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ON AVERAGE LOWER TOTAL INDEPENDENCE NUMBER OF A GRAPH

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Abstract: Computer or communication networks are so designed that they do not easily get disrupted under external attack and, moreover, these are easily reconstructible if they do get disrupted. These desirable properties of networks can be measured by various parameters like connectivity, toughness and integrity. In this paper we defined and examined the average lower total independence number of a connected graph as a new global graph parameter to describe the stability of communication networks. The average lower total independence number of a graph $i_{TAV}(G)$ is defined as $i_{TAV}(G)=\frac{1}{|V|}\sum_{v\in V} i_{Tv}(G)$, where $i_{Tv}(G)$ is the minimum cardinality of a maximal total independent set of G that contains v. In this paper, we consider the average lower total independence number of special graphs and a binomial tree.

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1. Introduction

The vulnerability of communication network, measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. Cable cuts, node interruptions, software errors or hardware

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failures and transmission failure at various points can be caused interrupt service for long periods of time. High levels of service dependability have traditionally characterized communication services. In a communication networks, requiring greater degrees of stability or less vulnerability. The stability and reliability of a network are of prime importance to network designers. If the communication network is modeled as a simple, undirected, connected and unweighted graph G deterministic measures tend to provide a worst-case analysis of some expects of the overall disconnection process.

A graph G is denoted by $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ are vertex and edge sets of G, respectively. n denotes the number of vertices and m denotes the number of edges of the graph G. The reliability of a graph can be measured by various parameters. The best known measure of reliability of a graph is its connectivity, defined to be the minimum number of vertices whose deletion results in a disconnected or trivial graph (the latter applying only to complete graphs). $k(G)$ denotes the connectivity of graph G. This parameter has been extensively studied.

As the connectivity is a worst-case measure, some of which are integrity (see [4]), toughness (see [7]), neighbour-integrity (see [3]-[11]), it does not always reflect what happens throughout the graph.

In this paper we investigate the average lower total independence number, a new measure for reliability and stability of a graph.

Other average parameters have been found to be more useful in some circumstances then the corresponding measures, such as average connectivity, average degree, average lower independence number and average distance of a graph (see [5]-[6]). For example, the average distance between vertices in graph was introduced as a tool in architecture and later turned out to be more valuable than the diameter when analyzing transportation networks.

We shall now define total independent number, lower total independent number and average lower total independence number for a graph.

Definition 1.1. Let $G(V,E)$ be a graph and $S \subseteq E \cup V$. Then S is called a total independent set of G if any two elements of S are neither adjacent nor incident. A maximal total independent set is a total independent set S with the property that any set properly containing S is not total independent. Let S_T be a family of maximal total independent sets. The total independent number $\beta_T(G)$ is the maximum cardinality of maximal total independent set of G and it is defined as $\beta_T(G)=\max\{|S|; |S| \in S_T\}$ (see [1], [2], [13]).

Definition 1.2. In this paper we define that the lower total independent

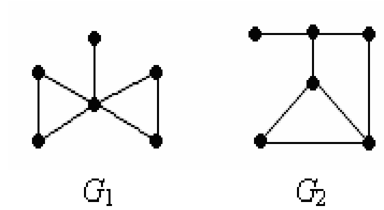


Figure 1: Graphs G_1 and G_2 with 6 vertices

number $i_T(G)$ is the minimum cardinality of maximal total independent set of G and it is defined as $i_T(G) = \min\{|S|; |S| \in S_T\}$. The lower total independent number containing v denoted by $i_{Tv}(G)$ is $i_{Tv}(G)$ =min$\{|S|; |S| \in S_T$ and $v\in S\}$. Then the set which has the minimum cardinality among all maximal total independent sets containing v of G, denoted S_{Tv} is called the lower total independent set that contains v.

For a vertex v of a graph G, the lower total independent number containing v, denoted $i_{Tv}(G)$ must be search for each one vertex v of graph G to compute average lower total independence number. Any two elements of the lower total independent set that contains v are neither adjacent nor incident. If a vertex is captured, then the adjacent vertices and incident edges are covered and become useless to the network as a whole or if an edge is captured, then incident vertices and adjacent edges are covered and these covered elements cannot be included in the lower total independent set S_{Tv} that contains v.

The average lower total independence number $i_{TAV}(G)$ of a graph $G(V,E)$ is defined as $i_{TAV}(G) = \frac{1}{|V|}\sum_{v\in V} i_{Tv}(G)$, where $i_{Tv}(G)$ is the minimum cardinality of a maximal total independent set of G that contains v.

As examples, we consider the two graphs in Figure 1. Both of these graphs with same number of vertices and egdes have connectivity 1 and $\beta_T(G_1) = \beta_T(G_2) = 4$. But the second would be a more reliable communication network than the first. This is reflected in the average lower total independence number since $i_{TAV}(G_1) = 3$ and $i_{TAV}(G_2) = \frac{17}{6}$. By using this example, we say that the stability of graph G_2 is more powerful than the stability of graph G_1. Because the average lower total independence number of the graph G_1 is greater than of the graph G_2. If we want to choose the more stable one from two graphs with the same number of vertices, it is enough to choose the one whose average lower total independence number is smaller. Consequently, whichever you study the graph parameter, if the result is small, then it means that the stability of the graph is big. In the vulnerability of the graph, we want this. And also, when we want to design a communications network, we wish that it is as possible as stable.

In Section 2, we consider the relationships between the average lower total independence number and other vulnerability parameters. In Section 3, we give some results between the average lower total independence number and some special graphs.

2. Relationships between the Average Lower Total Independence Number and Other Vulnerability Parameters

In this section, the relationships between the average lower total independence number and some other vulnerability parameters, namely the covering, dominating and accessibility numbers are established. Let u be a vertex of G. Let

$$N(u) = \{v \in V(G) | u \neq v,\ v \text{ and } u \text{ are adjacent}\}$$

denote the open neighborhood of u and let $N[u] = u \cup N(u)$ denote the closed neighborhood of u.

Definition 2.1. (see [12]) Two vertices are said to cover each other in a graph G if they are incident in G. A vertex cover in G is a set of vertices that covers all edges of G. The minimum cardinality of a vertex cover in a graph G is called the vertex *covering number* of G and is denoted by $\alpha(G)$.

Definition 2.2. (see [12]) An independent set of vertices of a graph G is a set of vertices of G whose elements are pair wise nonadjacent. The *independent number* $\beta(G)$ of G is the maximum cardinality among all independent sets of vertices of G.

Definition 2.3. (see [8]) A vertex dominating set for a graph G is a set S of vertices such that every vertex of G either belongs to S or is adjacent to a vertex of S. The minimum cardinality of a vertex dominating set in a graph G is called the *dominating number* of G and is denoted by $\sigma(G)$. For every graph G, $\sigma(G) \leq \beta(G)$.

Definition 2.4. (see [10]) A subset S of $V(G)$ is called an accessible set of the graph G if each vertex $u \in \{V(G) - S\}$ is adjacent to $N[S]$, where $N[S]$ is the closed neighborhood of S. The *accessibility number* of G is defined as the minimal number of vertices over all accessible sets of G and is denoted by $\eta(G)$.

We give some basic results about these parameters for some special graphs in the following Table 1.

Graph	$\alpha(G)$	$\sigma(G)$	$\eta(G)$
K_n	$n-1$	1	1
C_n	$\lceil\frac{n}{2}\rceil$	$\lceil\frac{n}{3}\rceil$	$\lceil\frac{n}{5}\rceil$
P_n	$\lceil\frac{n}{2}\rceil$	$\lceil\frac{n}{3}\rceil$	$\lceil\frac{n}{5}\rceil$
$W_{1,n-1}$	$1+\lceil\frac{n-1}{2}\rceil$	1	1
$S_{1,n-1}$	1	1	1

Table 1:

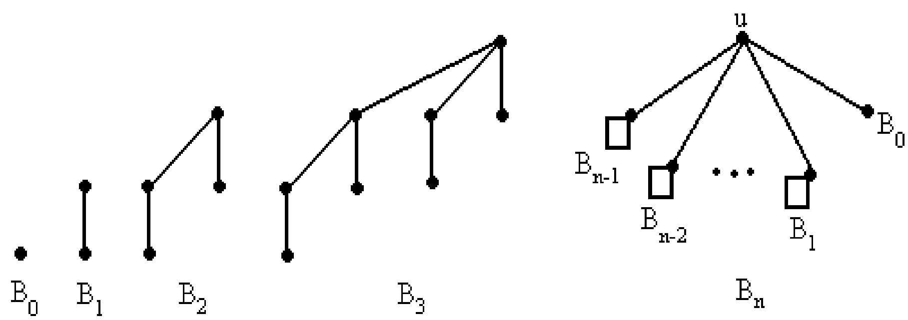


Figure 2: The general structure of the graphs B_0, B_1, B_2, B_3 and B_n

Corollary 2.1. *If G does not contain $2K_2$ as an induced subgraph and tree structure, then:*

a) $i_{TAV}(G) \leq \alpha(G) + 1$;

b) $i_{TAV}(G) \geq \sigma(G)$;

c) $i_{TAV}(G) \geq \eta(G)$.

3. Some Special Graphs and the Average Lower Total Independence Number

In this section, we give the average lower total independence number of special graphs and a binomial tree.

Definition 3.1. (see [9]) The binomial tree B_n is an ordered tree defined recursively. The binomial tree B_0 consists of a single vertex. The binomial tree B_n consists of two binomial trees B_{n-1} that are linked together: the root of the one is the leftmost child of the root of the other.

Theorem 3.1. *Let $n \geq 3$ and P_n be a path graph with n vertices. Then*

$$i_{TAV}(P_n) = \begin{cases} \frac{1}{n}\left(\lceil \frac{2n-1}{5} \rceil \ (n-1)+n\right), & n \mod 5 = 0, \\ \lceil \frac{2n-1}{5} \rceil, & n \mod 5 = 1, \\ \frac{1}{n}\left(n \lceil \frac{2n-1}{5} \rceil + \lfloor \frac{2n-1}{5} \rfloor\right), & n \mod 5 = 2, \\ \frac{2n+4}{5} - \frac{1}{n} \lceil \frac{n}{5} \rceil, & n \mod 5 = 3, \\ \frac{1}{n}\left(\lceil \frac{2n-1}{5} \rceil \ (n-2)+n\right), & n \mod 5 = 4. \end{cases}$$

Proof. S_{Tv} set (the lower total independent set that contains v) and the value of $i_{Tv}(G)$ must be search for each one vertex v of graph G to compute average lower total independence number. By the definition of parameter, when an element is included in S_{Tv} at most 5 elements (the number of elements of graph is the number of edges and vertices, $2n-1$) are covered which are vertex and edge in P_n path graph. This shows some differences according to n, the number of vertices of the graph. There are five cases.

Case 1. $n \mod 5 = 0$. If $n \mod 5 = 0$, then the number of the elements of S_{Tv} set including v is $\lceil \frac{2n-1}{5} \rceil$ for $\lceil \frac{2n-1}{5} \rceil$ vertex of graph. For the remaining vertices, this value is $\lceil \frac{2n-1}{5} \rceil + 1$. When we write this value in the definition of average lower total independence number, we obtain the following result:

$$i_{TAV}(P_n) = \frac{1}{n}(\lceil \frac{2n-1}{5} \rceil (n-1)+n).$$

Case 2. $n \mod 5 = 1$. The number of the elements of S_{Tv} set for every vertex in the graph P_n is the same. And this is $\lceil \frac{2n-1}{5} \rceil$. So the value of $i_{Tv}(G)$ for each one vertex is the same. Thus,

$$i_{TAV}(P_n) = \frac{1}{n} n \lceil \frac{2n-1}{5} \rceil = \lceil \frac{2n-1}{5} \rceil.$$

Case 3. $n \mod 5 = 2$. This situation is similar to Case 1. The value of $i_{Tv}(G)$ for $n - \lfloor \frac{2n-1}{5} \rfloor$ vertices is $\lceil \frac{2n-1}{5} \rceil$. For the remaining vertices, this value is $\lceil \frac{2n-1}{5} \rceil$ +1. Hence, we obtain

$$i_{TAV}(P_n) = \frac{1}{n}(n \lceil \frac{2n-1}{5} \rceil + \lfloor \frac{2n-1}{5} \rfloor).$$

Case 4. $n \mod 5 = 3$. If $n \mod 5 = 3$, then the number of the elements of S_{Tv} set for vertices, the number of which is $\lceil \frac{n}{5} \rceil$, in the graph P_n is $\lceil \frac{2n-1}{5} \rceil$. For the remaining vertices, this value is $\lceil \frac{2n-1}{5} \rceil + 1$. When we write this value in the definition of average lower total independence number, we obtain the following results:

$$i_{TAV}(P_n) = \frac{1}{n}(n\lceil\frac{2n-1}{5}\rceil + n - \lceil\frac{n}{5}\rceil) = \lceil\frac{2n-1}{5}\rceil + 1 - \frac{1}{n}\lceil\frac{n}{5}\rceil$$
$$= \frac{2n-1}{5} + 1 - \frac{1}{n}\lceil\frac{n}{5}\rceil = \frac{2n+4}{5} - \frac{1}{n}\lceil\frac{n}{5}\rceil.$$

Case 5. n mod $5 = 4$. In this situation the number of the elements of S_{Tv} set for vertices, the number of which is $2.\lceil\frac{2n-1}{5}\rceil$, in the graph P_n is $\lceil\frac{2n-1}{5}\rceil$. For the remaining vertices, this value is $\lceil\frac{2n-1}{5}\rceil + 1$. Hence, we obtain

$$i_{TAV}(P_n) = \frac{1}{n}(\lceil\frac{2n-1}{5}\rceil(n-2) + n).$$

The proof is completed. □

Theorem 3.2. *Let C_n be a cycle graph with n vertices. Then $i_{TAV}(C_n) = \lceil\frac{2n}{5}\rceil$.*

Proof. When we think of like previous theorem when an element is included in S_{Tv} at most 5 elements are covered which are vertex and edge in C_n cycle graph. The number of the elements of C_n graph is 2n and every vertex degree $\Delta(v)$ of graph is 2. Therefore, the number of the elements of S_{Tv} set for every vertices in the graph C_n is the same. Also, the value of for every vertices in the graph C_n is the same. And this is $\lceil\frac{2n}{5}\rceil$. Thus, we obtain

$$i_{TAV}(C_n) = \frac{1}{n}n\lceil\frac{2n}{5}\rceil = \lceil\frac{2n}{5}\rceil.$$

The proof is completed. □

Theorem 3.3. *Let $n \geq 3$ and $W_{1,n}$ be a wheel graph with $n+1$ vertices. Then $i_{TAV}(W_{1,n}) = \frac{1}{n+1}[(1 + \lceil\frac{n}{3}\rceil) + n(2 + \lceil\frac{2(n-4)}{5}\rceil)]$.*

Proof. S_{Tv} set and the value of $i_{Tv}(G)$ must be searched for each one vertex v of graph G to compute average lower total independence number. Here it will be enough to research two situations, the general structure of the graph $W_{1,n}$ is like Figure 3.

Case 1. For v_{n+1} vertex. Let us form $S_{Tv_{n+1}}$ for v_{n+1} vertex. v_{n+1} vertex is adjacent to all other vertices and incident to e_1, e_2,...,e_n edges. By the definition of parameter v_{n+1} vertex is included in $S_{Tv_{n+1}}$ and the vertices that are adjacent to v_{n+1} and the edge that are incident to v_{n+1} are covered. These covered elements cannot be included in $S_{Tv_{n+1}}$. The remaining graph structure will have d_1, d_2,..., d_n edges and no vertex. To cover these edges we must add an edge to $S_{Tv_{n+1}}$ set. When we add an edge to $S_{Tv_{n+1}}$ set at most 3 edges (selected edge and two other) are covered. Hence, the number of which is $\lceil\frac{n}{3}\rceil$

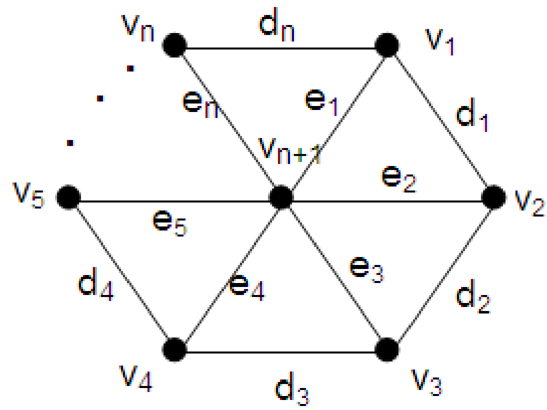
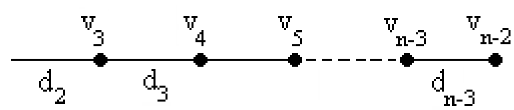


Figure 3: The general structure of the graph $W_{1,n}$



edges must be included in to $S_{Tv_{n+1}}$ cover all remaining graph structure d_1, d_2,...,d_n edges (this structure is a cycle without vertex). Therefore, the number of the elements of $S_{Tv_{n+1}}$ set for v_{n+1} vertex is $1+\lceil \frac{n}{3} \rceil$. So the value of $i_{Tv_{n+1}}(G)$ for v_{n+1} vertex is $1+\lceil \frac{n}{3} \rceil$.

Case 2. For v_1, v_2,...,v_n vertices. Let us form S_{Tv_1} for v_1 vertex. Vertex v_1 is adjacent v_2, v_n, v_{n+1} vertices. Therefore these vertices and the edges that are incident to v_1 are covered. The aim of parameter is to cover the graph with at least elements. To cover e_1, e_2,...,e_n edges it will be enough to add one of these edges in S_{Tv_1}. Let us select e_{n-1} edge. e_1, e_2,...,e_n edges, d_{n-1} and d_{n-2} edges and v_{n-1} vertex are covered. In this situation the remaining graph will have $n-4$ vertex and $n-4$ edge. Its structure will be like below.

When we continue to form S_{Tv_1} for v_1 vertex, with element vertex or edge of graph at most 5 elements are covered. $\lceil \frac{2(n-4)}{5} \rceil$ elements must be added in S_{Tv_1} to cover remaining graph structure. Therefore, the number of the elements of S_{Tv_1} set for v_1 vertex is $2+\lceil \frac{2(n-4)}{5} \rceil$. The number of the elements of S_{Tv} set for v_1, v_2,...,v_n vertices in the graph $W_{1,n}$ is the same. So the value of $i_{Tv}(G)$ for v_1, v_2,...,v_n vertices in the graph $W_{1,n}$ is the same and this value is $2+\lceil \frac{2(n-4)}{5} \rceil$. When we write this value in the definition of average lower total independence

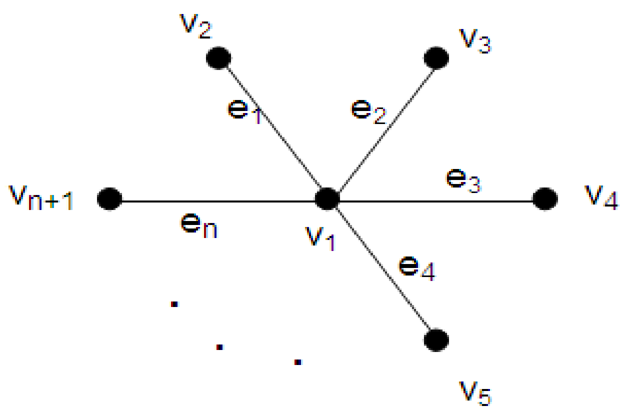


Figure 4: The general structure of the graph $S_{1,n}$

number, we obtain the following results:

$$i_{TAV}(W_{1,n}) = \frac{1}{n+1}[(1+\lceil\frac{n}{3}\rceil)+n(2+\lceil\frac{2(n-4)}{5}\rceil)].$$

The proof is completed. □

Theorem 3.4. *Let $S_{1,n}$ be a star graph with $n+1$ vertices. Then* $i_{TAV}(S_{1,n}) = \frac{n^2+1}{n+1}$.

Proof. The general structure of the graph $S_{1,n}$ is like Figure 4: We assume that v_1 vertex be center vertex in $S_{1,n}$ graph. S_{Tv} set and the value of $i_{Tv}(G)$ must be search for each one vertex v of graph G to compute average lower total independence number. Let us form S_{Tv_1} for v_1 vertex. v_1 vertex is adjacent to all other vertices and incident to all other edges. Therefore these elements are covered. Because of this; we obtain $S_{Tv_1} = \{v_1\}$ and $i_{Tv_1}(G)$ =1.

When we search S_{Tv_2} set for v_2 vertex; v_2 vertex covers v_1 vertex and e_1 edge. Therefore, when non covered any edge for example e_2 edge is included in S_{Tv_2} all the others edge and v_3 vertex are covered. The remaining graph structure will have only v_4, v_5,...,v_{n+1} vertices which are not adjacent. Because of definition, these vertices must be included in S_{Tv_2}. Therefore we obtain that S_{Tv_2} ={ v_2, e_2, v_4, v_5,...,v_{n+1} } and the number of the elements of S_{Tv_2} set for v_2 vertex is n. We obtain sets simillar to the other vertices of graph. So the number of the elements of sets for v_2, v_3,...,v_n, v_{n+1} vertices in the graph $S_{1,n}$ is the same. So the value of $i_{Tv}(G)$ for v_2, v_3,...,v_n, v_{n+1} vertices is the same and this value is n. Hence

$$i_{TAV}(S_{1,n}) = \frac{1}{n+1}(1.1 + n.n) = \frac{n^2+1}{n+1}.$$

The proof is completed. □

Theorem 3.5. *Let $n \geq 1$ and B_n be a binomial tree. Then $i_{TAV}(B_n) = 2^{n-1}$.*

Proof. The number of vertices of B_n binomial tree is 2^n. The value of $i_{Tv}(G)$ for every vertex in the graph B_n is the same. And this is 2^{n-1}. Hence, for $n = 0$, $i_{TAV}(B_0)$=1 and for $n \geq 1$

$$i_{TAV}(B_n) = \frac{1}{2^n} 2^n 2^{n-1} = 2^{n-1}.$$

The proof is completed. □

Corollary 3.1. *Let $n \geq 1$ and B_n be a binomial tree. Then $i_{TAV}(B_n) = 2 i_{TAV}(B_{n-1})$.*

4. Conclusion

If nodes and links of a communication network are destroyed, then its effectiveness decreases. Thus a communication network must be constructed so as to be as stable as possible, not only with respect to the initial disruption, but also with respect to a possible reconstruction of the network. The vulnerability of a communication network characterizes the resistively of the network to the disruption of some nodes or connection lines. In graph theory, many parameters measuring the vulnerability of communication networks have been defined. In this paper, we define a new stability parameter. We called it average lower total independence number. When we design two networks which have the same number of processors, if we want to choose the more stable one from two graphs with the same number of vertices and edges, it is enough to choose the one whose average lower total independence number is smaller.

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