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ON MARKOV-TYPE INEQUALITIES

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Abstract: We give analogies of Chebyshev's and Markov's inequalities without moment conditions. Some applications are also discussed.

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1. Introduction

Let $(\Omega, \mathcal{A}\ P)$ be a probability space and let X be a real-valued random variable. The classical Chebyshev's inequality states that for any $\varepsilon > 0$

$$P[|X| \geq \varepsilon] \leq \frac{EX^2}{\varepsilon^2},\ P[|X - EX| \geq \varepsilon] \leq \frac{\sigma^2 X}{\varepsilon^2}, \tag{1}$$

provided that $EX^2 < \infty$, and Markov's inequality yields

$$P[|X| \geq \varepsilon] \leq \frac{E|X|^r}{\varepsilon^r} \tag{2}$$

provided that $E|X|^r < \infty$, $0 < r < \infty$ (cf. Gut [3]). These inequalities, however, are informative (useful) only for those values of ε for which the right

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side is less than 1.

Chebyshev's inequality gives a lower bound for the probability of a symmetric interval around the mean. Let X be a random variable with mean μ and variance $\sigma^2 < \infty$. Then Chebyshev's inequality states that

$$P[\lambda_1 < X < \lambda_2] \geq 1 - \frac{4\sigma^2}{(\lambda_2 - \lambda_1)^2}, \tag{3}$$

where $\lambda_1 + \lambda_2 = 2\mu$, and this is useful for those intervals for which $\lambda_2 - \lambda_1 > 2\sigma^2$. The interval (λ_1, λ_2) is of the type $(\mu - t\sigma; \mu + t\sigma)$ with $t = (\lambda_2 - \lambda_1)/2\sigma$. There are many generalizations of the Chebyshev and Markov inequalities. For instance, a lower bound for the probability of any interval not necessarily symmetric around the mean is given by

$$P[\lambda_1 < X < \lambda_2] \geq \frac{4[(\mu - \lambda_1)(\lambda_2 - \mu) - \sigma^2]}{(\lambda_2 - \lambda_1)^2}, \tag{4}$$

and this is useful for intervals such that $\lambda_1 < \mu < \lambda_2$ and $(\mu - \lambda_1)(\lambda_2 - \mu) > \sigma^2$ (cf. Ferentinos [2]).

The aim of this paper is to extend the inequalities (1)-(4) to a class of random variables without moment conditions.

2. Univariate Random Variables

First assume that X is a positive random variable ($X \geq 0$ a.s.) and $\mathcal{G}$ is the class of all positive, strictly increasing functions g with $g(0) = 0$. $\mathbb{N}$ denotes the set of positive integers. We start with the following Chebyshev-Markov inequalities.

Proposition 1. *Let X be a positive random variable. Then for given $\varepsilon > 0$, g in $\mathcal{G}$ and k in $\mathbb{N}$*

$$E\frac{g(kX)}{g(kX) + g((k-1)X + \varepsilon)} I[X \geq \varepsilon] \leq P[X \geq \varepsilon]$$
$$\leq 2E\frac{g(kX)}{g(kX) + g((k-1)X + \varepsilon)} I[X \geq \varepsilon] \leq 2E\frac{g(kX)}{g(kX) + g((k-1)X + \varepsilon)}, \tag{5}$$

where $I[\cdot]$ denotes the indicator function.

Proof. The LHS of (5) follows from the fact that

$$g(kX)/[g(kX) + g((k-1)X + \varepsilon)] \leq 1 \text{ a.s.}$$

and the RHS of (5) follows since for $X \geq \varepsilon$

$$2g(kX)/[g(kX) + g((k-1)X + \varepsilon)] \geq 1 \text{ a.s.} \qquad \square$$

Now let X be an arbitrary real-valued random variable and $g \in \mathcal{G}$ (i.e. g is increasing in $|X|$). Then taking $|X|$ instead of X in (5) we have

Theorem 1. *Let X be an arbitrary random variable. Then for any $\varepsilon > 0$, g in $\mathcal{G}$ and k in $\mathbb{N}$*

$$\begin{aligned} &E\frac{g(k|X|)}{g(k|X|) + g((k-1)|X| + \varepsilon)} I[|X| \geq \varepsilon] \leq P[|X| \geq \varepsilon] \\ &\leq 2E\frac{g(k|X|)}{g(k|X|) + g((k-1)|X| + \varepsilon)} I[|X| \geq \varepsilon] \\ &\leq 2E\frac{g(k|X|)}{g(k|X|) + g((k-1)|X| + \varepsilon)}. \end{aligned} \tag{6}$$

We now consider the special cases:

$$g(|x|) = |x|^m, \quad m \in \mathbb{N}, \quad g(|x|) = |x|^r, \quad 0 < r < \infty.$$

Here we use the bounds:

$$(x+y)^r \leq \begin{cases} x^r + y^r, & \text{for } 0 < r \leq 1, \\ 2^{r-1}(x^r + y^r), & \text{for } r \geq 1, \end{cases} \tag{7}$$

whenever $x > 0$, $y > 0$ (cf. Gut [3], p. 562). Using Theorem 1 and (7) we obtain the following bounds.

Corollary 1. *Let X be a random variable, m a positive integer number ($m \geq 1$), and r a positive real number ($r > 0$). Then for any $\varepsilon > 0$ and all $k \geq 1$ we have the following inequalities:*

$$\begin{aligned} &k^r E\frac{|X|^r}{(k|X|)^r + [(k-1)|X| + \varepsilon]^r} I[|X| \geq \varepsilon] \leq P[|X| \geq \varepsilon] \\ &\leq 2k^r E\frac{|X|^r}{(k|X|)^r + [(k-1)|X| + \varepsilon]^r} I[|X| \geq \varepsilon] \\ &\leq 2k^r E\frac{|X|^r}{(k|X|)^r + [(k-1)|X| + \varepsilon]^r}, \quad r > 0, \end{aligned} \tag{8}$$

$$\begin{aligned} &k^m E\frac{|X|^m}{(2k^m - 1)|X|^m + \varepsilon^m} I[|X| \geq \varepsilon] \leq P[|X| \geq \varepsilon] \\ &\leq 2k^m E\frac{|X|^m}{(k^m + (k-1)^m)|X|^m + (k^m - (k-1)^m)\varepsilon^m} I[|X| \geq \varepsilon] \\ &\leq 2k^m E\frac{|X|^m}{(k^m + (k-1)^m)|X|^m + (k^m - (k-1)^m)\varepsilon^m}, \quad m \in \mathbb{N}, \end{aligned} \tag{9}$$

$$\begin{aligned} &k^r E\frac{|X|^r}{(k^r+2^{r-1}(k-1)^r)|X|^r+2^{r-1}\varepsilon^r}I[|X|\ge\varepsilon]\le P[|X|\ge\varepsilon]\\ &\le 2^r E\Big[\frac{k|X|}{(2k-1)|X|+\varepsilon}\Big]^r I[|X|\ge\varepsilon] \qquad (10)\\ &\le 2^r E\Big[\frac{k|X|}{(2k-1)|X|+\varepsilon}\Big]^r, \quad r\ge 1, \end{aligned}$$

$$\begin{aligned} &k^r E\frac{|X|^r}{(k^r+(k-1)^r)|X|^r+\varepsilon^r}I[|X|\ge\varepsilon]\le P[|X|\ge\varepsilon]\\ &\le 2E\Big[\frac{k|X|}{(2k-1)|X|+\varepsilon}\Big]^r I[|X|\ge\varepsilon] \qquad (11)\\ &\le 2E\Big[\frac{k|X|}{(2k-1)|X|+\varepsilon}\Big]^r, \quad 0<r\le 1. \end{aligned}$$

Proof. The bounds in (8) follow from (6). Now using (6) with $g(|X|)=|X|^m,\ m\in\mathbb{N}$, we have

$$\begin{aligned} &P[|X|\ge\varepsilon]\ge k^m E\frac{|X|^m}{(k|X|)^m+((k-1)|X|+\varepsilon)^m}I[|X|\ge\varepsilon]\\ &=k^m E\Big[|X|^m\Big/\Big(k^m|X|^m+(k-1)^m|X|^m\\ &+m(k-1)^{m-1}|X|^{m-1}\varepsilon+\dots\ +m(k-1)|X|\varepsilon^{m-1}+\varepsilon^m\Big)\Big]I[|X|\ge\varepsilon]\\ &\ge k^m E\Big[|X|^m\Big/\Big(k^m|X|^m+(k-1)^m|X|^m\\ &+m(k-1)^{m-1}|X|^m+\dots\ +m(k-1)|X|^m+\varepsilon^m\Big)\Big]I[|X|\ge\varepsilon]\\ &=k^m E\frac{|X|^m}{(2k^m-1)|X|^m+\varepsilon^m}I[|X|\ge\varepsilon], \end{aligned}$$

which leads to the LHS in (9).

Similarly, using the RHS in (6) we get:

$$\begin{aligned} &P[|X|\ge\varepsilon]\le 2E\frac{k^m|X|^m}{(k|X|)^m+((k-1)|X|+\varepsilon)^m}I[|X|\ge\varepsilon]\\ &=2k^m E\Big[|X|^m\Big/\Big(k^m|X|^m+(k-1)^m|X|^m\\ &+m(k-1)^{m-1}|X|^{m-1}\varepsilon+\dots\ +m(k-1)|X|\varepsilon^{m-1}+\varepsilon^m\Big)\Big]I[|X|\ge\varepsilon]\\ &\le 2k^m E\Big[|X|^m\Big/\Big(k^m|X|^m+(k-1)^m|X|^m\\ &+m(k-1)^{m-1}\varepsilon^m+\dots\ +m(k-1)\varepsilon^m+\varepsilon^m\Big)\Big]I[|X|\ge\varepsilon] \end{aligned}$$

$$= 2k^m E\frac{|X|^m}{(k^m + (k-1)^m)|X|^m + (k^m - (k-1)^m)\varepsilon^m} I[|X| \geq \varepsilon],$$

which leads to the RHS in (9).

For $r > 0$ and $g(|X|) = |X|^r,\ r > 0$, the inequality (6) gives (8). Now by (7) for $0 < r \leq 1$ we get

$$((2k-1)|X| + \varepsilon)^r \leq (k|X|)^r + ((k-1)|X| + \varepsilon)^r \leq (k^r + (k-1)^r)|X|^r + \varepsilon^r$$

which leads us via (8) to (11). Applying (7) for $r \geq 1$ we have

$$2^{-(r-1)}[(2k-1)|X| + \varepsilon]^r \leq (k|X|)^r + ((k-1)|X| + \varepsilon)^r$$
$$\leq (k^r + 2^{r-1}(k-1)^r)|X|^r + 2^{r-1}\varepsilon^r,$$

which leads us via (8) to (10). This completes the proof. □

Note that under the assumption $E|X|^r < \infty,\ r > 0$, we have from (8)

$$P[|X| \geq \varepsilon] \leq 2k^r E\frac{|X|^r}{(k|X|)^r + ((k-1)|X| + \varepsilon)^r} I[|X| \geq \varepsilon] \leq \frac{E|X|^r}{\varepsilon^r}$$

which for $r = 2$ gives the classical Chebyshev inequality.

DasGupta [1] gives the best constants in Chebyshev inequalities. For X with the specified distribution function F the author presents for a given $r \geq 0$ the exact best possible value (denoted there by $C^*(r)$) of a constant $C(r) := C(r, F)$ such that

$$P[|X| \geq \varepsilon] \leq C(r) E_F|X|^r/\varepsilon^r \text{ for any } \varepsilon > 0.$$

Taking $X \sim N(0,1)$ and $\varepsilon = 2$ for which $P[|X| \geq 2] = .04550$, the author lists $C(r)E_F|X|^r/\varepsilon^r$ for several choices of r. We compare those bounds with estimates studied in this paper.

Write

$$L_k^I(r;\varepsilon) = 2k^r E\frac{|X|^r}{(k|X|)^r + [(k-1)|X| + \varepsilon]^r} I[|X| \geq \varepsilon],$$

$$L_k(r;\varepsilon) = 2k^r E\frac{|X|^r}{(k|X|)^r + [(k-1)|X| + \varepsilon]^r}, \quad r > 0,$$

$$\overline{L}_k^I(m;\varepsilon) = 2k^m E\frac{|X|^m}{[k^m + (k-1)^m]|X|^m + [k^m - (k-1)^m]\varepsilon^m} I[|X| \geq \varepsilon],$$

$$\overline{L}_k(m;\varepsilon) = 2k^m E\frac{|X|^m}{[k^m + (k-1)^m]|X|^m + [k^m - (k-1)^m]\varepsilon^m}, \quad m \in \mathbb{N},$$

$$M_k^I(r;\varepsilon) = 2^r k^r E\Big[\frac{|X|}{(2k-1)|X| + \varepsilon}\Big]^r I[|X| \geq \varepsilon],$$

$$M_k(r;\varepsilon) = 2^r k^r E\Big[\frac{|X|}{(2k-1)|X| + \varepsilon}\Big]^r, \quad r \geq 1,$$

and

$$M_k^{*I}(r;\varepsilon) = 2k^r E\Big[\frac{|X|}{(2k-1)|X|+\varepsilon}\Big]^r I[|X| \geq \varepsilon],$$

$$M_k^{*}(r;\varepsilon) = 2k^r E\Big[\frac{|X|}{(2k-1)|X|+\varepsilon}\Big]^r, \quad 0 < r \leq 1.$$

The bounds of DasGupta [1] and $L_k^I(m,2) := L_k^I(m)$, $L_k(m,2) = L_k(m)$, $\overline{L}_k^I(m,2) := \overline{L}_k^I(m)$, $\overline{L}_k(m,2) := \overline{L}_k(m)$, $M_k^I(m,2) := M_k^I(m)$, $M_k(m,2) := M_k(m)$ are given in Table 1.

We see that many of our bounds are better than the best bound of DasGupta (.04715 when $r = 4$); these are shown in bold-face, and the best is .045591 which occurs when $m = 1$ and $k = 36$. In general it is difficult to evaluate the best constants. Our approach allows to omit such calculations and to give better bounds with choosing some parameters r and k. The role of these parameters in the estimations of $P[|X| \geq 2] = .04550$ is shown in Table 1 and Table 2.

3. Further Consequences

3.1. Some Bounds

First we discuss estimates of

$$P[\lambda_1 < X < \lambda_2],\ \lambda_1 < \lambda_2.$$

Write $\lambda = \lambda_1 + \lambda_2,\ \delta = \lambda_2 - \lambda_1$. Taking $\varepsilon := \delta/2$ in (6) we get

$$P[\lambda_1 < X < \lambda_2] = 1 - P\Big[\Big|X - \frac{\lambda_1+\lambda_2}{2}\Big| \geq \frac{\lambda_2-\lambda_1}{2}\Big]$$

$$\geq 1 - 2E\frac{g(k|X-\frac{\lambda}{2}|)}{g(k|X-\frac{\lambda}{2}|) + g((k-1)|X-\frac{\lambda}{2}|+\frac{\delta}{2})} I\Big[\Big|X - \frac{\lambda}{2}\Big| \geq \frac{\delta}{2}\Big]$$

$$\geq 1 - 2E\frac{g(k|X-\frac{\lambda}{2}|)}{g(k|X-\frac{\lambda}{2}|) + g((k-1)|X-\frac{\lambda}{2}|+\frac{\delta}{2})},$$

which for $g(|X|) = |X|^r$, $r > 0$, lead us to

$$P[\lambda_1 < X < \lambda_2] \geq 1 - 2k^r E\Big[|2X-\lambda|^r \Big/ \Big(k^r|2X-\lambda|^r + ((k-1)|2X-\lambda|+\delta)^r\Big)\Big] \cdot I[|2X-\lambda| \geq \delta]$$

	m	1	2	3	4	6	8	10
k	DasGupta's	.16997	.08286	.05672	.04715	.04962	-	-
1	$L_k^I(m)$	.04916	.05266	.05589	.05880	.06366	.06745	.07040
	$L_k(m)$	.50998	.31452	.22711	.18159	.13866	.12002	.11039
	$\overline{L}_k^I(m)$	.04916	.05266	.05589	.05880	.06366	.06745	.07041
	$\overline{L}_k(m)$	.50998	.31452	.22711	.18159	.13866	.12002	.11039
	$M_k^I(m)$	.04916	.05330	.05801	.06338	.07663	.09434	.11843
	$M_k(m)$	.50998	.34314	.26328	.21931	.17886	.16876	.17624
2	$L_k^I(m)$	.04722	.04892	.05059	.05222	.05531	.05814	.06069
	$L_k(m)$	.62480	.43664	.33355	.27068	.20071	.16456	.14349
	$\overline{L}_k^I(m)$	.04722	.05055	.05420	.05763	.06322	.06730	.07036
	$\overline{L}_k(m)$	.62480	.34851	.23702	.18450	.13883	.11998	.11037
	$M_k^I(m)$	.04722	.04904	.05097	.05302	.05751	.06261	.06841
	$M_k(m)$	.62480	.46375	.37209	.31313	.24313	.20488	.18264
4	$L_k^I(m)$	**.04633**	.04717	.04799	.04882	.05044	.05203	.05357
	$L_k(m)$	.72939	.56862	.46595	.39502	.30426	.24953	.21359
	$\overline{L}_k^I(m)$	**.04633**	.04825	.05074	.05350	.05907	.06404	.06809
	$\overline{L}_k(m)$	.72939	.41829	.27224	.20153	.14184	.11947	.10925
	$M_k^I(m)$	**.04633**	.04719	.04807	.04898	.05088	.05289	.05550
	$M_k(m)$	.72939	.59075	.50132	.43771	.35246	.29795	.26041
9	$L_k^I(m)$	**.04586**	**.04623**	**.04659**	**.04696**	.04768	.04841	.04912
	$L_k(m)$	.82762	.70993	.62454	.55890	.46360	.39729	.34843
	$\overline{L}_k^I(m)$	**.04586**	**.04676**	.04799	.04945	.05274	.05623	.05968
	$\overline{L}_k(m)$	.82762	.51901	.33868	.24286	.15691	.12309	.10750
	$M_k^I(m)$	**.04586**	**.04623**	**.04662**	**.04699**	.04776	.04855	.04936
	$M_k(m)$	.82762	.72461	.65082	.59325	.50824	.44720	.40079
16	$L_k^I(m)$	**.04570**	**.04591**	**.04611**	**.04632**	**.04672**	**.04713**	.04754
	$L_k(m)$	.87961	.79146	.72327	.66791	.58196	.51734	.46656
	$\overline{L}_k^I(m)$	**.04570**	**.04621**	**.04693**	.04778	.04976	.05193	.05419
	$\overline{L}_k(m)$	.87961	.59315	.39665	.28382	.17692	.13255	.11097
	$M_k^I(m)$	**.04570**	**.04591**	**.04612**	**.04632**	**.04675**	.04717	.04761
	$M_k(m)$	.87961	.80149	.74205	.69383	.61839	.56077	.51466
25	$L_k^I(m)$	**.04563**	**.04576**	**.04589**	**.04602**	**.04628**	**.04654**	**.04680**
	$L_k(m)$	.91059	.84230	.78742	.74138	.66692	.60821	.56016
	$\overline{L}_k^I(m)$	**.04563**	**.04596**	**.04642**	**.04698**	.04827	.04971	.05123
	$\overline{L}_k(m)$	.91059	.64843	.44495	.32051	.19715	.14396	.11726
	$M_k^I(m)$	**.04563**	**.04576**	**.04589**	**.04602**	**.04629**	**.04656**	**.04683**
	$M_k(m)$	.91059	.84948	.80128	.76105	.69591	.64420	.60145
36	$L_k^I(m)$	**.04559**	**.04568**	**.04577**	**.04586**	**.04604**	**.04622**	**.04640**
	$L_k(m)$	.93064	.87614	.83122	.79270	.72868	.67658	.63274
	$\overline{L}_k^I(m)$	**.04559**	**.04582**	**.04614**	**.04653**	.04744	.04846	.04953
	$\overline{L}_k(m)$	.93064	.69087	.48529	.35276	.21626	.15558	.12449
	$M_k^I(m)$	**.04559**	**.04568**	**.04577**	**.04586**	**.04605**	**.04623**	**.04642**
	$M_k(m)$	.93064	.88149	.84176	.80793	.75186	.70613	.66747

Table 1: Bounds of $P[|X| \geq 2] = 0.04550$

$$\geq 1 - 2k^r E \frac{|2X-\lambda|^r}{k^r|2X-\lambda|^r + ((k-1)|2X-\lambda| + \delta)^r}, \; r > 0.$$

k	r	3/4	1/2	1/4	1/9	1/16
1	$L_k^I(r)$	.04825	.04734	**.04642**	**.04591**	**.04573**
	$L_k(r)$	.59462	.70292	.83943	.92673	.95858
	$M_k^{*I}(r)$	.05732	.06685	.07799	.08497	.08756
2	$L_k^I(r)$	**.04679**	**.04636**	**.04593**	**.04569**	**.04561**
	$L_k(r)$	.69616	.78203	.88430	.94754	.97038
	$M_k^{*I}(r)$	.05563	.06554	.07723	.08460	.08734
4	$L_k^I(r)$	**.04613**	**.04592**	**.04571**	**.04559**	**.04555**
	$L_k(r)$	.78450	.84790	.92035	.96405	.97972
	$M_k^{*I}(r)$	.05485	.06493	.07687	.08442	.08724
9	$L_k^I(r)$	**.04577**	**.04568**	**.04559**	**.04554**	**.04552**
	$L_k(r)$	.86470	.90582	.95123	.97807	.98764
	$M_k^{*I}(r)$	.05443	.06460	.07667	.08433	.08719
16	$L_k^I(r)$	**.04565**	**.04560**	**.04555**	**.04552**	**.04551**
	$L_k(r)$	.90622	.93518	.96663	.98502	.99156
	$M_k^{*I}(r)$	.05429	.06449	.07661	.08430	.08717
25	$L_k^I(r)$	**.04560**	**.04557**	**.04553**	**.04551**	**.04551**
	$L_k(r)$	.93067	.95230	.97553	.98903	.99382
	$M_k^{*I}(r)$	.04902	.06444	.07658	.08428	.08716
36	$L_k^I(r)$	**.04557**	**.04555**	**.04552**	**.04551**	**.04551**
	$L_k(r)$	.94639	.96322	.98118	.99157	.99525
	$M_k^{*I}(r)$	.05419	.06441	.07656	.08427	.08715

Table 2: Bounds of $P[|X| \geq 2] = .04550$

Write

$$L_k^I(r;\lambda,\delta) = 2k^r E\frac{|2X-\lambda|^r}{k^r|2X-\lambda|^r + ((k-1)|2X-\lambda|+\delta)^r} \cdot I[|2X-\lambda| \geq \delta],$$

$$L_k(r;\lambda,\delta) = 2k^r E\frac{|2X-\lambda|^r}{k^r|2X-\lambda|^r + ((k-1)|2X-\lambda|+\delta)^r}, \quad r>0,$$

$$\overline{L}_k^I(m;\lambda,\delta) = 2k^m E\Big[|2X-\lambda|^m \Big/ \Big((k^m+(k-1)^m)|2X-\lambda|^m + (k^m-(k-1)^m)\delta^m\Big)\Big] I[|2X-\lambda| \geq \delta],$$

$$\overline{L}_k(m;\lambda,\delta) = k^m E\frac{|2X-\lambda|^m}{(k^m+(k-1)^m)|2X-\lambda|^m + (k^m-(k-1)^m)\delta^m}, \ m \in \mathbb{N},$$

$$M_k^I(r;\lambda,\delta) = 2^r k^r E\Big[\frac{|2X-\lambda|}{(2k-1)|2X-\lambda|+\delta}\Big]^r I[|2X-\lambda| \geq \delta],$$

$$M_k(r;\lambda,\delta) = 2^r k^r E\Big[\frac{|2X-\lambda|}{(2k-1)|2X-\lambda|+\delta}\Big]^r, \quad r \geq 1,$$

$$M_k^{*I}(r;\lambda,\delta) = 2k^r E\Big[\frac{|2X-\lambda|}{(2k-1)|2X-\lambda|+\delta}\Big]^r I[|2X-\lambda| \geq \delta],$$

$$M_k^*(r;\lambda,\delta) = 2k^r E\Big[\frac{|2X-\lambda|}{(2k-1)|2X-\lambda|+\delta}\Big]^r, \quad 0 < r \le 1.$$

We now study the following approximations

$$P[\lambda_1 < X < \lambda_2] \ge 1 - L_k^I(r;\lambda,\delta) := A_k^I(r;\lambda,\delta)$$
$$\ge 1 - L_k(r;\lambda,\delta) := A_k(r;\lambda,\delta), \quad r > 0, \quad (12)$$

$$P[\lambda_1 < X < \lambda_2] \ge 1 - \overline{L}_k^I(m;\lambda,\delta) := \overline{A}_k^I(r;\lambda,\delta)$$
$$\ge 1 - \overline{L}_k(m;\lambda,\delta) := \overline{A}_k(r;\lambda,\delta), \quad m \in \mathbb{N}, \quad (13)$$

$$P[\lambda_1 < X < \lambda_2] \ge 1 - M_k^I(r;\lambda,\delta) := B_k^I(r;\lambda,\delta)$$
$$\ge 1 - M_k(r;\lambda,\delta) := B_k(r;\lambda,\delta), \quad r \ge 1, \quad (14)$$

and

$$P[\lambda_1 < X < \lambda_2] \ge 1 - M_k^{*I}(r;\lambda,\delta) := B_k^{*I}(r;\lambda,\delta)$$
$$\ge 1 - M_k^*(r;\lambda,\delta) := B_k^*(r;\lambda,\delta), \quad 0 < r \le 1. \quad (15)$$

We note that for given $k,\ r,\ m$ the bounds in (12)-(15) are informative only for those intervals for which the RHS is positive.

The following examples give some comparisons of the above bounds and the Ferentinos [2] bound in (4).

Example 1. Pareto distribution. Let

$$F(x) = 1 - 1/x^3,\ f(x) = 3/x^4,\ x \ge 1, \quad (16)$$

and 0 otherwise. Then $\mu = EX = 3/2$, $\sigma^2 = \sigma^2 X = 3/4$, and the pth quantile $x_p = 1/\sqrt[3]{1-p}$.

Suppose first that $\lambda_1 = x_{1/4} = \sqrt[3]{4/3},\ \lambda_2 = 4$. Hence by (16) $P[\lambda_1 < X < \lambda_2] = .73437$, and the Ferentinos bound (4) of $P[\sqrt[3]{4/3} < X < 4]$ is .11819. The values of our bounds in (12) – (15) for selected values of $r,\ k$ (here $r = m$) are shown in Table 3.

k	m	1	2	3	4
	$A_k^I(m)$	.72517	.71765	.71133	.70567
	$A_k(m)$	.12965	.211067	.26730	.30899
1	$\overline{A}_k^I(m)$	.72517	.71765	.71133	.70567
	$\overline{A}_k(m)$	.12965	.21107	.26730	.30899
	$B_k^I(m)$	.72517	.71373	.69891	.67882
	$B_k(m)$	.12965	.19600	.23568	.25733
	$A_k^I(m)$	.73022	.72619	.72237	.71879
	$A_k(m)$	.08813	.14841	.19319	.22867
2	$\overline{A}_k^I(m)$	.73022	.72272	.71504	.70805
	$\overline{A}_k(m)$	.08813	.18976	.25783	.30495
	$B_k^I(m)$	.73022	.72570	.72076	.71531
	$B_k(m)$	.08813	.14057	.17818	.20696
	$A_k^I(m)$	.73238	.73039	.72843	.72650
	$A_k(m)$	.05713	.09906	.13203	.15939
4	$\overline{A}_k^I(m)$	.73238	.72805	.72260	.71660
	$\overline{A}_k(m)$	.05713	.15159	.22677	.28295
	$B_k^I(m)$	.73238	.73031	.72815	.72591
	$B_k(m)$	.05713	.09443	.12316	.14673
	$A_k^I(m)$	.73351	.73264	.73177	.73091
	$A_k(m)$	.03290	.05857	.07976	.09806
9	$\overline{A}_k^I(m)$	.73351	.73147	.72867	.72530
	$\overline{A}_k(m)$	.03290	.10815	.17911	.23846
	$B_k^I(m)$	.73351	.73262	.73172	.73081
	$B_k(m)$	.03290	.05613	.07498	.09113
	$A_k^I(m)$	.73389	.73340	.73292	.73243
	$A_k(m)$	.02172	.03924	.05409	.06719
16	$\overline{A}_k^I(m)$	.73389	.73273	.73108	.72904
	$\overline{A}_k(m)$	.02172	.08280	.14646	.20327
	$B_k^I(m)$	.73389	.73340	.73290	.73241
	$B_k(m)$	.02172	.03773	.05109	.06278
	$A_k^I(m)$	.73407	.73376	.73345	.73314
	$A_k(m)$	.01556	.02838	.03944	.04932
25	$\overline{A}_k^I(m)$	.73407	.73331	.73224	.73090
	$\overline{A}_k(m)$	.01556	.06670	.12392	.17719
	$B_k^I(m)$	.73407	.73375	.73344	.73313
	$B_k(m)$	.01556	.02735	.03737	.04626
	$A_k^I(m)$	.73416	.73395	.73373	.73352
	$A_k(m)$	.01177	.02161	.03021	.03797
36	$\overline{A}_k^I(m)$	.73416	.73364	.73288	.73194
	$\overline{A}_k(m)$	.01177	.05569	.10768	.15754
	$B_k^I(m)$	.73416	.73394	.73373	.73351
	$B_k(m)$	.01177	.02087	.02871	.03572

Table 3: Bounds of $P[\sqrt[3]{\frac{4}{3}} < X < 4] = .73437$ (X from Example 1, the bound of Ferentinos - .11819)

We now consider $\lambda_1 = x_{1/2} = \sqrt[3]{2}$, $\lambda_2 = x_{3/4} = \sqrt[3]{4}$. Hence by (16)

k	m	1	2	3	4
16	$A_k^I(m)$	.23778	.22557	.21338	.20122
	$A_k(m)$	.00782	.01080	.01077	.00880
	$\overline{A}_k^I(m)$	.23778	.21589	.19172	.16718
	$\overline{A}_k(m)$	.00782	.03670	.05361	.05765
	$B_k^I(m)$	.23778	.22532	.21263	.19969
	$B_k(m)$	.00782	.00897	.00699	.00300
25	$A_k^I(m)$	.24223	.23447	.22671	.21896
	$A_k(m)$	.00694	.01071	.01246	.01287
	$\overline{A}_k^I(m)$	.24223	.22830	.21290	.19723
	$\overline{A}_k(m)$	.00694	.03804	.06293	.07666
	$B_k^I(m)$	.24223	.23437	.22641	.21836
	$B_k(m)$	.00694	.00949	.00998	.00913
36	$A_k^I(m)$	.24463	.23925	.23388	.22851
	$A_k(m)$	.00594	.00965	.01192	.01322
	$\overline{A}_k^I(m)$	.24463	.23498	.22432	.21347
	$\overline{A}_k(m)$	.00594	.03661	.06506	.08390
	$B_k^I(m)$	.24463	.23921	.23374	.22823
	$B_k(m)$	.00594	.00878	.01014	.01055

Table 4: Bounds of $P[\sqrt[3]{2} < X < \sqrt[3]{4}] = .25$ (X from Example 1)

$P[\lambda_1 < X < \lambda_2] = 0.25$, but in this case the Ferentinos [2] bound in (4) is not informative. Examples of our bounds in (12) – (15) are shown in Table 4.

Example 2. Pareto distribution. Let

$$F(x) = 1 - 1/x,\ f(x) = 1/x^2, \quad x \geq 1, \tag{17}$$

and 0 otherwise. Then $\mu = EX = \infty$ and the pth quantile $x_p = 1/(1-p)$.

Here we consider $\lambda_1 = x_{1/2} = 2,\ \lambda_2 = x_{3/4} = 4$. Hence by (17) $P[\lambda_1 < X < \lambda_2] = 0.25$, but in this case the Ferentinos [2] bound (4), does not exist. Examples of our bounds in (12)-(15) are shown in Table 5 and Table 6.

3.2. Weak Law of Large Numbers

Now we study applications of the inequalities in Section 2 to the weak law of large numbers (WLLN). Let $\{X_n,\ n \geq 1\}$ be random variables with finite expectations $EX_n,\ n \geq 1$. Write

$$S_n = \sum_{j=1}^{n} X_j, \quad \overline{X}_n = S_n/n.$$

Then by (9), for any given $\varepsilon > 0$, each positive integer $m \in \mathbb{N}$ and all $k \geq 1$ we have

k	m	1	2	3	4
16	$A_k^I(m)$	.23858	.22716	.21577	.20441
	$A_k(m)$	.00744	.01038	.01052	.00886
	$\overline{A}_k^I(m)$	.23858	.21828	.19511	.17094
	$\overline{A}_k(m)$	.00744	.03525	.05167	.05564
	$B_k^I(m)$	.23858	.22693	.21506	.20296
	$B_k(m)$	.00744	.00867	.00697	.00340
25	$A_k^I(m)$	.24274	.23548	.22823	.22099
	$A_k(m)$	.00656	.01018	.01191	.01241
	$\overline{A}_k^I(m)$	.24274	.22981	.21502	.19956
	$\overline{A}_k(m)$	.00656	.03615	.05997	.07321
	$B_k^I(m)$	.24274	.23539	.22795	.22042
	$B_k(m)$	.00656	.00904	.00959	.00890
36	$A_k^I(m)$	.24498	.23996	.23494	.22992
	$A_k(m)$	.00560	.00912	.01131	.01260
	$\overline{A}_k^I(m)$	.24498	.23602	.22577	.21505
	$\overline{A}_k(m)$	.00560	.03461	.06167	.07978
	$B_k^I(m)$	.24498	.23991	.23480	.22965
	$B_k(m)$	.00560	.00832	.00965	.01010

Table 5: Bounds of $P[2 < X < 4] = .25$ (X from Example 2)

k	r	1/2	1/4	1/9	1/16
9	$A_k^I(r)$	.23969	.24484	.24771	.24871
	$A_k(r)$	.00461	.00254	.00116	.00067
16	$A_k^I(r)$	.24429	.24714	.24873	.24929
	$A_k(r)$	.00437	.00232	.00105	.00059
25	$A_k^I(r)$	.24637	.24819	.24919	.24955
	$A_k(r)$	.00370	.00194	.00087	.00049
36	$A_k^I(r)$	.24749	.24874	.24944	.24969
	$A_k(r)$	.00309	.00161	.00072	.00041

Table 6: Bounds of $P[2 < X < 4] = .25$ (X from Example 2)

$$P[|\overline{X}_n - E\overline{X}_n| \geq \varepsilon] \leq 2k^m E\Big[|S_n - ES_n|^m \Big/ \Big((k^m + (k-1)^m)|S_n - ES_n|^m + (k^m - (k-1)^m)n^m\varepsilon^m\Big)\Big]. \quad (18)$$

For any given $\varepsilon > 0$ and m we can choose k in (18) such that $(k^m - (k-1)^m)\varepsilon^m \geq 1$. So then we have

$$P[|\overline{X}_n - E\overline{X}_n| \geq \varepsilon] \leq 2k^m E\frac{|S_n - ES_n|^m}{(k^m + (k-1)^m)|S_n - ES_n|^m + n^m} \leq 2k^m E\frac{|S_n - ES_n|^m}{|S_n - ES_n|^m + n^m},$$

and we see that $\{X_n,\ n \geq 1\}$ satisfies the WLLN, i.e.

$$\overline{X}_n - E\overline{X}_n \overset{P.}{\to} 0, \quad n \to \infty \ (P. - \text{in probability}) \tag{19}$$

whenever

$$E\frac{|S_n - ES_n|^m}{|S_n - ES_n|^m + n^m} \to 0, \quad n \to \infty.$$

Referring now to (11) for $0 < r \leq 1$, we have

$$\begin{aligned} P[|\overline{X}_n - E\overline{X}_n| \geq \varepsilon] &\leq 2E\Big[\frac{k|\overline{X}_n - E\overline{X}_n|}{(2k-1)|\overline{X}_n - E\overline{X}_n| + \varepsilon}\Big]^r I[|\overline{X}_n - E\overline{X}_n| \geq \varepsilon] \\ &\leq 2E\Big[\frac{k|\overline{X}_n - E\overline{X}_n|}{(k-1)|\overline{X}_n - E\overline{X}_n| + (k+1)\varepsilon}\Big]^r I[|\overline{X}_n - E\overline{X}_n| \geq \varepsilon] \\ &\leq 2E\Big[\frac{k|S_n - ES_n|}{(k-1)|S_n - ES_n| + n(k+1)\varepsilon}\Big]^r. \end{aligned} \tag{20}$$

For any given $\varepsilon > 0$ we can choose k in (20) such that $(k+1)\varepsilon \geq 1$. So then we have

$$P[|\overline{X}_n - E\overline{X}_n| \geq \varepsilon] \leq 2E\Big[\frac{k|S_n - ES_n|}{(k-1)|S_n - ES_n| + n}\Big]^r$$

and we see that the condition

$$E\Big[\frac{|S_n - ES_n|}{|S_n - ES_n| + n}\Big]^r \to 0, \quad n \to \infty,$$

implies the WLLN. A similar discussion can be done for $r \geq 1$. The rate of convergence in (19) can be estimated by the presented bounds. For instance one can use the inequalities:

$$\begin{aligned} P\Big[\Big|\frac{S_n}{n} - E\frac{S_n}{n}\Big| \geq \varepsilon\Big] \leq 2\inf_{k\geq 1} E\big[(k|S_n - ES_n|)^m \big/ \big((k^m + (k-1)^m)|S_n - ES_n|^m \\ + (k^m - (k-1)^m)(n\varepsilon)^m\big)\big], \end{aligned}$$

$$P\Big[\Big|\frac{S_n}{n} - E\frac{S_n}{n}\Big| \geq \varepsilon\Big] \leq 2 \inf_{k\geq 1,\ r\in(0,1]} E\Big[\frac{k|S_n - ES_n|}{(k-1)|S_n - ES_n| + n(k+1)\varepsilon}\Big]^r,$$

$$P\Big[\Big|\frac{S_n}{n} - E\frac{S_n}{n}\Big| \geq \varepsilon\Big] \leq \inf_{k\geq 1,\ r\geq 1} E\Big[\frac{2k|S_n - ES_n|}{(2k-1)|S_n - ES_n| + n\varepsilon}\Big]^r,$$

$$P\Big[\Big|\frac{S_n}{n} - E\frac{S_n}{n}\Big| \geq \varepsilon\Big] \leq 2 \inf_{k\geq 1,\ r>0} E\frac{(k|S_n - ES_n|)^r}{(k|S_n - ES_n|)^r + [(k-1)|S_n - ES_n| + n\varepsilon]^r}.$$

4. Bivariate Case

First we recall a result of Ferentinos [2]. Let $X_1,\ X_2$ be two random variables with means $\mu_1,\ \mu_2$ and finite variances $\sigma_1^2,\ \sigma_2^2$, respectively. Then

$$P[\lambda_{11} < X_1 < \lambda_{12},\ \lambda_{21} < X_2 < \lambda_{22}] \geq 1 - \sum_{i=1}^{2} T_i, \tag{21}$$

where

$$T_i = \frac{4\sigma_i^2 + [2\mu_i - (\lambda_{i1} + \lambda_{i2})]^2}{(\lambda_{i2} - \lambda_{i1})}, \quad i = 1, 2.$$

We give similar bounds for the probability in (21) for a large class of random variables. Write

$$A_1 = [\lambda_{11} < X_1 < \lambda_{12}],\ A_2 = [\lambda_{21} < X_2 < \lambda_{22}].$$

Using Boole's inequality

$$P[A_1 \cap A_2] \geq 1 - \sum_{i=1}^{2} P(\overline{A_i})$$

we get the following generalization of (21).

Theorem 2. *Let $X_1,\ X_2$ be random variables, $g \in \mathcal{G}$, and $\lambda_{11},\ \lambda_{12},\ \lambda_{21}$ and λ_{22} be real numbers such that $\lambda_{11} \leq \lambda_{12}$ and $\lambda_{21} \leq \lambda_{22}$. Write*

$$\lambda_1 = \lambda_{11} + \lambda_{12},\ \delta_1 = \lambda_{12} - \lambda_{11},$$
$$\lambda_2 = \lambda_{21} + \lambda_{22},\ \delta_2 = \lambda_{22} - \lambda_{21}.$$

Then for all $k \geq 1$,

$$P[\lambda_{11} < X_1 < \lambda_{12},\ \lambda_{21} < X_2 < \lambda_{22}]$$
$$\geq 1 - 2\sum_{i=1}^{2} E \frac{g(k|X_i - \frac{\lambda_i}{2}|)}{g(k|X_i - \frac{\lambda_i}{2}|) + g((k-1)|X_i - \frac{\lambda_i}{2}| + \frac{\delta_i}{2})} I[|X_i - \frac{\lambda_i}{2}| \geq \frac{\delta_i}{2}]$$
$$\geq 1 - 2\sum_{i=1}^{2} E \frac{g(k|X_i - \frac{\lambda_i}{2}|)}{g(k|X_i - \frac{\lambda_i}{2}|) + g((k-1)|X_i - \frac{\lambda_i}{2}| + \frac{\delta_i}{2})}.$$

Corollary 2. *Suppose that the assumptions of Theorem 2 are satisfied with $g(|X|) = |X|^r$, $r > 0$. Then for all $k \geq 1$ we have the following bounds:*

$$P[\lambda_{11} < X_1 < \lambda_{12},\ \lambda_{21} < X_2 < \lambda_{22}] \geq 1 - 2\sum_{i=1}^{2} L_k^I(r; \lambda_i, \delta_i)$$

$$\geq 1 - 2\sum_{i=1}^{2} L_k(r;\lambda_i,\delta_i), \quad r > 0,$$

$$P[\lambda_{11} < X_1 < \lambda_{12},\ \lambda_{21} < X_2 < \lambda_{22}] \geq 1 - 2\sum_{i=1}^{2} \overline{L}_k^I(m;\lambda_i,\delta_i)$$

$$\geq 1 - 2\sum_{i=1}^{2} \overline{L}_k(m;\lambda_i,\delta_i), \quad m \in \mathbb{N},$$

$$P[\lambda_{11} < X_1 < \lambda_{12},\ \lambda_{21} < X_2 < \lambda_{22}] \geq 1 - 2\sum_{i=1}^{2} M_k^I(r;\lambda_i,\delta_i)$$

$$\geq 1 - 2\sum_{i=1}^{2} M_k(r;\lambda_i,\delta_i), \quad r \geq 1,$$

$$P[\lambda_{11} < X_1 < \lambda_{12},\ \lambda_{21} < X_2 < \lambda_{22}] \geq 1 - 2\sum_{i=1}^{2} M_k^{*I}(r;\lambda_i,\delta_i)$$

$$\geq 1 - 2\sum_{i=1}^{2} M_k^*(r;\lambda_i,\delta_i), \quad 0 < r \leq 1,$$

where

$$L_k^I(r;\lambda_i,\delta_i) = 2k^r E\Big[|2X_i - \lambda_i|^r \Big/ \Big(k^r|2X_i - \lambda_i|^r + ((k-1)|2X_i - \lambda_i| + \delta_i)^r\Big)\Big] \cdot I[|2X_i - \lambda_i| \geq \delta_i],$$

$$L_k(r;\lambda_i,\delta_i) = 2k^r E\Big[\frac{|2X_i - \lambda_i|^r}{k^r|2X_i - \lambda_i|^r + ((k-1)|2X_i - \lambda_i| + \delta_i)^r}\Big], \quad r > 0,$$

$$\overline{L}_k^I(m;\lambda_i,\delta_i) = 2k^m E\Big[|2X_i - \lambda_i|^m \Big/ \Big((k^m + (k-1)^m)|2X_i - \lambda_i|^m + (k^m - (k-1)^m)\delta^m\Big)\Big] \cdot I[|2X_i - \lambda_i| \geq \delta_i],$$

$$\overline{L}_k(m;\lambda_i,\delta_i) = 2k^m E\Big[|2X_i - \lambda_i|^m \Big/ \Big((k^m + (k-1)^m)|2X_i - \lambda_i|^m + (k^m - (k-1)^m)\delta^m\Big)\Big], \ m \in \mathbb{N},$$

$$M_k^I(r;\lambda,\delta) = 2^r k^r E\Big[\frac{|2X_i - \lambda_i|}{(2k-1)|2X_i - \lambda_i| + \delta_i}\Big]^r \cdot I[|2X_i - \lambda_i| \geq \delta_i],$$

$$M_k(r;\lambda,\delta) = 2^r k^r E\Big[\frac{|2X_i-\lambda_i|}{(2k-1)|2X_i-\lambda_i|+\delta_i}\Big]^r, \quad r \geq 1,$$
$$M_k^{*I}(r;\lambda,\delta) = 2k^r E\Big[\frac{|2X_i-\lambda_i|}{(2k-1)|2X_i-\lambda_i|+\delta_i}\Big]^r \cdot I[|2X_i-\lambda_i| \geq \delta_i],$$
$$M_k^{*}(r;\lambda,\delta) = 2k^r E\Big[\frac{|2X_i-\lambda_i|}{(2k-1)|2X_i-\lambda_i|+\delta_i}\Big]^r, \quad 0 < r \leq 1.$$

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