
$\|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial t^\beta}(f_{m,h;n,d}-\varphi)\|_{L_p(\Omega)}\le\|f_{r,h;r,d}-\varphi\|_{W_p^r(\Omega)}\le\varepsilon\quad\alpha,\beta=0,1,\dots;\alpha+\beta\le r.$
Therefore, Lemma 1 $\Rightarrow$

$$\begin{aligned}\|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial t^\beta}\varphi\|_{L_p(\Omega)}\le&\\ &\le\|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial t^\beta}(f_{r,h;r,f}-\varphi)\|_{L_p(\Omega)}+\|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial t^\beta}f_{r,h;r,f}\|_{L_p(\Omega)}\le\\ &\le\|f_{r,h;r,f}-\varphi\|_{W_p^r\Omega}+ch^{-\alpha}d^{-\beta}\omega_{(\alpha,\beta)}(f;(h,d))_{L_p(\Omega)}<\\ &<\varepsilon+ch^{-\alpha}d^{-\beta}\omega_{(\alpha,\beta)}(f;(h,d))_{L_p(\Omega)}\end{aligned}$$

Let us now show (8).

Lemma 2 $\Rightarrow$

$$\begin{aligned}\|\,\|f-\varphi\|_{A_{p,h}[0,1]}\|_{A_{p,d}[0,t]}\le&\\ &\le\|\,\|f-f_{r,h;r,d}\|_{A_{p,h}[0,1]}\|_{A_{p,d}[0,t]}+\|\,\|f_{r,h;r,d}-\varphi\|_{A_{p,h}[0,1]}\|_{A_{p,d}[0,t]}\le\\ &\le c\Big(\|\tau_m(f;h)_{L_p[0,1]}\|_{A_{p,d}[0,t]}+\|\tau_n(f;d)_{L_p[0,t]}\|_{A_{p,h}[0,1]}\\ &\qquad+\|\,\|f_{r,h;r,d}-\varphi\|_{A_{p,h}[0,1]}\|_{A_{p,d}[0,t]}\Big).\end{aligned}$$

Let us estimate the last summand. By [17, Lemma 8],

$$\begin{aligned}\|\,\|f_{r,h;r,d}-\varphi\|_{A_{p,h}[0,1]}\|_{A_{p,d}[0,t]}\le&\\ &\le\|\,\|f_{r,h;r,d}-\varphi\|_{L_p[0,1]}\|_{A_{p,d}[0,t]}+h\|\,\|\frac{\partial}{\partial x}(f_{r,h;r,d}-\varphi)\|_{L_p[0,1]}\|_{A_{p,d}[0,t]}.\end{aligned}$$

By Fubini's theorem and by [17, Lemma 8],

$$\begin{aligned}\|\,\|f_{r,h;r,d}-\varphi\|_{L_p[0,1]}\|_{A_{p,d}[0,t]}\le&\\ &\le\|\,\|f_{r,h;r,d}-\varphi\|_{A_{p,d}[0,t]}\|_{L_p[0,1]}\le\\ &\le\|\,\|f_{r,h;r,d}-\varphi\|_{L_p[0,t]}\|_{L_p[0,1]}+d\|\,\|\frac{\partial}{\partial x}(f_{r,h;r,d}-\varphi)\|_{L_p[0,t]}\|_{L_p[0,1]}\le\\ &\le(1-d)\|f_{r,h;r,d}-\varphi\|_{W_p^r(\Omega)}<c\varepsilon.\end{aligned}$$

Analogously,

$$h\| \, \|\frac{\partial}{\partial x}(f_{r,h;r,d}-\varphi)\|_{L_p[0,1]}\|_{A_{p,d}[0,t]} \leq$$

$$\leq h\| \, \|\frac{\partial}{\partial x}(f_{r,h;r,d}-\varphi)\|_{A_{p,d}[0,t]}\|_{L_p[0,1]} \leq$$

$$\leq h\| \, \|\frac{\partial}{\partial x}(f_{r,h;r,d}-\varphi)\|_{L_p[0,1]}\|_{L_p[0,t]} + hd\| \, \|\frac{\partial^2}{\partial x \partial t}(f_{r,h;r,d}-\varphi)\|_{L_p[0,1]}\|_{L_p[0,t]} \leq$$

$$\leq (1+d)\|f_{r,h;r,d}-\varphi\|_{W_p^r(\Omega)} \leq ch\varepsilon.$$

□

4.2.2. A Priori Estimates

Lemma 5. (K.) *(See [7, p. 84].) For the solution u of problem (1), with homogeneous initial and boundary conditions, it holds true that*

$$\|u\|_{l^\infty(\Sigma_{hd}^j)} \leq \sup\{|u(x,t)|; (x,t) \in \Omega_t\} \leq \frac{1}{2\sqrt{2}}\|f\|_{L_2(\Omega_t)}, \quad t = jd.$$

Lemma 6. *For the solution v of problem (2), with homogeneous initial and boundary conditions (or, equivalently, for the solution z of problem (3) with exact approximation on the boundary), assume that*

1. $\max\{0, \frac{1}{2} - (1-\varepsilon_0)\frac{h^2}{4d}\} \leq \sigma \leq 1$;
2. $\varepsilon_0 \in (0,1]$.

Then,

$$\|v\|_{l^\infty(\Sigma_{hd}^j)} \leq \frac{c}{2\sqrt{2\varepsilon_0}}\| \, \|f\|_{A_{2,h}[0,1]}\|_{A_{2,d}[0,t]}, \quad t = jd,$$

where c is an absolute constant.

Proof. By [25, Lemmata 2 and 4], in view of the homogeneity,

$$\begin{aligned}\|v\|_{l^\infty(\Sigma_{hd}^j)} &\leq \tfrac{1}{2\sqrt{2\varepsilon_0}}\left(\sum_{\mu=0}^{j-1} d\|f(\cdot,t_{j+\frac{1}{2}})\|^2_{A_{2,h}[0,1]}\right)^{\frac{1}{2}} \leq \\ &\leq \tfrac{1}{2\sqrt{2\varepsilon_0}}\left(\sum_{\mu=0}^{j-1} dS(\tfrac{d}{2}, \|f\|_{A_{2,h}[0,1]}; t_{j+\frac{1}{2}})^2\right)^{\frac{1}{2}} \leq \\ &\leq \tfrac{c}{2\sqrt{2\varepsilon_0}}\| \, \|f\|_{A_{2,h}[0,1]}\|_{A_{2,d}[0,t]},\end{aligned}$$

taking in consideration also the basic properties of the spaces $A_{p,t}$. □

Lemma 7. *Assume that*

1. *the conditions of Lemma 5 hold;*

2. $u(x,0) \equiv 0$ a.e. $x \in [0,1]$.

Then, $\forall f \in L_2(\Omega) \Rightarrow$

$$\|\tfrac{\partial u}{\partial t}\|_{L_2(\Omega)} \leq \|f\|_{L_2(\Omega)},$$
$$\|\tfrac{\partial^2 u}{\partial t^2}\|_{L_2(\Omega)} \leq \|f\|_{L_2(\Omega)}.$$

Proof. Let $f \in L_2(\Omega)$. We shall apply the method of energetic inequalities. It is known (see [7, pp. 83-84]), that ($u(x,t)$ being a real-valued function)

$$-\int_0^1 \frac{\partial u}{\partial t}(x,t)\frac{\partial^2 u}{\partial x^2}(x,t)dx = \frac{\partial}{\partial t}\int_0^1 \frac{\partial u}{\partial x}(x,t)^2 dx \leq \frac{1}{2}\|f(\cdot,t)\|_{L_2[0,1]}. \tag{9}$$

The following identities hold true:

$$\int_0^1 \frac{\partial u}{\partial t}(\xi,\tau)^2 d\xi - \int_0^1 \frac{\partial u}{\partial t}(\xi,\tau)\frac{\partial^2 u}{\partial x^2}(\xi,\tau)d\xi = \int_0^1 \frac{\partial u}{\partial t}(\xi,\tau)f(\xi,\tau)d\xi \tag{10}$$

$$\int_0^1 \frac{\partial u}{\partial t}(\xi,\tau)\frac{\partial^2 u}{\partial x^2}(\xi,\tau)d\xi - \int_0^1 \frac{\partial^2 u}{\partial x^2}(\xi,\tau)^2 d\xi = \int_0^1 \frac{\partial^2 u}{\partial x^2}(\xi,\tau)f(\xi,\tau)d\xi \tag{11}$$

Subtract the left-hand and right-hand sides (LHS and RHS) of (11) from these of (10) and use (9):

$$\int_0^1 |\tfrac{\partial u}{\partial t}(\xi,\tau)|^2 d\xi + 2\tfrac{\partial}{\partial t}\int_0^1 \tfrac{\partial u}{\partial x}(\xi,\tau)^2 d\xi + \int_0^1 |\tfrac{\partial^2 u}{\partial x^2}(\xi,\tau)|^2 d\xi =$$
$$= \int_0^1 \left(\tfrac{\partial u}{\partial t}(\xi,\tau) + \tfrac{\partial^2 u}{\partial x^2}(\xi,\tau)\right) f(\xi,\tau)d\xi =$$
$$= \int_0^1 |f(\xi,\tau)|^2 d\xi$$

Integrate in τ from 0 to t to obtain:

$$\|\frac{\partial u}{\partial t}\|^2_{L_2(\Omega_t)} + 2\left(\|\frac{\partial u}{\partial x}(\cdot,t)\|^2_{L_2[0,1]} + \|\frac{\partial u}{\partial x}(\cdot,0)\|^2_{L_2[0,1]}\right) + \|\frac{\partial^2 u}{\partial x^2}\|^2_{L_2(\Omega_t)} = \|f\|^2_{L_2(\Omega_t)}$$

Under the conditions of the lemma $\|\frac{\partial u}{\partial x}(\cdot,0)\|_{L_2[0,1]} = 0$, therefore,

$$\|\frac{\partial u}{\partial t}\|^2_{L_2(\Omega_t)} + 2\|\frac{\partial u}{\partial x}(\cdot,t)\|^2_{L_2[0,1]} + \|\frac{\partial^2 u}{\partial x^2}\|^2_{L_2(\Omega_t)} = \|f\|^2_{L_2(\Omega_t)} \Rightarrow$$

$$\Rightarrow \|\frac{\partial u}{\partial t}\|^2_{L_2(\Omega_t)} + \|\frac{\partial^2 u}{\partial x^2}\|^2_{L_2(\Omega_t)} = \|f\|^2_{L_2(\Omega_t)},$$

from where the lemma follows immediately. In fact, we proved something more than was claimed:

$$\begin{aligned} &\|\tfrac{\partial u}{\partial t}\|_{L_2(\Omega_t)} \le \|f\|_{L_2(\Omega_t)}, \\ &\|\tfrac{\partial^2 u}{\partial t^2}\|_{L_2(\Omega_t)} \le \|f\|_{L_2(\Omega_t)}, \end{aligned} \tag{12}$$

$\forall t:\ 0 < t \le T$. □

Lemma 8. *Assume that*

1. *the conditions and the notation of Lemma 7 hold;*
2. $f \in C_0^\infty(\Omega)$.

Then, for $t : 0 < t \le T$,

$$\|\frac{\partial^3 u}{\partial x^3}\|_{L_2(\Omega_t)} \le 2\sqrt{2}\left(\|f\|_{L_2(\Omega_t)} + \|\frac{\partial f}{\partial x}\|_{L_2(\Omega_t)}\right); \tag{13}$$

$$\|\frac{\partial^4 u}{\partial x^4}\|_{L_2(\Omega_t)} \le \|\frac{\partial^2 f}{\partial x^2}\|_{L_2(\Omega_t)}; \tag{14}$$

$$\|\frac{\partial^2 u}{\partial t^2}\|_{L_2(\Omega_t)} \le \|\frac{\partial f}{\partial t}\|_{L_2(\Omega_t)}; \tag{15}$$

$$\|\frac{\partial^3 u}{\partial t^3}\|_{L_2(\Omega_t)} \le 2\|\frac{\partial^2 f}{\partial t^2}\|_{L_2(\Omega_t)} \tag{16}$$

$$\|\frac{\partial^5 u}{\partial x^4 \partial t}\|_{L_2(\Omega_t)} \le \|\frac{\partial^3 f}{\partial x^2 \partial t}\|_{L_2(\Omega_t)}; \tag{17}$$

$$\|\frac{\partial^3 u}{\partial x \partial t^2}\|_{L_2(\Omega_t)} \le (1+2\sqrt{2})\left(\|\frac{\partial f}{\partial t}\|_{L_2(\Omega_t)} + \|\frac{\partial^2 f}{\partial x \partial t}\|_{L_2(\Omega_t)}\right); \tag{18}$$

$$\|\frac{\partial^4 u}{\partial x \partial t^3}\|_{L_2(\Omega_t)} \le (1+2\sqrt{2})\left(\|\frac{\partial^2 f}{\partial t^2}\|_{L_2(\Omega_t)} + \|\frac{\partial^3 f}{\partial x \partial t^2}\|_{L_2(\Omega_t)}\right). \tag{19}$$

For the proof of Lemma 8, some additional preparation is needed.

Definition 4. Let $1 \le p < \infty;\ \mu,\nu \in \mathbb{N}\cup\{0\}$.

$\dot{W}_p^{(\mu,\nu)}(\Omega)$

$$= \left\{ f : f \in M(\Omega), \frac{\partial^{\mu+\nu} f}{\partial x^\mu \partial t^\nu} \in L_p(\Omega), \|f\|_{\dot{W}_p^{(\mu,\nu)}(\Omega)} = \|\frac{\partial^{\mu+\nu} f}{\partial x^\mu \partial t^\nu}\|_{L_p(\Omega)} < \infty \right\}.$$

Let $r \in \mathbb{N}$, $I_r = \{(\mu,\nu) : \mu,\nu \in \mathbb{N} \cup \{0\},\ \mu+\nu \le r\}$.

Lemma 9. *Let $1 \le p < \infty$; $k, r \in \mathbb{N}$.*

Then, $W_p^r(\Omega)$ is dense on $L_p(\Omega) \curvearrowright \dot{W}_p^{(k,0)}(\Omega)$.

Proof. Consider

$$\bar{W}_p^{(k,0)}(\Omega)$$
$$= \{f : f \in \dot{W}_p^{(k,0)}(\Omega),\ \frac{\partial^k f}{\partial x^k} - \text{bounded everywhere on } \Omega, \|\cdot\|_{\bar{W}_p^{(k,0)}} = \|\cdot\|_{\dot{W}_p^{(k,0)}}\}.$$

We shall show that $L_p \curvearrowright \bar{W}_p^{(k,0)}$ is a dense subspace of $L_p \curvearrowright \dot{W}_p^{(k,0)}$.

Let $f \in L_p \curvearrowright \bar{W}_p^{(k,0)}$ and $\varepsilon > 0$. Clearly, $\|f(x,\cdot)\|_{L_p[0,T]} < \infty$, a.e. $x \in [0,1]$. Let $x_0 : \|f(x_0,\cdot)\|_{L_p[0,t]} < \infty$. Expand in Taylor expansion with integral remainder:

$$f(x,\tau) = \sum_{\nu=0}^{k-1} \frac{(x-x_0)^\nu}{\nu!} \frac{\partial^\nu}{\partial x^\nu} f(x_0,\tau)$$
$$+ \frac{1}{(k-1)!} \int_{x_0}^{x} (x-\zeta)^{k-1} \frac{\partial^k}{\partial \zeta^k} f(\zeta,\tau) d\zeta,\ \tau \in [0,T].$$

Choose

$$\varphi_{k,\varepsilon} : \|\varphi_{k,\varepsilon} - \frac{\partial^k f}{\partial x^k}\|_{L_p(\Omega)} < \varepsilon,$$

$\varphi_{k,\varepsilon}$ – bounded everywhere on Ω;

$$\varphi_\nu,\ \nu = 0,1,\dots,k-1 : \varphi_\nu(\tau) = \frac{\partial^\nu}{\partial x^\nu} f(x_0,\tau).$$

Set f_ε:

$$f_\varepsilon(x,\tau) = \sum_{\nu=0}^{k-1} \frac{(x-x_0)^\nu}{\nu!} \varphi_\nu(\tau) + \frac{1}{(k-1)!} \int_{x_0}^{x} (x-\zeta) \varphi_{k,\varepsilon}(\zeta,\tau) d\zeta.$$

It is clear that

$$\|f - f_\varepsilon\|_{L_p} < c\varepsilon,\ \|f - f_\varepsilon\|_{\dot{W}_p^{(k,0)}} < c\varepsilon.$$

Obviously, $f_\varepsilon \in \bar{W}_p^{(k,0)}$. It remains to verify that $f_\varepsilon \in L_p$. This is implied by

Minkowski's inequality, the selection of x_0 and the inequalities

$$\|(x-x_0)^\nu \varphi_\nu(\tau)\|_{L_p(\Omega)} \leq c\left(\|f\|_{L_p(\Omega)} + \|f\|_{\dot{W}_p^{(k,0)}}\right),\ \nu = 1,\dots,k-1,$$

which follow from the continuity of $(x-x_0)^\nu$, the compactness of $[0,1]$ and the standard technique for estimation of the intermediate derivatives of f via f itself and its highest-order derivative(s) (see [5, pp. 195-198]). With this, the density of $L_p \curvearrowright \bar{W}_p^{(k,0)}$ is proved.

In order to prove the lemma, now it suffices to show that $W_p^r \cap \bar{W}_p^{(k,0)}$ is dense on $L_p \curvearrowright \bar{W}_p^{(k,0)}$, for $r \geq k$.

It is known that (see, e.g., [5, p. 175, (4)]), that (in our notation)

$\forall g \in L_p(\Omega),\ 1 \leq p < \infty \Rightarrow$

$$\lim_{\delta\to+0} \sup_{|\xi|\leq\delta} \left(\int_0^t\int_0^1 \chi_\xi(\zeta)|g(\zeta+\xi,\tau)-g(\zeta,\tau)|^p d\zeta d\tau\right)^{\frac{1}{p}} = \lim_{\delta\to+0} \omega_{(0,1)}(g;(\delta,\delta))_{L_p(\Omega)} = 0,$$

$$\lim_{\delta\to+0} \sup_{|\nu|\leq\delta} \left(\int_0^t\int_0^1 \chi_\nu(\tau)|g(\zeta,t+\tau)-g(\zeta,\tau)|^p d\zeta d\tau\right)^{\frac{1}{p}} = \lim_{\delta\to+0} \omega_{(0,1)}(g;(\delta,\delta))_{L_p(\Omega)} = 0,$$

where χ_ξ, χ_ν are the characteristic functions of

$$\{\zeta : \zeta, \zeta+\xi \in [0,1]\},\ \{\tau : \tau, \tau+\nu \in [0,T]\},$$

respectively.

Let $f \in L_p \curvearrowright \bar{W}_p^{(k,0)}$. Clearly,

$$\tilde{f}_\delta = f_{r+1,\frac{\delta}{r+1};r+1,\frac{\delta T}{r+1}} \in W_p^r \cap W_p^{(k,0)},\ \delta \leq 1.$$

Lemma 2, (6) $\Rightarrow$

$$\|f-\tilde{f}_s\|_{L_p(\Omega)} \leq$$

$$\leq c\left(\omega_{(r+1,0)}(f;(\tfrac{\delta}{r+1},\tfrac{\delta T}{r+1}))_{L_p(\Omega)} + \omega_{(0,r+1)}(f;(\tfrac{\delta}{r+1},\tfrac{\delta T}{r+1}))_{L_p(\Omega)}\right) \leq$$

$$\leq c\left(\omega_{(1,0)}(f;(\tfrac{\delta}{r+1},\tfrac{\delta T}{r+1}))_{L_p} + \omega_{(0,1)}(f;(\tfrac{\delta}{r+1},\tfrac{\delta T}{r+1}))_{L_p}\right) \to$$

$$\to +0, \quad \delta \to +0.$$

Since $f \in \bar{W}_p^{(k,0)}$, from [10, p. 748] (see also Theorem 7.1 therein) it follows that differentiation in x and t commutes with the integrals in the explicit expression for $\tilde{f}_\delta$. Hence, after simple computations, making also use of the continuity of the derivatives of Θ from the definition of the Steklov mean (in the sense of Brudnyi – see Section 3), as well as using Lemma 2 (6), we get, analogously to the previous inequality,

$$\|f - \tilde{f}_\delta\|_{\bar{W}_p^{(k,0)}} =$$

$$= \|\frac{\partial^k f}{\partial x^k} - (\frac{\partial^k \tilde{f}}{\partial x^k})_\delta\|_{L_p} + c\sum_{\nu=1}^{k} \|\frac{d^\nu \Theta}{dx^\nu}\|_{c[0,1]} \binom{k}{\nu} \omega_{(1,0)}(\frac{\partial^{k-\nu} f}{\partial x^{k-\nu}}; (\frac{2\delta}{r+1}, \frac{\delta T}{r+1}))_{L_p(\Omega)} \leq$$

$$\leq c\left(\omega_{(1,0)}(\frac{\partial^{k-\nu} f}{\partial x^{k-\nu}}; (\frac{\delta}{r+1}, \frac{\delta T}{r+1}))_{L_p(\Omega)} + \sum_{\nu=0}^{k} \binom{k}{\nu} \omega_{(1,0)}(\frac{\partial^{k-\nu} f}{\partial x^{k-\nu}}; (\frac{2\delta}{r+1}, \frac{\delta T}{r+1}))_{L_p(\Omega)}\right) \to$$

$$\to +0, \quad \delta \to +0.$$

□

Lemma 10. *Let* $1 \leq p < \infty,\ k \in \mathbb{N}$.
Then, $C_0^\infty(\Omega)$ *is dense on* $L_p(\Omega) \curvearrowright \dot{W}_p^{(k,0)}(\Omega)$.

Proof. This is an immediate implication of Lemma 3 and Lemma 9. □

Now we are ready to proceed to the proof of Lemma 8, as follows.

Proof. (of Lemma 8.) Let $f \in C_0^\infty(\Omega)$. From the general theory of the parabolic equation in consideration it is clear that

$$u(x,t) = 2\sum_{n=1}^{\infty} \sin \pi n x \int_0^t \exp^{-\pi^2 n^2 \tau} \int_0^1 f(\xi, t-\tau) \sin \pi n \xi d\xi d\tau \qquad (20)$$

(the equality being in the sense of uniform convergence in Ω), where the series in the right-hand side is twice term-by-term differentiable, and

$$\frac{\partial^2 u}{\partial x^2}(x,t) = 2\sum_{n=1}^{\infty} \pi^2 n^2 \sin \pi n x \int_0^t \exp^{-\pi^2 n^2 \tau} \int_0^1 f(\xi, t-\tau) \sin \pi n \xi d\xi d\tau \qquad (21)$$

where the series (21) is also uniformly convergent, as well as (20). Moreover, since the derivatives of f of arbitrary order are in $C_0^\infty(\Omega)$, it follows that $\forall k \in \mathbb{Z}$ the series

$$S_k(f) = 2\sum_{n=1}^{\infty} \pi n \sin \pi n x \int_0^t \exp^{-\pi^2 n^2 \tau} \int_0^1 f(\xi, t-\tau) \sin \pi n \xi d\xi d\tau$$

$$C_k(f) = 2\sum_{n=1}^{\infty} \pi n \cos \pi n x \int_0^t \exp^{-\pi^2 n^2 \tau} \int_0^1 f(\xi, t-\tau) \sin \pi n \xi d\xi d\tau$$

are uniformly convergent in Ω. (The verification is via integrating by parts in $\int\limits_0^1 f(\xi, t-\tau)\sin \pi n\xi d\xi$ and $\int\limits_0^1 f(\xi, t-\tau)\cos \pi n\xi d\xi$.)

For $f \in C_0^\infty(\Omega)$, after a term-by-term differentiation of (20) in x and t we get

$$\begin{aligned}
&\tfrac{\partial u}{\partial t} = f - P(f);\\
&\tfrac{\partial^2 u}{\partial x^2} = P(f);\\
&\tfrac{\partial^3 u}{\partial x^3} = Q(\tfrac{\partial f}{\partial x});\\
&\tfrac{\partial^4 u}{\partial x^4} = P(\tfrac{\partial^2 f}{\partial x^2});\\
&\tfrac{\partial^2 u}{\partial t^2} = \tfrac{\partial f}{\partial t} + P(\tfrac{\partial f}{\partial t});\\
&\tfrac{\partial^3 u}{\partial t^3} = \tfrac{\partial^2 f}{\partial t^2} + P(\tfrac{\partial^2 f}{\partial t^2});\\
&\tfrac{\partial^4 u}{\partial x\partial t^3} = \tfrac{\partial^3 f}{\partial x\partial t^2} + Q(\tfrac{\partial^3 f}{\partial x\partial t^2});\\
&\tfrac{\partial^3 u}{\partial x\partial t^2} = \tfrac{\partial^2 f}{\partial x\partial t} + Q(\tfrac{\partial^2 f}{\partial x\partial t});\\
&\tfrac{\partial^5 u}{\partial x^4\partial t} = Q(\tfrac{\partial^3 f}{\partial x^2\partial t});
\end{aligned} \tag{22}$$

where the linear operators $P, Q : C_0^\infty(\Omega) \to L_2(\Omega)$ are defined by

$$P : f \mapsto -S_2(f),\ Q : f \mapsto -C_2(f).$$

Let $0 < t \leq 1$. Clearly,

$$C_0^\infty(\Omega_t) \subset C_0^\infty(\Omega)$$

(as functions whose definition is extended by 0 to $\Omega \setminus \Omega_t$).

Lemma 7 (12) $\Rightarrow$

$$\|P(f)\|_{L_2(\Omega_t)} = \|f\|_{L_2(\Omega_t)},\ \forall f \in C_0^\infty(\Omega_t). \tag{23}$$

If $f \in C_0^\infty(\Omega_t)$, then also the derivatives of f belong to $C_0^\infty(\Omega_t)$. Therefore, by (23),

$$\|\tfrac{\partial^4 u}{\partial x^4}\|_{L_2(\Omega_t)} = \|P(\tfrac{\partial^2 f}{\partial x^2})\|_{L_2(\Omega_t)} \leq \|\tfrac{\partial^2 f}{\partial x^2}\|_{L_2(\Omega_t)} \Rightarrow (14),$$

$$\|\tfrac{\partial^2 u}{\partial t^2}\|_{L_2(\Omega_t)} = \|\tfrac{\partial f}{\partial t}\|_{L_2(\Omega_t)} + \|P(\tfrac{\partial f}{\partial t})\|_{L_2(\Omega_t)} \leq 2\|\tfrac{\partial f}{\partial t}\|_{L_2(\Omega_t)} \Rightarrow (15),$$

$$\|\tfrac{\partial^3 u}{\partial t^3}\|_{L_2(\Omega_t)} = \|\tfrac{\partial^2 f}{\partial t^2}\|_{L_2(\Omega_t)} + \|P(\tfrac{\partial^2 f}{\partial t^2})\|_{L_2(\Omega_t)} \leq 2\|\tfrac{\partial^2 f}{\partial t^2}\|_{L_2(\Omega_t)} \Rightarrow (16),$$

$$\|\tfrac{\partial^5 u}{\partial x^4\partial t}\|_{L_2(\Omega_t)} = \|P(\tfrac{\partial^3 f}{\partial x^2\partial t})\|_{L_2(\Omega_t)} \leq \|\tfrac{\partial^3 f}{\partial x^2\partial t}\|_{L_2(\Omega_t)} \Rightarrow (17).$$

In order to prove in an analogous way the remaining inequalities in the lemma, we need an analogue of (23) for Q. We shall show that the following is fulfilled:

$$\|Q(\frac{\partial f}{\partial x})\|_{L_2(\Omega_t)} \leq 2\sqrt{2}\left(\|f\|_{L_2(\Omega_t)} + \|\frac{\partial f}{\partial x}\|_{L_2(\Omega_t)}\right), \ \forall f \in C_0^\infty(\Omega_t). \qquad (24)$$

Using term-by-term differentiation and integration by parts in the series defining $P(f)$ when $f \in C_0^\infty(\Omega)$, we establish that

$$\frac{\partial}{\partial x}P(f) = Q(\frac{\partial f}{\partial x}).$$

Therefore, in order to prove (24), it suffices to show that

$$\|P(f)\|_{L_2(\Omega_t)} + \|P(f)\|_{\dot{W}_2^{(1,0)}(\Omega_t)} \leq 2\sqrt{2}\left(\|f\|_{L_2(\Omega_t)} + \|f\|_{\dot{W}_2^{(1,0)}(\Omega_t)}\right), \ \forall f \in C_0^\infty(\Omega_t). \qquad (25)$$

The norms in the left-hand and right-hand sides of (25) are equivalent to the norms of $P(f)$ and f, respectively, in

$$L_2(\Omega_t) \curvearrowright \dot{W}_2^{(1,0)}(\Omega_t), \qquad (26)$$

as well as to

$$\| \, |P(f)| + |\frac{\partial}{\partial x}P(f)| \, \|_{L_2(\Omega_t)}, \ \| \, |f| + |\frac{\partial}{\partial x}f| \, \|_{L_2(\Omega_t)}, \qquad (27)$$

respectively (provided that $P(f)$ and f belong to the respective spaces).

It can be directly verified that ($k \in \mathbb{N} \cup \{0\}$, $\dot{W}_2^{(0,0)}(\Omega_t) = L_2(\Omega_t)$)

$$L_2(\Omega_t) \curvearrowright \dot{W}_2^{(k,0)}(\Omega_t)$$

(with the equivalent norm

$$\| \, |g| + |\frac{\partial^k}{\partial x^k}g| \, \|_{L_2(\Omega_t)})$$

and the vector-valued L_2-space

$$L_2([0,t]; W_2^k[0,1])$$

with norm

$$\left(\int_0^t \|g(\tau)\|_{W_2^k[0,1]} d\tau\right)^{\frac{1}{2}}$$

are isomorphic and isometric:

$$L_2(\Omega_t) \curvearrowright \dot{W}_2^{(,0)}(\Omega_t) \rightleftharpoons L_2([0,t]; W_2^k[0,1]). \qquad (28)$$

(The verification is simple: assume that g belongs to one of the two spaces;

then Fubini's theorem implies that g belongs also to the other space, with equality of the norms.)

We shall prove (25) using interpolation of vector-valued L_2-spaces. Because of (28), (25) will follow from

$$\|Pf\|_{L_2([0,t];W_2^1[0,1])} \leq \|f\|_{L_2([0,t];W_2^1[0,1])}, \ \forall f \in C_0^\infty(\Omega_t), \tag{29}$$

taking account also for the embedding constant in (27).

Analogously, (23), (28) $\Rightarrow$

$$\|Pf\|_{L_2([0,t];L_2[0,1])} \leq \|f\|_{L_2([0,t];L_2[0,1])}, \ \forall f \in C_0^\infty(\Omega_t). \tag{30}$$

In the proof of (14) it was also proved that

$$\|P(\frac{\partial^2 f}{\partial x^2})\|_{L_2(\Omega_t)} \leq \|\frac{\partial^2 f}{\partial x^2}\|_{L_2(\Omega_t)}, \ \forall f \in C_0^\infty(\Omega_t).$$

From here and from

$$\frac{\partial^2}{\partial x^2} P(f) = P(\frac{\partial^2 f}{\partial x^2}), \ \forall f \in C_0^\infty(\Omega_t),$$

cf. the proof of

$$\frac{\partial}{\partial x} P(f) = Q\left(\frac{\partial f}{\partial x}\right),$$

it follows that

$$\|P(f)\|_{\dot{W}_2^{2,0}(\Omega_t)} \leq \|f\|_{\dot{W}_2^{2,0}(\Omega_t)}, \ \forall f \in C_0^\infty(\Omega_t). \tag{31}$$

(23), (31), (28) $\Rightarrow$

$$\|P(f)\|_{L_2([0,t];W_2^2[0,1])} \leq 2\|f\|_{L_2([0,t];W_2^2[0,1])}, \ \forall f \in C_0^\infty(\Omega_t). \tag{32}$$

Lemmata 3, 10, (28) $\Rightarrow C_0^\infty(\Omega_t)$ is dense on $L_2([0,t];L_2[0,1])$ and in $L_2([0,t];$ $W_2^2[0,1])$.

Therefore, from (30), (32), by the theorem about extension of a linear (i.e., continuous) operator with dense definition domain (see, e.g., [6, p. 22]), there exist bounded linear operators

$$\tilde{p}, \tilde{p}_1 : \tilde{p} \in L\left(L_2([0,t];L_2[0,1])\right), \tilde{p}_1 \in L\left(L_2([0,t];W_2^2[0,1])\right),$$

such that

$$\tilde{p}\mid_{C_0^\infty(\Omega_t)} = \tilde{p}_1 \mid_{C_0^\infty(\Omega_t)} = p$$

and the norm of $\tilde{p}$ is ≤ 1, while that of $\tilde{p}_1$ is ≤ 2.

Clearly,

$$L_2([0,t];W_2^2[0,1]) \hookrightarrow L_2([0,t];L_2[0,1]). \tag{33}$$

We shall show that

$$\tilde{p}\,|_{L_2([0,t];W_2^2[0,1])} = \tilde{p}_1$$

in $L_2([0,t];L_2[0,1])$, i.e., that

$$\|\tilde{p}(f) - \tilde{p}_1(f)\_L_2([0,t];L_2[0,1]) = 0,\ \forall f \in L_2([0,t];W_2^2[0,1]). \tag{34}$$

Let $f \in L_2([0,t];W_2^2[0,1])$, $\varepsilon > 0$. From the density of $C_0^\infty(\Omega_t)$ in $L_2([0,t];W_2^2[0,1])$ and (33),

$$\exists f_\varepsilon \in C_0^\infty(\Omega_t) : \|f - f_\varepsilon\|_{L_2([0,t];W_2^2[0,1])} < \varepsilon.$$

Therefore,

$$\|\tilde{p}(f) - \tilde{p}_1(f)\_L_2([0,t];L_2[0,1]) \leq$$

$$\leq \|\tilde{p}(f) - \tilde{p}(f_\varepsilon)\|_{L_2([0,t];L_2[0,1])} + \|\tilde{p}(f_\varepsilon) - \tilde{p}_1(f_\varepsilon)\|_{L_2([0,t];L_2[0,1])}$$

$$+ \|\tilde{p}_1(f_\epsilon) - \tilde{p}_1(f)\|_{L_2([0,t];L_2[0,1])} \leq$$

$$\leq \|f - f_\varepsilon\|_{L_2([0,t];L_2[0,1])} + 0 + \|\tilde{p}_1(f_\epsilon) - \tilde{p}_1(f)\|_{L_2([0,t];L_2[0,1])} \leq$$

$$\leq 3\|f - f_\varepsilon\|_{L_2([0,t];W_2^2[0,1])} < 3\varepsilon; \quad \varepsilon \to +0 \Rightarrow (34).$$

Therefore, $\tilde{p} \in L_2([0,t];L_2[0,1])$ with norm ≤ 1, $\tilde{p}_1 \in L_2([0,t];W_2^2[0,1])$ with norm ≤ 2. Therefore, by interpolation with the $C_{[\frac{1}{2}]}$-method (see [1, p. 140])

$$\|\tilde{p}f\|_{L_2([0,t];W_2^1[0,1])} \leq \sqrt{2}\|f\|_{L_2([0,t];W_2^1[0,1])},\ \forall f \in L_2([0,t];W_2^1[0,1]). \tag{35}$$

From here and from (28), since

$$\frac{\partial}{\partial x}\tilde{p}\,|_{C_0^\infty(\Omega_t)} = Q(\frac{\partial}{\partial x}),$$

it follows (25), which implies (24). Now (13), (18), (19) follow analogously to (14) – (17), by using (22). □

Remark 3. (See also Remarks 1, 4, 6, 12.) In the special case of higher order of accuracy in [25, Theorem 2], it is possible to also formulate appropriate modifications of the *a priori* estimates considered in this section which, together with the modified error estimate in terms of properties of the solution mentioned in Remark 1 (corresponding to Stages 5 and 4 of the extended Lax principle, respectively), would lead to an error estimate of higher order of accuracy in terms of properties of the RHS of problem (1) in the case when this RHS is sufficiently smooth (corresponding to Stage 6 of the extended Lax principle).

4.3. Error Estimates Directly in Terms of Properties of the Right-Hand Side

4.3.1. Main Result

Theorem 2. *Assume that*

1. *problem (1) is with homogeneous initial and boundary conditions;*
2. *problem (1) is being solved numerically via (2) with exact approximation of the initial and boundary conditions, i.e., the discrete problems (2) and (3) are also with homogeneous initial and boundary conditions;*
3. $\max\left\{0, \frac{1}{2} - (1-\varepsilon_0)\frac{h^2}{4d}\right\} \le \sigma \le 1$;
4. $\varepsilon_0 \in (0,1]$.

Then, for the error z*, determined as the solution of (3), the following estimate holds true*

$$\begin{aligned}\|z\|_{l^\infty(\Sigma^j_{hd})} \le c\,(\quad & (h^2 + d(|\tfrac{1}{2}-\sigma| + d))\|f\|_{L_2(\Omega_t)} + \\ & + \|\tau_2(f;h)_{L_2[0,1]}\|_{A_{2,d}[0,t]} + \\ & + \|\tau_2(f;h)_{L_2[0,t]}\|_{A_{2,h}[0,1]} + \\ & + |\tfrac{1}{2}-\sigma| . \|\omega_1(f;d)_{L_2[0,1]}\|_{L_2[0,1]}\ ),\end{aligned}$$

where $0 < t = jd \le T,\ c = c(\varepsilon_0) > 0$.

Proof. By the conditions of the theorem, f is bounded and measurable in Ω, therefore, $f \in L_2(\Omega_t)$.

Let $\varepsilon > 0$. Choose $\varphi_\varepsilon \in C_0^\infty(\Omega_t)$, satisfying (7, 8) in Lemma 4, where in the notation of this lemma we may choose, e.g., $r = 5$.

Consider the following boundary problems of the type of (1):

$$(P_j)\begin{cases} \frac{\partial}{\partial t}U_j(x,t) - \frac{\partial^2}{\partial x^2}U_j(x,t) = f_j(x,t), & (x,t) \in \Omega, \\ U_j(x,0) = 0, & \text{a.e. } x \in [0,1], \\ U_j(k,t) = 0, & \text{a.e. } t \in [0,T], \end{cases}$$

where $f_0 = f - \varphi_\varepsilon,\ f_1 = \varphi_\varepsilon,\ u_j$ are continuous on Ω, $j = 0,1$.

There is a 1-1 correspondence between the problems (P_j), $j = 0,1$, and respective problems of the type of (2), (3), with solutions V_j, Z_j, $j = 0,1$, respectively. By the linearity,

$$u = U_0 + U_1,\ v = V_0 + V_1,\ z = Z_0 + Z_1.$$

It remains to estimate Z_0 and Z_1.

Since the approximation of the boundary and initial conditions is exact, (i.e., $Z_{0,0\mu} = Z_{0,N\mu} = Z_{0,\nu 0} = 0;\ \nu = 0,1,\ldots,N;\ \mu = 0,1,\ldots,j$), it is fulfilled that

$$\|Z_0\|_{l^\infty(\Sigma^j_{hd})} \le \|U_0\|_C + \|V_0\|_{l^\infty(\Sigma^j_{hd})}. \tag{36}$$

Lemmata 4 – 6 ⇒

$$\begin{aligned}\|Z_0\|_{l^\infty(\Sigma^j_{hd})} &\le \tfrac{1}{2\sqrt{2}}\|f_0\|_{L_2(\Omega_t)} + \tfrac{c}{2\sqrt{\varepsilon_0}}\|\ \|f_0\|_{A_{2,h}[0,1]}\|_{A_{2,d}[0,1]} \le \\ &\le c\|\ \|f_0\|_{A_{2,h}[0,1]}\|_{A_{2,d}[0,1]} = \\ &= c\|\ \|f-\varphi_\varepsilon\|_{A_{2,h}[0,1]}\|_{A_{2,d}[0,1]} \le \\ &\le c\left(\|\tau_5(f;h)_{L_2[0,1]}\|_{A_{2,d}[0,t]} + \|\tau_5(f;d)_{L_2[0,t]}\|_{A_{2,h}[0,1]}\right) + \varepsilon \le \\ &\le c\left(\|\tau_2(f;h)_{L_2[0,1]}\|_{A_{2,d}[0,t]} + \|\tau_2(f;d)_{L_2[0,t]}\|_{A_{2,h}[0,1]}\right) + \varepsilon.\end{aligned} \tag{37}$$

(5) and the properties of τ-moduli yield

$$\begin{aligned}\|Z_1\|_{l^\infty(\Sigma^j_{hd})} \le \tfrac{c}{2\sqrt{2\varepsilon_0}}\Big(& h^2\|\ \|\tfrac{\partial^4 U_1}{\partial x^4}\|_{L_2[0,1]}\|_{A_{2,d}[0,t]} + \\ & + d^2\|\ \|\tfrac{\partial^3 U_1}{\partial t^3}\|_{L_2[0,t]}\|_{A_{2,h}[0,1]} + \\ & + |\tfrac{1}{2} - \sigma|d\|\ \|\tfrac{\partial^2 U_1}{\partial t^2}\|_{L_2[0,t]}\|_{A_{2,h}[0,1]}\ \Big).\end{aligned} \tag{38}$$

Let us estimate $\|\ \|\frac{\partial^4 U_1}{\partial x^4}\|_{L_2[0,1]}\|_{A_{2,d}[0,t]}$.

From the properties of the upper Baire's function, Fubini's theorem and [17, Lemma 8] it follows

$$\begin{aligned}\|\ \|\tfrac{\partial^4 U_1}{\partial x^4}\|_{L_2[0,1]}\|_{A_{2,d}[0,t]} &\le \|\ \|\tfrac{\partial^4 U_1}{\partial x^4}\|_{A_{2,d}[0,t]}\|_{L_2[0,1]} \\ &\le \|\ \|\tfrac{\partial^4 U_1}{\partial x^4}\|_{L_2[0,t]}\|_{L_2[0,1]} + \|\tau_1(\tfrac{\partial^4 U_1}{\partial x^4};d)_{L_2[0,t]}\|_{L_2[0,1]} \le \\ &\le \|\ \|\tfrac{\partial^4 U_1}{\partial x^4}\|_{L_2[0,t]}\|_{L_2[0,1]} + cd\|\ \|\tfrac{\partial^5 U_1}{\partial x^4 \partial t}\|_{L_2[0,t]}\|_{L_2[0,1]}\end{aligned} \tag{39}$$

Fubini's theorem and Lemmata 8, 4 yield

$$\begin{aligned} \| \|\tfrac{\partial^4 U_1}{\partial x^4}\|_{L_2[0,t]}\|_{L_2[0,1]} &= \|\tfrac{\partial^4 U_1}{\partial x^4}\|_{L_2(\Omega_t)} \le \\ &\le \|\tfrac{\partial^2 \varphi_\varepsilon}{\partial x^2}\|_{L_2(\Omega_t)} \le \\ &\le \tfrac{c}{h^2}\omega_{(2,0)}(f;(h,d))_{L_2(\Omega_t)} + \varepsilon. \end{aligned} \tag{40}$$

Analogously to (40), we get

$$\| \|\frac{\partial^5 U_1}{\partial x^4 \partial t}\|_{L_2[0,t]}\|_{L_2[0,1]} \le \frac{c}{h^2 d}\omega_{(2,1)}(f;(h,d))_{L_2(\Omega_t)} + \varepsilon. \tag{41}$$

(39–41), (6) $\Rightarrow$

$$\begin{aligned} &h^2 \| \|\tfrac{\partial^4 U_1}{\partial x^4}\|_{L_2[0,1]}\|_{A_{2,d}[0,t]} \le \\ &\le c\left(\omega_{(2,0)}(f;(h,d))_{L_2(\Omega_t)} + \omega_{(2,1)}(f;(h,d))_{L_2(\Omega_t)} + \varepsilon\right) \le \\ &\le c\left(\omega_{(2,0)}(f;(h,d))_{L_2(\Omega_t)} + \varepsilon\right). \end{aligned} \tag{42}$$

The other two summands in (38) are being estimated analogously to (39–42) by using the respective statements in Lemma 8.

Summation of the estimates obtained for the three summands in (38) yields

$$\begin{aligned} \|Z_1\|_{l^\infty(\Sigma^j_{hd})} \le c\,(\quad & (h^2 + d(|\tfrac{1}{2} - \sigma| + d)\|f\|_{L_2(\Omega_1)} + \\ & + \|\omega_2(f;h)_{L_2[0,1]}\|_{L_2[0,t]} + \\ & + \|\omega_2(f;h)_{L_2[0,t]}\|_{L_2[0,1]} + \\ & + |\tfrac{1}{2} - \sigma|.\|\omega_1(f;d)_{L_2[0,t]}\|_{L_2[0,1]}\ ) + c\varepsilon. \end{aligned} \tag{43}$$

(37), (43), $\varepsilon \to +0 \Rightarrow$ Theorem 2. □

Remark 4. (See also Remarks 1, 3, 6, 12.) In the special case of higher order of accuracy in [25, Theorem 2], it is possible to also formulate appropriate error estimate of higher order of accuracy in terms of properties of the RHS of problem (1) in the case when this RHS is sufficiently smooth (corresponding to Stage 6 of the extended Lax principle). To this end, at Stages 4 and 5 it will be necessary to invoke the corresponding modifications of the results in Sections 4.1, 4.2.2, as described in Remarks 1, 3, respectively.

4.3.2. Some Implications

From Theorem 2 and the properties of the τ-moduli we can straightforwardly obtain very sharp new error estimates in terms of properties of the right-hand side. Here we shall give only some of them. In their formulation we shall be implicitly assuming that the conditions of Theorem 2 are fulfilled (and that, with no loss of generality, $h, d \le 1$). Corollary 11 follows from Corollary 10 and the isomorphism between A-spaces and Besov spaces for smoothness index higher than the index of Sobolev embedding (see [19, Section 2.3, item 3] and the references to [4, 17] therein).

Corollary 1.

$$\|z\|_{l^\infty(\Sigma^j_{hd})} \le c_1 \left(\sqrt{h} \| \bigvee_0^1 {}_2\, f \|_{A_{2,d}[0,t]} + \sqrt{d} \| \bigvee_0^t {}_2\, f \|_{A_{2,h}[0,1]} + (\sqrt{h} + \sqrt{d}) \|f\|_{L_2(\Omega_1)} \right).$$

Corollary 2.

$$\|z\|_{l^\infty(\Sigma^j_{hd})} \le c_2 \left(h \| \, \|\frac{\partial f}{\partial x}\|_{L_2[0,1]} \|_{A_{2,d}[0,t]} + \sqrt{d} \| \bigvee_0^t {}_2\, f \|_{A_{2,h}[0,1]} + (h + \sqrt{d}) \|f\|_{L_2(\Omega_1)} \right).$$

Corollary 3.

$$\|z\|_{l^\infty(\Sigma^j_{hd})} \le c_3 \left(h \| \, \|\frac{\partial f}{\partial x}\|_{L_2[0,1]} \|_{A_{2,d}[0,t]} + d \| \, \|\frac{\partial f}{\partial x}\|_{L_2[0,t]} \|_{A_{2,h}[0,1]} + (h + d) \|f\|_{L_2(\Omega_1)} \right).$$

Corollary 4.

$$\|z\|_{l^\infty(\Sigma^j_{hd})} \le c_4 \left(h^{\frac{3}{2}} \| \bigvee_0^1 {}_2\, \frac{\partial f}{\partial x} \|_{A_{2,d}[0,t]} + d \| \, \|\frac{\partial f}{\partial x}\|_{L_2[0,t]} \|_{A_{2,h}[0,1]} + (h\sqrt{h} + d) \|f\|_{L_2(\Omega_1)} \right).$$

Corollary 5.

$$\|z\|_{l^\infty(\Sigma^j_{hd})} \le c_5 \left(h^2 \| \, \|\frac{\partial^2 f}{\partial x^2}\|_{L_2[0,1]} \|_{A_{2,d}[0,t]} + d \| \, \|\frac{\partial f}{\partial x}\|_{L_2[0,t]} \|_{A_{2,h}[0,1]} + (h^2 + d) \|f\|_{L_2(\Omega_1)} \right).$$

Corollary 6. *In the Crank-Nicolson case* $\sigma = \frac{1}{2}$,

$$\|z\|_{l^\infty(\Sigma_{hd}^j)}$$
$$\leq c_6 \left(h^2 \| \, \|\frac{\partial^2 f}{\partial x^2}\|_{L_2[0,1]}\|_{A_{2,d}[0,t]} + d\sqrt{d}\| \bigvee_0^t{}_2 \, \frac{\partial f}{\partial x}\|_{A_{2,h}[0,1]} + (h^2 + d\sqrt{d})\|f\|_{L_2(\Omega_1)} \right).$$

Corollary 7. *In the Crank-Nicolson case* $\sigma = \frac{1}{2}$,

$$\|z\|_{l^\infty(\Sigma_{hd}^j)}$$
$$\leq c_7 \left(h^2 \| \, \|\frac{\partial^2 f}{\partial x^2}\|_{L_2[0,1]}\|_{A_{2,d}[0,t]} + d^2 \| \, \|\frac{\partial^2 f}{\partial x^2}\|_{L_2[0,t]}\|_{A_{2,h}[0,1]} + (h^2 + d^2)\|f\|_{L_2(\Omega_1)} \right).$$

Corollary 8. *Let*

1. $0 < c_{8,0} < \infty$;
2. $d \leq c_{8,0} h^{\frac{3}{2}}$;
3. $\max\{0, \frac{1}{2} - (1 - \varepsilon_0)\frac{\sqrt{h}}{4c}\} \leq \sigma \leq 1$;
4. $\bigvee_0^1{}_2 \, \frac{\partial f}{\partial t}(\cdot, \tau) \in BM[0,t]$;
5. $\|\frac{\partial f}{\partial t}(\cdot, \tau)\|_{L_2[0,t]} \in BM[0,1]$.

Then, $\exists c_8 : 0 < c_8 < \infty$, *depending on* $c_{8,0}$, *such that*

$$\|z\|_{l^\infty(\Sigma_{hd}^j)} \leq c_8 h^{\frac{3}{2}}.$$

Corollary 9. *Let*

1. $0 < c_{9,0} < \infty$;
2. $d \leq c_{9,0} h^{\frac{4}{3}}$;
3. $\sigma = \frac{1}{2}$ *(the Crank-Nicolson scheme);*
4. $\|\frac{\partial^2 f}{\partial x^2}(\cdot, \tau)\|_{L_2[0,1]} \in BM[0,t]$;
5. $\bigvee_0^1{}_2 \, \frac{\partial f}{\partial t}(\cdot, \tau) \in BM[0,1]$.

Then, $\exists c_9 : 0 < c_9 < \infty$, *depending on* $c_{9,0}$, *such that*

$$\|z\|_{l^\infty(\Sigma^j_{hd})} \le c_9 h^2.$$

Corollary 10. *Let*

1. $0 < s < 2$;
2. (a) *either* $0 < r < 1,\ \sigma \neq \frac{1}{2}$,
   (b) *or* $0 < r < 2,\ \sigma = \frac{1}{2}$ *(the Crank-Nicolson scheme);*

Then, $\exists c_{10} : 0 < c_{10} < \infty$, *such that*

$$\|z\|_{l^\infty(\Sigma^j_{hd})} \le c_{10}\left(h^s\| \, \|f\|_{A^s_{2\infty}[0,1]}\|_{A_{2,d}[0,t]} + d^r\| \, \|f\|_{A^r_{2\infty}[0,t]}\|_{A_{2,h}[0,1]}\right).$$

Corollary 11. *Under the conditions of Corollary 10,*
$\exists c_{11} : 0 < c_{11} < \infty$, *such that*

$$\|z\|_{l^\infty(\Sigma^j_{hd})} \le c_{11}\left(h^s\| \, \|f\|_{B^s_{p(s)q(s)}[0,1]}\|_{A_{2,d}[0,t]} + d^r\| \, \|f\|_{B^r_{p(r)q(r)}[0,t]}\|_{A_{2,h}[0,1]}\right),$$

where

$$p(\alpha) = \begin{cases} \alpha^{-1}, & 0 < \alpha < \frac{1}{2}, \\ 2, & \alpha = \frac{1}{2}, \\ 2, & \frac{1}{2} < \alpha < 2, \end{cases}$$

$$q(\alpha) = \begin{cases} 1, & 0 < \alpha < \frac{1}{2}, \\ 1, & \alpha = \frac{1}{2}, \\ \infty, & \frac{1}{2} < \alpha < 2, \end{cases}$$

and where c_{11} *can be selected independently of* s *and* r.

Remark 5. The values of the constants c_j, $j = 1, \dots, 11$, in Corollaries 1–11 can be estimated via Theorem 2.

Remark 6. (See also Remarks 1, 3, 4, 12.) In the special case of higher order of accuracy for the explicit scheme with $\sigma = 0$ in [25, Theorem 2], as discussed in Remarks 1, 3, 4 above, it is possible to obtain a variety of additional corollaries proving higher order of convergence (up to order 4) in terms of regularity properties of the RHS f similar to the ones considered in Corollaries 1–11 but referring to respective partial derivatives of f which are two orders higher than the ones considered in Corollaries 1–11.

Remark 7. In Corollaries 1–11 we considered the case of continually changing fractional (non-integer) partial smoothness indices s and r only in Corollaries 10, 11, in relevance to A-spaces and Besov spaces. It is possible to generalize all Corollaries 1–9 from the case of integer partial smoothness indices in x and t, as is currently the case, to the more general case of continually changing non-integer partial smoothness indices s and r, similarly to Corollaries 10, 11, where the L_2-norm in the RHSs of the estimates will be upgraded to Triebel-Lizorkin F^{ϱ}_{22}-norm for appropriate non-integer ϱ, the current L_2-case corresponding to $\varrho = 0$. In some of the upgraded corollaries, the integer-order partial derivatives can be replaced by respective fractional derivatives (e.g., two-sided Marchaud fractional derivatives in x or t living on [0,1] or [0,T], respectively). It would be too spacious to include the technical elaboration of these upgrades here.

Remark 8. Another way to obtain upgrades of Corollaries 1, 2, 4, 8 and 9 involving intermediate orders of convergence which are not multiples of 1/2 is to consider Wiener-Young p-variation for $p \neq 2$ in the place of the quadratic variation (corresponding to $p = 2$) considered everywhere in these corollaries.

5. Concluding Remarks

Remark 9. It is well-known that Ω is a domain with Lipschitz boundary (see, e.g., [33, 30, 1, 11]) and

$$\|f\|_{W_p^r(\Omega)} \sim \|f\|_{L_p(\Omega)} + \|\frac{\partial^r f}{\partial x^r}\|_{L_p(\Omega)} + \|\frac{\partial^r f}{\partial t^r}\|_{L_p(\Omega)}.$$

Remark 10. Attempts to obtain the a priori estimates in quest via direct estimation of the series in (20), (21) and their derivatives using Plancherel's theorem in L_2 lead to the appearance of $\int\limits_0^t \frac{(\cdot)}{\tau} d\tau$ in the right-hand sides of (13–19), which, on its part, leads to loss of logarithmic order of approximation in Section 4.3.

Remark 11. (Cf. also [4, Chapter 3], [24, Remark 12].) The Cauchy-Dirichlet problem (1) with initial condition g and boundary conditions g_0, g_1, can be reduced to a problem of the same type, but with homogeneous initial and boundary conditions and a new right-hand side $\tilde{f} = \tilde{f}(f, g, g_0, g_1)$ under very general assumptions on f, g, g_0, g_1, for example: g_0, g_1, g are measurable; g_0, g_1, g may have points of discontinuity of only the first kind; these points form sets with measure zero; consistency conditions $g(0+) = g_0(0+)$, $g(1-) = g_1(0+)$ hold true. (Here we use standard notation $\rho(\xi_0+)$, $\rho(\xi_0-)$ for the right-hand and left-hand limit of the univariate function ρ, respectively, at its argument value ξ_0.)

Indeed, assume that under these conditions it is possible to find a function $\Phi = \Phi(f, g, g_0, g_1)$, defined and continuous on the interior of Ω, such that:

- $\frac{\partial\Phi}{\partial t}, \frac{\partial^2\Phi}{\partial x^2}$ exist and have finite values a.e. on the interior of Ω;
- On the boundary,
  - $\Phi(x,t) \to g(x_0)$ for $(x,t) \to (x_0, 0)$, $\Phi(x,t) \to g_j(t_0)$ for $(x,t) \to (j, t_0)$, $j = 0, 1$, if x_0, t_0 are continuity points of g, g_j, respectively;
  - $\Phi(x,t) \to g(0+)$, $(x,t) \to (0,0)$;
  - $\Phi(x,t) \to g(1-)$, $(x,t) \to (1,0)$.

Then, the general Dirichlet problem (i.e., with inhomogeneous initial and boundary conditions) is being reduced to

$$\begin{cases} \frac{\partial\tilde{u}}{\partial t} - \frac{\partial^2\tilde{u}}{\partial x^2} = \tilde{f} = f - \frac{\partial\Phi}{\partial t} + \frac{\partial^2\Phi}{\partial x^2}, & \text{a.e. in } \Omega, \\ \tilde{u}(x,0) = 0, & \text{a.e. in } [0,1], \\ \tilde{u}(j,t) = 0, & \text{a.e. in } [0,t]. \end{cases}$$

The solution $\tilde{u}$ is continuous on the interior of Ω, because $\tilde{u} = u - \Phi$ and the functions u, Φ are continuous there.

This approach was proposed in 1986 during the work on [4, Chapter 3] as a PDE-related analogue to, and upgrade of, the approach to reducing the inhomogeneous boundary-value problem for second-order ODEs to the homogeneous case, proposed in [24]. In [4, Chapter 3] we went further to propose the following example of a way to define Φ with the above-said properties:

$$\Phi(x,t) = \Theta(x)(g)_{2,\frac{t}{6T}}(x) + (1-\Theta(x))(g)_{2,-\frac{t}{6T}}(x) +$$

$$+ \Theta(\frac{t}{T})\Theta(x)(g_0)_{2,\frac{Tx}{6}}(t) + (1-\Theta(\frac{t}{T}))\Theta(x)(g_0)_{2,-\frac{Tx}{6}}(t) +$$

$$+ \Theta(\frac{t}{T})(1-\Theta(x))(g_1)_{2,\frac{T(1-x)}{6}}(t) + (1-\Theta(\frac{t}{T}))(1-\Theta(x))(g_1)_{2,-\frac{T(1-x)}{6}}(t) -$$

$$- \Theta(x)(g_0)_{2,\frac{Tx}{6}}(0) + (1-\Theta(x))(g_1)_{2,-\frac{T(1-x)}{6}}(0) -$$

$$- \Theta(x)(g)_{2,\frac{t}{6T}}(0) - (1-\Theta(x))(g)_{2,-\frac{t}{6T}}(1) +$$

$$+ \Theta(x)g(0+) + (1-\Theta(x))g(1-), \quad (44)$$

where

- Θ is the smooth blending function in Brudnyi's definition for Steklov-means on an interval (see Section 3 and the references therein):

  * $\Theta \in C^\infty[0,1]$;
  * $\Theta\,|_{[0,\frac{1}{3}]} \equiv 1;\ \ \Theta\,|_{[\frac{2}{3},1]} \equiv 0,\ \ 0 < \Theta(x) < 1,\ x \in (\frac{1}{3}, \frac{2}{3})$;

- $(\varphi)_{2,\alpha}(\beta)$ is the (non-modified) Steklov means of order 2, with step α:

$$\begin{aligned}(\varphi)_{2,\alpha}(\beta) &= \tfrac{1}{\alpha^2}\int_0^\alpha\int_0^\alpha(-\varphi(\beta+\xi_1+\xi_2) + 2\varphi(\beta+\tfrac{\xi_1+\xi_1}{2}))d\xi_1 d\xi_2 = \\ &= -\tfrac{1}{\alpha^2}\int_\beta^{\beta+\frac{\alpha}{2}}\int_{\eta_2}^{\eta_2+\frac{\alpha}{2}}\varphi(\eta_1)d\eta_1 d\eta_2 + \tfrac{8}{\alpha^2}\int_\beta^{\beta+\alpha}\int_{\eta_2}^{\eta_2+\alpha}\varphi(\eta_1)d\eta_1 d\eta_2\end{aligned}$$

It can be verified that Φ in (44) is correctly defined on the interior of Ω and is continuous there. Based on the continuity of Φ and the properties of the Steklov mean, it can be also established that the initial and boundary conditions are satisfied. Finally, it can be checked that $\frac{\partial\Phi}{\partial t}, \frac{\partial^2\Phi}{\partial x^2}$ exist, are measurable, and have finite values on the interior of Ω, whenever $g,\ g_0,\ g_1$ have the same properties in $[0,1], [0,T]$, respectively.

We can apply Theorem 2 to the resulting problem with homogeneous initial and boundary conditions, provided that $f - \frac{\partial\Phi}{\partial t} + \frac{\partial^2\Phi}{\partial x^2}$ is bounded in Ω_t. In connection with this, it is interesting to note that this construction opens the opportunity to construct examples of $f,\ g,\ g_0,\ g_1$, for which $\frac{\partial\Phi}{\partial t} - \frac{\partial^2\Phi}{\partial x^2}, f \notin L_2(\Omega_t)$ but, nevertheless, the conditions of Theorem 2 continue to hold.

Finally, we note that the model construction (44) was proposed as early as in 1986. Since then the author found out that it is possible to design a simpler construction based on a combination of Boolean-sum interpolation on the boundary of the (hyper)rectangular domain and simpler, one-sided, Steklov means (see also [24, Remark 12]). It will be this simpler construction that we shall be considering in the next paper [26] in this sequence.

Remark 12. (ODEs, PDEs and the 'curse of dimensionality'. See also Remarks 1, 3, 4, 6, 11.)

- The extended Lax principle allows analogous treatment of boundary problems for ODEs and PDEs alike.

- However, the analogous structures in the PDE case are considerably more complex, even in the case of the relatively simple cartesian product domains which are a natural class of domains to consider when studying finite difference methods.

- Not surprisingly, the respective methods needed for the design of these constructions become much more technically involved in the PDE case.

- Ultimately, in the PDE case the exposition of the results of the application of the extended Lax principle becomes much more spacious and considerably more complex.

  * In the ODE case, we saw in [23, 24] that already the case of a relatively simple second-order linear ODE with variable coefficients required a fairly spacious exposition of the intermediate results at Stage 5 and the final results at Stage 6 of the application of the extended Lax principle.
  * This is the main reason which motivated us to consider in the model application of the extended Lax principle to PDEs a second-order linear PDE with constant coefficients and only a variable right-hand side. The resulting exposition, as we saw in the present paper, was comparable in size with the exposition in the analogous paper [23] for the ODE case, but with variable coefficients and right-hand side.
  * The same refers also to another important relevant topic: the reduction of inhomogeneous boundary-value problems for linear differential equations to homogeneous boundary-value problems for the same linear differential equations with a new right-hand side.
    + For the ODE case such a study was conducted in [24] where we saw that the reduction was relatively simple to implement, due to the full ordering of $\mathbb{R}^1$.
    + In the analogous paper [26] dedicated to the PDE case (cf. also Remark 11 for an early version) we shall see that already in the case of a simple rectangular domain in $\mathbb{R}^2$ the above-said reduction is considerably more technically challenging to implement.
  * Another aspect in which the higher multidimensional complexity of the PDE case immediately makes an impact, is the study of special cases of increased order of convergence.
    + In the ODE case, in [22, 23, 24] we considered the special case of order of convergence of 'the best' scheme in full detail.

+ In the PDE case, in [25], here and in [26] we study a simpler differential equation - with constant coefficients and only variable RHS.

  - Yet, even in this relatively simpler case, we had to limit our investigation to only one special case (see [25, Theorem 2]) for the explicit scheme $\sigma = 0$ only, for a special ratio between the step h in the space variable x and the step d in the time variable t, and for increased smoothness of the RHS. Even under these limitations, the study was not as comprehensive as in the ODE case. The additional limitations are as, follows.

    a. We conducted the study in full detail only up to Stage 4 of the extended Lax principle.

    b. A conduction of the study to the same level of detail at Stages 5 and 6 of the extended Lax paradigm would lead to doubling of the volume of the present paper and [26]. What is needed, is a separate study of Stages 5 and 6 of this special case in two subsequent publications.

  - It remains to study the same problem for increased order of smoothness in the case of implicit schemes $0 < \sigma \leq 1$, with further additional special considerations for the Crank-Nicolson scheme $\sigma = 1/2$. Complementing the study in [25], the present paper and [26] with consideration of all these cases in full detail would essentially result in quadrupling the volume of the study, while still not investigating the case of variable coefficients in the PDE.

Remark 13. A preliminary announcement, without proofs or details, of part of the results of [4] and the present paper was published in [16].

Remark 14. The method developed and the results obtained in this paper (as well as in the previous paper [25] and in the next paper [26] on this topic) are an application of the general theory developed in [17, 18].

Remark 15. The material in this paper (as well as in the previous paper [25] and in the next paper [26] on this topic) covers the part of the previously unpublished results in [4, Chapter 3] related to PDEs and complementary to the unpublished results in [4, Chapter 3] about ODEs which appear in [22, 23, 24].

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