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SOME CONVERGENT SEQUENCES AND SERIES

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Abstract: The purpose of this paper is to present a special class of convergent sequences and to find their limits, as well as to present a special class of series and to find their sums.

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1. Introduction

Pârşan [3] proposed in 1978, in the oldest Romanian mathematical journal Gazeta Matematică, the following problem.

Problem 1. Let $a, r \in (0, +\infty)$ and let $(a_n)_{n\in\mathbb{N}}$ be the sequence defined by $a_n = a + (n-1)r$, for each $n \in \mathbb{N}$. Evaluate $\lim\limits_{n\to\infty} \frac{a_{2n+1}a_{2n+3}\cdots a_{4n+1}}{a_{2n}a_{2n+2}\cdots a_{4n}}$. (The limit is $\sqrt{2}$.)

Sîntămărian [4, problems 442–444, pp. 63], [5], [6, problems 59–61, 63, 64, 67, pp. 19–22] gave generalizations of this problem together with other related interesting problems.

In Section 2 we shall present our results regarding the above-mentioned theme.

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Vernescu [7] obtained the sums of some convergent series, from which we mention, for example, the following two:

$$\sum_{n=1}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\cdot\frac{1}{n+1}=1$$

and

$$\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\cdot\frac{1}{n+1}=2\sqrt{2}-3.$$

The way of getting these sums inspired us in obtaining the results which we shall give in Section 3.

2. Some Convergent Sequences

Proposition 2. (see Sîntămărian [5], [6, problems 59–61, pp. 19, 20]) *Let* $k\in\mathbb{N}\cup\{0\}$, $p\in\mathbb{N}\setminus\{1\}$, $q,s\in\mathbb{N}$ *and* $a,r\in(0,+\infty)$. *We consider the sequence* $(a_n)_{n\in\mathbb{N}}$ *defined by* $a_n=a+(n-1)r$, *for each* $n\in\mathbb{N}$.

Then:

(i) $\displaystyle\lim_{n\to\infty}\frac{a_{qn+k+1}a_{qn+k+1+p}\cdots a_{qn+k+1+s(n-1)p}}{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}}=\sqrt[p]{\frac{ps+q}{q}}$;

(ii) $\displaystyle\lim_{n\to\infty}\sqrt[n]{\frac{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}}{(n!)^s}}=\left(\sqrt[p]{\frac{ps+q}{q}}\right)^q[(ps+q)r]^s$;

(iii) $\displaystyle\lim_{n\to\infty}\frac{\sqrt[n]{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}}}{n^s}=\left(\sqrt[p]{\frac{ps+q}{q}}\right)^q\left[\frac{(ps+q)r}{\mathrm{e}}\right]^s$.

Proof. (i) We have $a_na_{n+2}<a_{n+1}^2$, for each $n\in\mathbb{N}$. Hence

$$\prod_{i=1}^{p-1}a_{n+i-1}^{i}a_{n+i+1}^{i}<\prod_{i=1}^{p-1}a_{n+i}^{2i}$$

and

$$\prod_{i=1}^{p-1}a_{n+i-1}^{p-i}a_{n+i+1}^{p-i}<\prod_{i=1}^{p-1}a_{n+i}^{2(p-i)},$$

for any $n\in\mathbb{N}$. We get

$$a_na_{n+p}^{p-1}<a_{n+p-1}^{p}$$

and

$$a_n^{p-1}a_{n+p}<a_{n+1}^{p},$$

for each $n \in \mathbb{N}$.

Let $x_n = \frac{a_{qn+k+1}a_{qn+k+1+p}\cdots a_{qn+k+1+s(n-1)p}}{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}}$, for each $n \in \mathbb{N}$. We have

$$\begin{aligned} x_n^p &= a_{qn+k+1}^{p-1}(a_{qn+k+1}a_{qn+k+1+p}^{p-1})(a_{qn+k+1+p}a_{qn+k+1+2p}^{p-1})\cdots \\ &\quad \times(a_{qn+k+1+(s(n-1)-1)p}a_{qn+k+1+s(n-1)p}^{p-1})a_{qn+k+1+s(n-1)p} \\ &\quad /(a_{qn+k}^p a_{qn+k+p}^p a_{qn+k+2p}^p \cdots a_{qn+k+s(n-1)p}^p) \\ &< \frac{a_{qn+k+1}^{p-1}a_{qn+k+p}^p a_{qn+k+2p}^p \cdots a_{qn+k+s(n-1)p}^p a_{qn+k+1+s(n-1)p}}{a_{qn+k}^p a_{qn+k+p}^p a_{qn+k+2p}^p \cdots a_{qn+k+s(n-1)p}^p} \\ &= \frac{a_{qn+k+1}^{p-1}a_{qn+k+1+s(n-1)p}}{a_{qn+k}^p}, \end{aligned}$$

for any $n \in \mathbb{N} \setminus \{1\}$. Also, we have

$$\begin{aligned} x_n^p &= a_{qn+k+1}^p a_{qn+k+1+p}^p \cdots a_{qn+k+1+(s(n-1)-1)p}^p a_{qn+k+1+s(n-1)p}^p \\ &\quad /[a_{qn+k}(a_{qn+k}^{p-1}a_{qn+k+p})(a_{qn+k+p}^{p-1}a_{qn+k+2p})\cdots \\ &\quad \times(a_{qn+k+(s(n-1)-1)p}^{p-1}a_{qn+k+s(n-1)p})a_{qn+k+s(n-1)p}^{p-1}] \\ &> \frac{a_{qn+k+1}^p a_{qn+k+1+p}^p \cdots a_{qn+k+1+(s(n-1)-1)p}^p a_{qn+k+1+s(n-1)p}^p}{a_{qn+k}a_{qn+k+1}^p a_{qn+k+1+p}^p \cdots a_{qn+k+1+(s(n-1)-1)p}^p a_{qn+k+s(n-1)p}^{p-1}} \\ &= \frac{a_{qn+k+1+s(n-1)p}^p}{a_{qn+k}a_{qn+k+s(n-1)p}^{p-1}}, \end{aligned}$$

for any $n \in \mathbb{N} \setminus \{1\}$. So, we are able to write

$$\frac{a_{qn+k+1+s(n-1)p}^p}{a_{qn+k}a_{qn+k+s(n-1)p}^{p-1}} < x_n^p < \frac{a_{qn+k+1}^{p-1}a_{qn+k+1+s(n-1)p}}{a_{qn+k}^p},$$

for each $n \in \mathbb{N} \setminus \{1\}$. It follows that $\lim_{n\to\infty} x_n^p = \frac{ps+q}{q}$, because

$$\lim_{n\to\infty} \frac{a_{qn+k+1+s(n-1)p}^p}{a_{qn+k}a_{qn+k+s(n-1)p}^{p-1}} = \frac{ps+q}{q}$$

and

$$\lim_{n\to\infty} \frac{a_{qn+k+1}^{p-1}a_{qn+k+1+s(n-1)p}}{a_{qn+k}^p} = \frac{ps+q}{q}.$$

Therefore $\lim_{n\to\infty} x_n = \sqrt[p]{\frac{ps+q}{q}}$.

(*ii*) Let $y_n = \frac{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}}{(n!)^s}$, for each $n \in \mathbb{N}$. We have

$$\begin{aligned}
\frac{y_{n+1}}{y_n} &= \frac{a_{q(n+1)+k}a_{q(n+1)+k+p}\cdots a_{q(n+1)+k+snp}}{[(n+1)!]^s} \\
&\quad \times \frac{(n!)^s}{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}} \\
&= \frac{a_{qn+q+k}a_{qn+q+k+p}\cdots a_{qn+q+k+s(n-1)p}}{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}} \\
&\quad \times \frac{a_{qn+q+k+s(n-1)p+p}a_{qn+q+k+s(n-1)p+2p}\cdots a_{qn+q+k+snp}}{(n+1)^s} \\
&= \frac{a_{qn+q+k}a_{qn+q+k+p}\cdots a_{qn+q+k+s(n-1)p}}{a_{qn+q+k-1}a_{qn+q+k-1+p}\cdots a_{qn+q+k-1+s(n-1)p}} \\
&\quad \times \frac{a_{qn+q+k-1}a_{qn+q+k-1+p}\cdots a_{qn+q+k-1+s(n-1)p}}{a_{qn+q+k-2}a_{qn+q+k-2+p}\cdots a_{qn+q+k-2+s(n-1)p}}\cdots \\
&\quad \times \frac{a_{qn+k+1}a_{qn+k+1+p}\cdots a_{qn+k+1+s(n-1)p}}{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}} \\
&\quad \times \frac{a_{qn+q+k+s(n-1)p+p}}{n+1}\cdot\frac{a_{qn+q+k+s(n-1)p+2p}}{n+1}\cdots \\
&\quad \times \frac{a_{qn+q+k+s(n-1)p+sp}}{n+1},
\end{aligned}$$

for any $n \in \mathbb{N}$. Using the limit from part (*i*), it follows that

$$\lim_{n\to\infty}\frac{y_{n+1}}{y_n} = \left(\sqrt[p]{\frac{ps+q}{q}}\right)^q [(ps+q)r]^s.$$

Now, according to the Cauchy-d'Alembert criterion, we are able to write

$$\lim_{n\to\infty}\sqrt[n]{y_n} = \lim_{n\to\infty}\frac{y_{n+1}}{y_n} = \left(\sqrt[p]{\frac{ps+q}{q}}\right)^q [(ps+q)r]^s.$$

(*iii*) For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\frac{\sqrt[n]{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}}}{n^s} \\
&\qquad = \sqrt[n]{\frac{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}}{n^{ns}}}.
\end{aligned}$$

Let $z_n = \frac{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}}{n^{ns}}$, for each $n \in \mathbb{N}$. We have

$$\begin{aligned}
\frac{z_{n+1}}{z_n} &= \frac{a_{q(n+1)+k}a_{q(n+1)+k+p}\cdots a_{q(n+1)+k+snp}}{(n+1)^{(n+1)s}} \\
&\quad \times \frac{n^{ns}}{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}}
\end{aligned}$$

$$= \frac{a_{qn+q+k}a_{qn+q+k+p}\cdots a_{qn+q+k+s(n-1)p}}{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}} \times \frac{a_{qn+q+k+s(n-1)p+p}a_{qn+q+k+s(n-1)p+2p}\cdots a_{qn+q+k+snp}}{(n+1)^s} \times \frac{1}{\left[\left(1+\frac{1}{n}\right)^n\right]^s},$$

for any $n \in \mathbb{N}$.

Having in view the proof of part (*ii*), it follows that

$$\lim_{n\to\infty} \frac{z_{n+1}}{z_n} = \left(\sqrt[p]{\frac{ps+q}{q}}\right)^q \left[\frac{(ps+q)r}{\mathrm{e}}\right]^s.$$

Now, according to the Cauchy-d'Alembert criterion, we are able to write

$$\lim_{n\to\infty} \sqrt[n]{z_n} = \lim_{n\to\infty} \frac{z_{n+1}}{z_n} = \left(\sqrt[p]{\frac{ps+q}{q}}\right)^q \left[\frac{(ps+q)r}{\mathrm{e}}\right]^s. \qquad \square$$

Remark 3. We present a few particular cases of the results given in Proposition 2.

(i_1) For $a = r = 1$, $k = 0$, $p = 3$ and $q = s = 2$ we get

$$\lim_{n\to\infty} \frac{(2n+1)(2n+4)\cdots(8n-5)}{(2n)(2n+3)\cdots(8n-6)} = \sqrt[3]{4}.$$

(i_2) For $a = r = 1$, $k = 0$, $p = 3$, $q = 2$ and $s = 3$ we get

$$\lim_{n\to\infty} \frac{(2n+1)(2n+4)\cdots(11n-8)}{(2n)(2n+3)\cdots(11n-9)} = \sqrt[3]{\frac{11}{2}}.$$

(i_3) For $a = r = 1$, $k = 0$, $p = 5$, $q = 3$ and $s = 2$ we get

$$\lim_{n\to\infty} \frac{(3n+1)(3n+6)\cdots(13n-9)}{(3n)(3n+5)\cdots(13n-10)} = \sqrt[5]{\frac{13}{3}}.$$

(i_4) For $a = r = 1$, $k = 1$, $p = 7$ and $q = s = 1$ we get

$$\lim_{n\to\infty} \frac{(n+2)(n+9)\cdots(8n-5)}{(n+1)(n+8)\cdots(8n-6)} = \sqrt[7]{8}.$$

(ii_1) For $a = r = 1$, $k = 0$, $p = 3$ and $q = s = 2$ we get

$$\lim_{n\to\infty} \sqrt[n]{\frac{(2n)(2n+3)\cdots(8n-6)}{(n!)^2}} = 128\sqrt[3]{2}.$$

(ii_2) For $a = r = 1$, $k = 0$, $p = 3$, $q = 2$ and $s = 3$ we get

$$\lim_{n\to\infty} \sqrt[n]{\frac{(2n)(2n+3)\cdots(11n-9)}{(n!)^3}} = 11^3 \left(\sqrt[3]{\frac{11}{2}}\right)^2.$$

(ii_3) For $a = r = 1$, $k = 0$, $p = 5$, $q = 3$ and $s = 2$ we get

$$\lim_{n\to\infty} \sqrt[n]{\frac{(3n)(3n+5)\cdots(13n-10)}{(n!)^2}} = 13^2 \left(\sqrt[5]{\frac{13}{3}}\right)^3.$$

(ii_4) For $a = r = 1$, $k = 1$, $p = 7$ and $q = s = 1$ we get

$$\lim_{n\to\infty} \sqrt[n]{\frac{(n+1)(n+8)\cdots(8n-6)}{n!}} = 8\sqrt[7]{8}.$$

(iii_1) For $a = r = 1$, $k = 0$, $p = 3$ and $q = s = 2$ we get

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n)(2n+3)\cdots(8n-6)}}{n^2} = \frac{128\sqrt[3]{2}}{e^2}.$$

(iii_2) For $a = r = 1$, $k = 0$, $p = 3$, $q = 2$ and $s = 3$ we get

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n)(2n+3)\cdots(11n-9)}}{n^3} = \left(\frac{11}{e}\right)^3 \left(\sqrt[3]{\frac{11}{2}}\right)^2.$$

(iii_3) For $a = r = 1$, $k = 0$, $p = 5$, $q = 3$ and $s = 2$ we get

$$\lim_{n\to\infty} \frac{\sqrt[n]{(3n)(3n+5)\cdots(13n-10)}}{n^2} = \left(\frac{13}{e}\right)^2 \left(\sqrt[5]{\frac{13}{3}}\right)^3.$$

(iii_4) For $a = r = 1$, $k = 1$, $p = 7$ and $q = s = 1$ we get

$$\lim_{n\to\infty} \frac{\sqrt[n]{(n+1)(n+8)\cdots(8n-6)}}{n} = \frac{8\sqrt[7]{8}}{e}.$$

Remark 4. (see Sîntămărian, [4, Problem 442, p. 63]) proposed the following problem, which is a particular case of the result given in part (i) of Proposition 2.

Let $p \in \mathbb{N} \setminus \{1\}$ and $q \in \mathbb{N}$. Evaluate

$$\lim_{n\to\infty} \left(\frac{qn+1}{qn} \cdot \frac{qn+p+1}{qn+p} \cdots \frac{qn+np+1}{qn+np} \right).$$

On February 12-th, 2008, Professor Dorian Popa communicated with the author about a beautiful solution of this problem, a solution which we shall present further on.

Let $x_n = \frac{qn+1}{qn} \cdot \frac{qn+p+1}{qn+p} \cdots \frac{qn+np+1}{qn+np}$, for each $n \in \mathbb{N}$. We have

$$\ln x_n = \sum_{k=0}^{n} \ln \frac{qn+kp+1}{qn+kp} = \sum_{k=0}^{n} \frac{1}{qn+kp} \cdot \frac{\ln\left(1+\frac{1}{qn+kp}\right)}{\frac{1}{qn+kp}},$$

for any $n \in \mathbb{N}$.

We consider the function $f : (0, +\infty) \to \mathbb{R}$, defined by $f(x) = \frac{\ln(1+x)}{x}$, for each $x \in (0, +\infty)$. Because f is strictly decreasing on $(0, +\infty)$, we are able to write

$$\frac{\ln\left(1 + \frac{1}{qn}\right)}{\frac{1}{qn}} \le \frac{\ln\left(1 + \frac{1}{qn+kp}\right)}{\frac{1}{qn+kp}} \le \frac{\ln\left(1 + \frac{1}{qn+np}\right)}{\frac{1}{qn+np}},$$

for any $k \in \{0, 1, \dots, n\}$, for each $n \in \mathbb{N}$. So, we have

$$\frac{\ln\left(1 + \frac{1}{qn}\right)}{\frac{1}{qn}} \sum_{k=0}^{n} \frac{1}{qn + kp} \le \ln x_n \le \frac{\ln\left(1 + \frac{1}{qn+np}\right)}{\frac{1}{qn+np}} \sum_{k=0}^{n} \frac{1}{qn + kp},$$

for each $n \in \mathbb{N}$. It follows that

$$\lim_{n\to\infty} \ln x_n = \frac{1}{p} \ln \frac{p + q}{q},$$

because

$$\lim_{n\to\infty} \frac{\ln\left(1 + \frac{1}{qn}\right)}{\frac{1}{qn}} = 1, \qquad \lim_{n\to\infty} \frac{\ln\left(1 + \frac{1}{qn+np}\right)}{\frac{1}{qn+np}} = 1,$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{qn + kp} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \frac{1}{q + p\frac{k}{n}} = \int_0^1 \frac{1}{q + px}\, dx = \frac{1}{p} \ln \frac{q + p}{q}.$$

Therefore $\lim\limits_{n\to\infty} x_n = \sqrt[p]{\frac{p+q}{q}}$.

Further on we shall give problems in the solving of which the limit from part (i) of Proposition 2 can be applied.

Problem 5. (see Sîntămărian [4, Problem 443, p. 63], [6, Problem 63, p. 20]) Evaluate $\lim\limits_{n\to\infty} \frac{\binom{4n}{2n}}{4^n \binom{2n}{n}}$.

Solution. We have

$$x_n := \frac{\binom{4n}{2n}}{4^n \binom{2n}{n}} = \frac{(2n + 1)(2n + 3) \cdots (4n - 1)}{(2n + 2)(2n + 4) \cdots (4n)},$$

for each $n \in \mathbb{N}$. Choosing $a = r = 1$, $k = 1$, $p = 2$, $q = 2$ and $s = 1$ in part (i)

of Proposition 2, we obtain

$$\lim_{n\to\infty} \frac{(2n+2)(2n+4)\cdots(4n)}{(2n+1)(2n+3)\cdots(4n-1)} = \sqrt{2}.$$

So, $\lim_{n\to\infty} x_n = \frac{\sqrt{2}}{2}$.

Remark 6. We can solve Problem 5 using the Wallis formula (see Knopp, [2, p. 384])

$$\lim_{n\to\infty} \left(\frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)} \right)^2 \cdot \frac{1}{2n+1} = \frac{\pi}{2}.$$

We have

$$\begin{aligned} &\lim_{n\to\infty} \frac{(2n+1)(2n+3)\cdots(4n-1)}{(2n+2)(2n+4)\cdots(4n)} \\ &= \lim_{n\to\infty} \left(\frac{1\cdot 3\cdots(4n-1)}{2\cdot 4\cdots(4n)} \cdot \sqrt{4n+1} \times \frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)} \cdot \frac{1}{\sqrt{2n+1}} \cdot \frac{\sqrt{2n+1}}{\sqrt{4n+1}} \right) \\ &= \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{\pi}{2}} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}. \end{aligned}$$

Problem 7. (see Sîntămărian, [4, Problem 444, p. 63]) Evaluate

$$\lim_{n\to\infty} \frac{4^n \binom{4n}{2n}^2}{\binom{2n}{n}\binom{8n}{4n}}.$$

Solution. We have

$$y_n := \frac{4^n \binom{4n}{2n}^2}{\binom{2n}{n}\binom{8n}{4n}} = \frac{(4n+2)(4n+4)\cdots(8n)}{(4n+1)(4n+3)\cdots(8n-1)} \cdot x_n,$$

for each $n \in \mathbb{N}$, where $(x_n)_{n\in\mathbb{N}}$ is the sequence from (the solution of) Problem 5. Choosing $a = r = 1$, $k = 1$, $p = 2$, $q = 4$ and $s = 2$ in part (i) of Proposition 2, we obtain

$$\lim_{n\to\infty} \frac{(4n+2)(4n+4)\cdots(8n-2)}{(4n+1)(4n+3)\cdots(8n-3)} = \sqrt{2}.$$

So, $\lim_{n\to\infty} y_n = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 1$.

Remark 8. The limit $\lim_{n\to\infty} \frac{(4n+2)(4n+4)\cdots(8n-2)}{(4n+1)(4n+3)\cdots(8n-3)} = \sqrt{2}$ can be also found using the Wallis formula.

Problem 9. (see Sîntămărian, [6, Problem 64, p. 20]) Evaluate

$$\lim_{n\to\infty} \frac{3^{3n}\binom{2n}{n}^2}{\binom{3n}{n}\binom{6n}{3n}}.$$

Solution. We have

$$z_n := \frac{3^{3n}\binom{2n}{n}^2}{\binom{3n}{n}\binom{6n}{3n}}$$

$$= \left[\frac{(3n+3)(3n+6)\cdots(6n)}{(3n+2)(3n+5)\cdots(6n-1)}\right]^2 \frac{(3n+2)(3n+5)\cdots(6n-1)}{(3n+1)(3n+4)\cdots(6n-2)},$$

for each $n \in \mathbb{N}$. Choosing $a = r = 1$, $k = 2$, $p = 3$, $q = 3$ and $s = 1$ in part (i) of Proposition 2, we obtain

$$\lim_{n\to\infty} \frac{(3n+3)(3n+6)\cdots(6n)}{(3n+2)(3n+5)\cdots(6n-1)} = \sqrt[3]{2}.$$

Choosing $a = r = 1$, $k = 1$, $p = 3$, $q = 3$ and $s = 1$ in part (i) of Proposition 2, we obtain

$$\lim_{n\to\infty} \frac{(3n+2)(3n+5)\cdots(6n-1)}{(3n+1)(3n+4)\cdots(6n-2)} = \sqrt[3]{2}.$$

So, $\lim_{n\to\infty} z_n = \left(\sqrt[3]{2}\right)^2 \cdot \sqrt[3]{2} = 2$.

Alexandrescu in [1], proposed in Gazeta Matematică the following problem.

Problem 10. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be the sequences defined by the recurrence relations

$$nx_{n+1} = \left(n + \frac{1}{2}\right) x_n \quad \text{and} \quad ny_{n+1} = (n+1)y_n,$$

for each $n \in \mathbb{N}$, with $y_1 \neq 0$.

Evaluate $\lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{y_1+y_2+\cdots+y_n}$. (The limit is 0.)

Remark 11. For the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ given in Problem 10, with $x_1 \neq 0$, Sîntămărian (see [6, Problem 66, part 2°, p. 21]) proposed to be found the limit

$$\lim_{n\to\infty} \frac{y_n(x_n + x_{n+1} + \cdots + x_{2n})}{x_n(y_n + y_{n+1} + \cdots + y_{2n})}.$$

We have

$$\frac{y_n(x_n+x_{n+1}+\cdots+x_{2n})}{x_n(y_n+y_{n+1}+\cdots+y_{2n})} = \frac{8n}{9(n+1)}\cdot\frac{(2n+1)(2n+3)(2n+5)\cdots(4n-1)}{(2n)(2n+2)(2n+4)\cdots(4n-2)}\cdot\frac{4n+1}{4n}-\frac{4(n-1)}{9(n+1)},$$

for each $n \in \mathbb{N}$. Choosing $a = r = 1$, $k = 0$, $p = 2$, $q = 2$ and $s = 1$ in part (i) of Proposition *2*, we obtain

$$\lim_{n\to\infty}\frac{(2n+1)(2n+3)\cdots(4n-1)}{(2n)(2n+2)\cdots(4n-2)} = \sqrt{2}.$$

So,

$$\lim_{n\to\infty}\frac{y_n(x_n+x_{n+1}+\cdots+x_{2n})}{x_n(y_n+y_{n+1}+\cdots+y_{2n})} = \frac{8}{9}\cdot\sqrt{2}\cdot 1-\frac{4}{9} = \frac{4(2\sqrt{2}-1)}{9}.$$

Also, Sîntămărian [6] gave the following problem.

Problem 12. (see Sîntămărian, [6, Problem 67, pp. 21, 22]) We consider the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, with $x_1 \neq 0$ and $y_1 \neq 0$, defined by the recurrence relations

$$nx_{n+1} = \left(n+\frac{1}{3}\right)x_n \quad\text{and}\quad ny_{n+1} = (n+1)y_n,$$

for each $n \in \mathbb{N}$.

Evaluate:

1° $\lim\limits_{n\to\infty}\frac{x_1+x_2+\cdots+x_n}{y_1+y_2+\cdots+y_n}$;

2° $\lim\limits_{n\to\infty}\frac{y_n(x_n+x_{n+1}+\cdots+x_{2n})}{x_n(y_n+y_{n+1}+\cdots+y_{2n})}$;

3° $\lim\limits_{n\to\infty}\frac{y_n(x_n+x_{n+1}+\cdots+x_{3n})}{x_n(y_n+y_{n+1}+\cdots+y_{3n})}$;

4° $\lim\limits_{n\to\infty}\frac{y_{2n}(x_{2n}+x_{2n+1}+\cdots+x_{3n})}{x_{2n}(y_{2n}+y_{2n+1}+\cdots+y_{3n})}$.

Solution. We specify that a detailed solution can be found in our book Sîntămărian [6, pp. 93–99]. We present here a sketch of it.

1° Using the recurrence relations we get

$$\frac{x_1+x_2+\cdots+x_n}{y_1+y_2+\cdots+y_n} = \frac{3(3n+1)}{2(n+1)}\cdot\frac{1\cdot 4\cdot 7\cdots(3n-2)}{3\cdot 6\cdot 9\cdots(3n)}\cdot\frac{x_1}{y_1},$$

for each $n \in \mathbb{N}$. Because

$$0 < \frac{1\cdot 4\cdot 7\cdots(3n-2)}{3\cdot 6\cdot 9\cdots(3n)} < \frac{\sqrt[3]{3n-2}}{3n},$$

for each $n \in \mathbb{N}\setminus\{1\}$, we can write that $\lim\limits_{n\to\infty}\frac{x_1+x_2+\cdots+x_n}{y_1+y_2+\cdots+y_n} = 0$.

2° We have

$$\frac{y_n(x_n + x_{n+1} + \cdots + x_{2n})}{x_n(y_n + y_{n+1} + \cdots + y_{2n})} = \frac{n}{n+1} \cdot \frac{(3n+1)(3n+4)(3n+7)\cdots(6n-2)}{(3n)(3n+3)(3n+6)\cdots(6n-3)} \cdot \frac{6n+1}{6n} - \frac{n-1}{2(n+1)},$$

for each $n \in \mathbb{N}$. Choosing $a = r = 1$, $k = 0$, $p = 3$, $q = 3$ and $s = 1$ in part (i) of Proposition 2, we obtain

$$\lim_{n\to\infty} \frac{(3n+1)(3n+4)\cdots(6n-2)}{(3n)(3n+3)\cdots(6n-3)} = \sqrt[3]{2}.$$

So,

$$\lim_{n\to\infty} \frac{y_n(x_n + x_{n+1} + \cdots + x_{2n})}{x_n(y_n + y_{n+1} + \cdots + y_{2n})} = 1 \cdot \sqrt[3]{2} \cdot 1 - \frac{1}{2} = \sqrt[3]{2} - \frac{1}{2}.$$

3° We have

$$\frac{y_n(x_n + x_{n+1} + \cdots + x_{3n})}{x_n(y_n + y_{n+1} + \cdots + y_{3n})} = \frac{9n}{8(2n+1)} \cdot \frac{(3n+1)(3n+4)(3n+7)\cdots(9n-5)}{(3n)(3n+3)(3n+6)\cdots(9n-6)} \times \frac{(9n-2)(9n+1)}{(9n-3)(9n)} - \frac{3(n-1)}{8(2n+1)},$$

for each $n \in \mathbb{N}$. Choosing $a = r = 1$, $k = 0$, $p = 3$, $q = 3$ and $s = 2$ in part (i) of Proposition 2, we obtain

$$\lim_{n\to\infty} \frac{(3n+1)(3n+4)\cdots(9n-5)}{(3n)(3n+3)\cdots(9n-6)} = \sqrt[3]{3}.$$

So,

$$\lim_{n\to\infty} \frac{y_n(x_n + x_{n+1} + \cdots + x_{3n})}{x_n(y_n + y_{n+1} + \cdots + y_{3n})} = \frac{9}{16} \cdot \sqrt[3]{3} \cdot 1 - \frac{3}{16} = \frac{3(3\sqrt[3]{3} - 1)}{16}.$$

4° We have

$$\frac{y_{2n}(x_{2n} + x_{2n+1} + \cdots + x_{3n})}{x_{2n}(y_{2n} + y_{2n+1} + \cdots + y_{3n})} = \frac{9n}{5(n+1)} \cdot \frac{(6n+1)(6n+4)(6n+7)\cdots(9n-2)}{(6n)(6n+3)(6n+6)\cdots(9n-3)} \cdot \frac{9n+1}{9n} - \frac{3(2n-1)}{5(n+1)},$$

for each $n \in \mathbb{N}$. Choosing $a = r = 1$, $k = 0$, $p = 3$, $q = 6$ and $s = 1$ in part (i)

of Proposition 2, we obtain

$$\lim_{n\to\infty} \frac{(6n+1)(6n+4)\cdots(9n-2)}{(6n)(6n+3)\cdots(9n-3)} = \sqrt[3]{\frac{3}{2}}.$$

So,

$$\lim_{n\to\infty} \frac{y_{2n}(x_{2n}+x_{2n+1}+\cdots+x_{3n})}{x_{2n}(y_{2n}+y_{2n+1}+\cdots+y_{3n})} = \frac{9}{5}\cdot\sqrt[3]{\frac{3}{2}}\cdot 1 - \frac{6}{5} = \frac{3(3\sqrt[3]{3}-2\sqrt[3]{2})}{5\sqrt[3]{2}}.$$

3. Some Convergent Series

Proposition 13. *Let $p \in \mathbb{N}\setminus\{1\}$ and $q \in \{1,2,\ldots,p-1\}$. Then:*
(i) $\sum_{n=1}^{\infty} \frac{q(q+p)\ldots(q+np-p)}{p(2p)\cdots(np)} \cdot \frac{1}{n+1} = \frac{q}{p-q}$;
(ii) $\sum_{n=1}^{\infty} (-1)^n \frac{q(q+p)\ldots(q+np-p)}{p(2p)\cdots(np)} \cdot \frac{1}{n+1} = \frac{p}{p-q}\sqrt[p]{2^{p-q}} - \frac{q}{p-q} - 2.$

Proof. (i) Using the Binomial Theorem (Knopp [2, p. 209]) we obtain

$$\begin{aligned}(1-x)^{-\frac{q}{p}} &= 1 + \frac{-\frac{q}{p}}{1!}(-x) + \frac{-\frac{q}{p}\left(-\frac{q}{p}-1\right)}{2!}(-x)^2 + \cdots \\ &\quad + \frac{-\frac{q}{p}\left(-\frac{q}{p}-1\right)\cdots\left(-\frac{q}{p}-n+1\right)}{n!}(-x)^n + \cdots \\ &= 1 + \frac{q}{p}x + \frac{q(q+p)}{p^2\cdot 2!}x^2 + \cdots \\ &\quad + \frac{q(q+p)\cdots(q+np-p)}{p^n\cdot n!}x^n + \cdots,\end{aligned}$$

for any $x \in (-1,1)$. So,

$$\frac{1}{\sqrt[p]{(1-x)^q}} = 1 + \sum_{n=1}^{\infty} \frac{q(q+p)\ldots(q+np-p)}{p(2p)\cdots(np)} \cdot x^n,$$

for each $x \in (-1,1)$. Integrating from 0 to x, where $x \in (-1,1)$, we get

$$\begin{aligned}&-\frac{p}{p-q}\sqrt[p]{(1-x)^{p-q}} + \frac{p}{p-q} \\ &\qquad = x + \sum_{n=1}^{\infty} \frac{q(q+p)\ldots(q+np-p)}{p(2p)\cdots(np)} \cdot \frac{x^{n+1}}{n+1}.\end{aligned}$$

It is not difficult to verify that the series

$$\sum_{n=1}^{\infty} \frac{q(q+p)\ldots(q+np-p)}{p(2p)\cdots(np)} \cdot \frac{1}{n+1}$$

is convergent. According to Abel's Theorem (Knopp [2, p. 177]), it follows that

$$\sum_{n=1}^{\infty} \frac{q(q+p)\ldots(q+np-p)}{p(2p)\cdots(np)} \cdot \frac{1}{n+1} = \frac{q}{p-q}.$$

(*ii*) Using the Binomial Theorem (Knopp [2, p. 209]) we obtain

$$\begin{aligned}
(1+x)^{-\frac{q}{p}} &= 1 + \frac{-\frac{q}{p}}{1!}x + \frac{-\frac{q}{p}\left(-\frac{q}{p}-1\right)}{2!}x^2 + \cdots \\
&\quad + \frac{-\frac{q}{p}\left(-\frac{q}{p}-1\right)\cdots\left(-\frac{q}{p}-n+1\right)}{n!}x^n + \cdots \\
&= 1 - \frac{q}{p}x + \frac{q(q+p)}{p^2 \cdot 2!}x^2 - \cdots \\
&\quad + (-1)^n \frac{q(q+p)\cdots(q+np-p)}{p^n \cdot n!}x^n + \cdots,
\end{aligned}$$

for any $x \in (-1,1)$. So,

$$\frac{1}{\sqrt[p]{(1+x)^q}} = 1 + \sum_{n=1}^{\infty}(-1)^n \frac{q(q+p)\ldots(q+np-p)}{p(2p)\cdots(np)} \cdot x^n,$$

for each $x \in (-1,1)$. Integrating from 0 to x, where $x \in (-1,1)$, we get

$$\begin{aligned}
&\frac{p}{p-q}\sqrt[p]{(1+x)^{p-q}} - \frac{p}{p-q} \\
&\quad = x + \sum_{n=1}^{\infty}(-1)^n \frac{q(q+p)\ldots(q+np-p)}{p(2p)\cdots(np)} \cdot \frac{x^{n+1}}{n+1}.
\end{aligned}$$

Because the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{q(q+p)\ldots(q+np-p)}{p(2p)\cdots(np)} \cdot \frac{1}{n+1}$$

is convergent, it follows, according to Abel's Theorem (Knopp [2, p. 177]), that

$$\sum_{n=1}^{\infty}(-1)^n \frac{q(q+p)\ldots(q+np-p)}{p(2p)\cdots(np)} \cdot \frac{1}{n+1} = \frac{p}{p-q}\sqrt[p]{2^{p-q}} - \frac{q}{p-q} - 2. \quad \square$$

Remark 14. We present a few particular cases of the results given in Proposition *13*.

(i_1) For $p = 2$ and $q = 1$ we get $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} \cdot \frac{1}{n+1} = 1$.

(i_2) For $p = 3$ and $q = 1$ we get $\sum_{n=1}^{\infty} \frac{1\cdot 4\cdots(3n-2)}{3\cdot 6\cdots(3n)} \cdot \frac{1}{n+1} = \frac{1}{2}$.

(i_3) For $p = 3$ and $q = 2$ we get $\sum_{n=1}^{\infty} \frac{2\cdot 5\cdots(3n-1)}{3\cdot 6\cdots(3n)} \cdot \frac{1}{n+1} = 2$.

(i_4) For $p=4$ and $q=1$ we get $\sum_{n=1}^{\infty}\frac{1\cdot5\cdots(4n-3)}{4\cdot8\cdots(4n)}\cdot\frac{1}{n+1}=\frac{1}{3}$.

(i_5) For $p=4$ and $q=3$ we get $\sum_{n=1}^{\infty}\frac{3\cdot7\cdots(4n-1)}{4\cdot8\cdots(4n)}\cdot\frac{1}{n+1}=3$.

(i_6) For $p\in\mathbb{N}\setminus\{1\}$ and $q=1$ we get

$$\sum_{n=1}^{\infty}\frac{1\cdot(p+1)\ldots(np-p+1)}{p(2p)\cdots(np)}\cdot\frac{1}{n+1}=\frac{1}{p-1}.$$

(i_7) For $p\in\mathbb{N}\setminus\{1\}$ and $q=p-1$ we get

$$\sum_{n=1}^{\infty}\frac{(p-1)(2p-1)\ldots(np-1)}{p(2p)\cdots(np)}\cdot\frac{1}{n+1}=p-1.$$

(ii_1) For $p=2$ and $q=1$ we get

$$\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)}\cdot\frac{1}{n+1}=2\sqrt{2}-3.$$

(ii_2) For $p=3$ and $q=1$ we get

$$\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot4\cdots(3n-2)}{3\cdot6\cdots(3n)}\cdot\frac{1}{n+1}=\frac{3}{2}\sqrt[3]{4}-\frac{5}{2}.$$

(ii_3) For $p=3$ and $q=2$ we get

$$\sum_{n=1}^{\infty}(-1)^n\frac{2\cdot5\cdots(3n-1)}{3\cdot6\cdots(3n)}\cdot\frac{1}{n+1}=3\sqrt[3]{2}-4.$$

(ii_4) For $p=4$ and $q=1$ we get

$$\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot5\cdots(4n-3)}{4\cdot8\cdots(4n)}\cdot\frac{1}{n+1}=\frac{4}{3}\sqrt[4]{8}-\frac{7}{3}.$$

(ii_5) For $p=4$ and $q=3$ we get

$$\sum_{n=1}^{\infty}(-1)^n\frac{3\cdot7\cdots(4n-1)}{4\cdot8\cdots(4n)}\cdot\frac{1}{n+1}=4\sqrt[4]{2}-5.$$

(ii_6) For $p\in\mathbb{N}\setminus\{1\}$ and $q=1$ we get

$$\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot(p+1)\ldots(np-p+1)}{p(2p)\cdots(np)}\cdot\frac{1}{n+1}=\frac{p}{p-1}\sqrt[p]{2^{p-1}}-\frac{2p-1}{p-1}.$$

(ii_7) For $p\in\mathbb{N}\setminus\{1\}$ and $q=p-1$ we get

$$\sum_{n=1}^{\infty}(-1)^n\frac{(p-1)(2p-1)\ldots(np-1)}{p(2p)\cdots(np)}\cdot\frac{1}{n+1}=p\sqrt[p]{2}-p-1.$$

Corollary 15. *Let $p \in \mathbb{N} \setminus \{1\}$ and $q \in \{1, 2, \dots, p-1\}$. Then:*

(*i*) $\sum_{n=1}^{\infty} \frac{q(q+p)\dots(q+2np-p)}{p(2p)\cdots(2np)} \cdot \frac{1}{2n+1} = \frac{p}{2(p-q)} \sqrt[p]{2^{p-q}} - 1$;

(*ii*) $\sum_{n=1}^{\infty} \frac{q(q+p)\dots(q+2np-2p)}{p(2p)\cdots(2np-p)} \cdot \frac{1}{2n} = \frac{p}{p-q} - \frac{p}{2(p-q)} \sqrt[p]{2^{p-q}}$.

Proof. (*i*) We add the series from parts (*i*) and (*ii*) of Proposition 13.

(*ii*) We subtract the series from parts (*i*) and (*ii*) of Proposition 13. □

Remark 16. We present a few particular cases of the results given in Corollary 15.

(i_1) For $p = 2$ and $q = 1$ we get

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (4n-1)}{2 \cdot 4 \cdots (4n)} \cdot \frac{1}{2n+1} = \sqrt{2} - 1.$$

(i_2) For $p = 3$ and $q = 1$ we get

$$\sum_{n=1}^{\infty} \frac{1 \cdot 4 \cdots (6n-2)}{3 \cdot 6 \cdots (6n)} \cdot \frac{1}{2n+1} = \frac{3}{4} \sqrt[3]{4} - 1.$$

(i_3) For $p = 3$ and $q = 2$ we get

$$\sum_{n=1}^{\infty} \frac{2 \cdot 5 \cdots (6n-1)}{3 \cdot 6 \cdots (6n)} \cdot \frac{1}{2n+1} = \frac{3}{2} \sqrt[3]{2} - 1.$$

(i_4) For $p = 4$ and $q = 1$ we get

$$\sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdots (8n-3)}{4 \cdot 8 \cdots (8n)} \cdot \frac{1}{2n+1} = \frac{2}{3} \sqrt[4]{8} - 1.$$

(i_5) For $p = 4$ and $q = 3$ we get

$$\sum_{n=1}^{\infty} \frac{3 \cdot 7 \cdots (8n-1)}{4 \cdot 8 \cdots (8n)} \cdot \frac{1}{2n+1} = 2\sqrt[4]{2} - 1.$$

(ii_1) For $p = 2$ and $q = 1$ we get

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (4n-3)}{2 \cdot 4 \cdots (4n-2)} \cdot \frac{1}{2n} = 2 - \sqrt{2}.$$

(ii_2) For $p = 3$ and $q = 1$ we get

$$\sum_{n=1}^{\infty} \frac{1 \cdot 4 \cdots (6n-5)}{3 \cdot 6 \cdots (6n-3)} \cdot \frac{1}{2n} = \frac{3}{2} - \frac{3}{4} \sqrt[3]{4}.$$

(ii_3) For $p=3$ and $q=2$ we get

$$\sum_{n=1}^{\infty}\frac{2\cdot 5\cdots(6n-4)}{3\cdot 6\cdots(6n-3)}\cdot\frac{1}{2n}=3-\frac{3}{2}\sqrt[3]{2}.$$

(ii_4) For $p=4$ and $q=1$ we get

$$\sum_{n=1}^{\infty}\frac{1\cdot 5\cdots(8n-7)}{4\cdot 8\cdots(8n-4)}\cdot\frac{1}{2n}=\frac{4}{3}-\frac{2}{3}\sqrt[4]{8}.$$

(ii_5) For $p=4$ and $q=3$ we get

$$\sum_{n=1}^{\infty}\frac{3\cdot 7\cdots(8n-5)}{4\cdot 8\cdots(8n-4)}\cdot\frac{1}{2n}=4-2\sqrt[4]{2}.$$

As can be seen, from Proposition 13 we do not obtain the series of the form

$$\sum_{n=1}^{\infty}\frac{1\cdot 4\cdots(3n-2)}{2\cdot 5\cdots(3n-1)}\cdot\frac{1}{n+1},\quad \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 4\cdots(3n-2)}{2\cdot 5\cdots(3n-1)}\cdot\frac{1}{n+1},$$

$$\sum_{n=1}^{\infty}\frac{1\cdot 5\cdots(4n-3)}{2\cdot 6\cdots(4n-2)}\cdot\frac{1}{n+1},\quad \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 5\cdots(4n-3)}{2\cdot 6\cdots(4n-2)}\cdot\frac{1}{n+1},$$

$$\sum_{n=1}^{\infty}\frac{1\cdot 5\cdots(4n-3)}{3\cdot 7\cdots(4n-1)}\cdot\frac{1}{n+1},\quad \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 5\cdots(4n-3)}{3\cdot 7\cdots(4n-1)}\cdot\frac{1}{n+1},$$

$$\sum_{n=1}^{\infty}\frac{2\cdot 6\cdots(4n-2)}{3\cdot 7\cdots(4n-1)}\cdot\frac{1}{n+1},\quad \sum_{n=1}^{\infty}(-1)^n\frac{2\cdot 6\cdots(4n-2)}{3\cdot 7\cdots(4n-1)}\cdot\frac{1}{n+1},$$

and so on. It is not difficult to verify that these series are convergent.

Let $\alpha\in\mathbb{R}$ and $m\in\mathbb{N}$. We introduce the notation

$$(\alpha)_0:=1,\ (\alpha)_m:=\alpha(\alpha+1)\cdots(\alpha+m-1),$$

known as Pochhammer's symbol.

Also, we shall use a generalized hypergeometric series

$${}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;x)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k(\alpha_2)_k(\alpha_3)_k}{k!(\beta_1)_k(\beta_2)_k}\cdot x^k.$$

We can write

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1\cdot 4\cdots(3n-2)}{2\cdot 5\cdots(3n-1)}\cdot\frac{1}{n+1} &= \sum_{k=0}^{\infty}\frac{1\cdot 4\cdots(3k+1)}{2\cdot 5\cdots(3k+2)}\cdot\frac{1}{k+2}\\ &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{4}{3}\right)_k}{\left(\frac{5}{3}\right)_k}\cdot\frac{1}{k+2}\end{aligned}$$

$$= \frac{1}{4}\sum_{k=0}^{\infty}\frac{(1)_k\left(\frac{4}{3}\right)_k(2)_k}{k!\left(\frac{5}{3}\right)_k(3)_k}\cdot 1^k$$
$$= \frac{1}{4}\,{}_3F_2\left(1,\frac{4}{3},2;\frac{5}{3},3;1\right)$$

and

$$\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 4\cdots(3n-2)}{2\cdot 5\cdots(3n-1)}\cdot\frac{1}{n+1} = -\sum_{k=0}^{\infty}(-1)^k\frac{1\cdot 4\cdots(3k+1)}{2\cdot 5\cdots(3k+2)}\cdot\frac{1}{k+2}$$
$$= -\frac{1}{4}\,{}_3F_2\left(1,\frac{4}{3},2;\frac{5}{3},3;-1\right).$$

In the same manner we obtain

$$\sum_{n=1}^{\infty}\frac{1\cdot 5\cdots(4n-3)}{2\cdot 6\cdots(4n-2)}\cdot\frac{1}{n+1} = \frac{1}{4}\,{}_3F_2\left(1,\frac{5}{4},2;\frac{3}{2},3;1\right),$$
$$\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 5\cdots(4n-3)}{2\cdot 6\cdots(4n-2)}\cdot\frac{1}{n+1} = -\frac{1}{4}\,{}_3F_2\left(1,\frac{5}{4},2;\frac{3}{2},3;-1\right),$$
$$\sum_{n=1}^{\infty}\frac{1\cdot 5\cdots(4n-3)}{3\cdot 7\cdots(4n-1)}\cdot\frac{1}{n+1} = \frac{1}{6}\,{}_3F_2\left(1,\frac{5}{4},2;\frac{7}{4},3;1\right),$$
$$\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 5\cdots(4n-3)}{3\cdot 7\cdots(4n-1)}\cdot\frac{1}{n+1} = -\frac{1}{6}\,{}_3F_2\left(1,\frac{5}{4},2;\frac{7}{4},3;-1\right),$$
$$\sum_{n=1}^{\infty}\frac{2\cdot 6\cdots(4n-2)}{3\cdot 7\cdots(4n-1)}\cdot\frac{1}{n+1} = \frac{1}{3}\,{}_3F_2\left(1,\frac{3}{2},2;\frac{7}{4},3;1\right),$$
$$\sum_{n=1}^{\infty}(-1)^n\frac{2\cdot 6\cdots(4n-2)}{3\cdot 7\cdots(4n-1)}\cdot\frac{1}{n+1} = -\frac{1}{3}\,{}_3F_2\left(1,\frac{3}{2},2;\frac{7}{4},3;-1\right).$$

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