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RELATIONS BETWEEN THE ULTRAHYPERBOLIC OPERATOR AND $(k-1)$-TH DERIVATIVE OF DIRAC'S DELTA IN $P(x)$ AND $P(x)-c^2$

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Abstract: In this article we obtain formulae between the operator ultrahyperbolic iterated $j-$times defined by (13) and the $(k-1)$-th derivative of Dirac's delta defined in $P(x)=x_1^2+\ldots x_\mu^2-x_{\mu+1}^2\cdots-x_{\mu+\nu}^2$ and $P(x)-c^2$. In particular we obtain that $\delta^{(k-1)}(P_+)$ is homogeneous solution of the ultrahyperbolic operator iterated l times if $\frac{n}{2}-l\leq k<\frac{n}{2}$and $E_{n,r,\mu,\nu}$ defined in (23) is elementary solution of ultrahyperbolic operator iterated l times. By putting $k=s+1, l=1$ in (18) we have that $\delta^{(s)}(P_+)$ is homogeneous solution of the ultrahyperbolic operator if $s=\frac{n-4}{2}$.

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1. Introduction

Let $x=(x_1,x_2,\ldots x_n)$ be a point of the $n-$dimensional Euclidean space R^n.

Consider a quadratic form in n variables defined by

$$P=P(x)=x_1^2+\ldots x_\mu^2-x_{\mu+1}^2\cdots-x_{\mu+\nu}^2, \tag{1}$$

where $\mu+\nu=n$ is the dimension of the space.

We note that the following results are valid:

$$\delta^{(k)}(P_+)=(-1)^k k!\mathrm{Res}_{\lambda=-k-1}P_+^\lambda \tag{2}$$

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(see [7], pp. 256 and 278),

$$\left\{\delta^{(k)}(P_+)\right\}^{\Lambda} = C(-k,n)(-1)^k \left\{e^{-\frac{\nu\pi i}{2}}(Q-i0)^{k-\frac{n}{2}} - e^{\frac{\nu\pi i}{2}}(Q+i0)^{k-\frac{n}{2}}\right\} \quad (3)$$

(see [1], p. 126, formula (99)), where

$$\left(\delta^{(k)}(P),\varphi\right) = \int_0^\infty \left[\left(\frac{\partial}{2s\partial s}\right)^k \left\{s^{q-2}\frac{\Psi(r,s)}{2}\right\}\right]_{s=r} r^{p-1}dr, \quad (4)$$

$$\Psi(r,s) = \int \varphi d\Omega_\mu d\Omega_\nu, \quad (5)$$

(see [7], p. 249),

$$(Q \pm i0)^\lambda = \lim\,(Q \pm i\varepsilon \mid y \mid^2)^\lambda, \quad (6)$$

$$\mid y \mid^2 = y_1^2 + \dots y_n^2\ , \quad (7)$$

$$C(-k,n) = 2^{n-2k}\pi^{\frac{n-2}{2}}\Gamma(\frac{n}{2}-k)(2i)^{-1}, \quad (8)$$

$$Q = Q(y) = y_1^2 + \dots y_\mu^2 - y_{\mu+1}^2 \cdots - y_{\mu+\nu}^2$$

and the symbol Λ we means the Fourier transform,

$$\overset{\Lambda}{f} = \int\limits_{R^n} f(x)e^{-ixy}dx. \quad (9)$$

On the other hand, we have the following properties:

$$\left\{L^l\delta\right\}^\Lambda = (-1)^l Q^l, \quad (10)$$

(see [10], p. 40)

$$(Q \pm i0)^\lambda.(Q \pm i0)^\mu = (Q \pm i0)^{\lambda+\mu} \quad (11)$$

(see [10], p. 23) if λ, μ and $\lambda + \mu \neq -\frac{n}{2} - r, r = 0,1,2,...$ and

$$(Q+i0)^s = (Q-i0)^s = Q^s \quad (12)$$

(see [7], p. 276) for $s = 1,2,3,\dots,$ where L^j is the ultrahyperbolic operator iterated j times defined by

$$L^j = \left\{\frac{\partial^2}{\partial x_1^2} + \dots \frac{\partial^2}{\partial x_\mu^2} - \frac{\partial^2}{\partial x_{\mu+1}^2} - \dots - \frac{\partial^2}{\partial x_{\mu+\nu}^2}\right\}^j. \quad (13)$$

2. $L^l \left\{\delta^{(k-1)}(P_+)\right\}$

Lemma 1. *Let $\delta^{(k-1)}(P_+)$ be the distributional defined by (4) and L^j the ultrahyperbolic operator iterated j times defined by (13), then the following formula is valid*

$$L^l \left\{\delta^{(k-1)}(P_+)\right\} = \frac{2^{2l}\Gamma(\frac{n}{2}-k)}{\Gamma(\frac{n}{2}-k-l)}\delta^{(k+l-1)}(P_+) \tag{14}$$

for n odd and $k+l < \frac{n}{2}$ if n is even.

Proof. We know that $L^j\delta$ is a finite lineal combination of δ and its derivatives, therefore $L^j\delta$ is a convolutor of the space $D^{'}$(space of the distributions), that is $L^j\delta$ is a distribution of the class $O_c^{'}$, where $O_c^{'}$ is the dual of space O_c (see [11], p. 244). On the other hand $\delta^{(k-1)}(P_+)$ is a homogeneous distributions and using (see [6]), $\delta^{(k-1)}(P_+)$ is a temperated distribution, therefore $\delta^{(k-1)}(P_+) \in S^{'}$where $S^{'}$ is the dual of Schwartz space S, and we conclude by appealing to classic theorem of L. Schwartz (see [11], p. 268, formula (II,8,5)) that the following formula is valid

$$\left\{L^l\delta * \delta^{(k-1)}(P_+)\right\}^\Lambda = \left\{L^l\delta\right\}^\Lambda . \left\{\delta^{(k-1)}(P_+)\right\}^\Lambda, \tag{15}$$

where the symbol $*$ means convolution. From (15) and using the formulae (3), (10), (11) and (12) we have

$$\left\{L^l\delta * \delta^{(k-1)}(P_+)\right\}^\Lambda = \frac{(-1)^{k+l}C(-k,n)}{C(-k-l,n)(-1)^{k+l}}\left\{\delta^{(k+l-1)}(P_+)\right\}^\Lambda \tag{16}$$

for n odd and $k+l < \frac{n}{2}$ if n is even. From (16) and using (8) we have

$$L^l\delta * \delta^{(k-1)}(P_+) = \frac{2^{2l}\Gamma(\frac{n}{2}-k)}{\Gamma(\frac{n}{2}-k-l)}\delta^{(k+l-1)}(P_+) \tag{17}$$

for n odd and $k+l < \frac{n}{2}$ if n is even. From (17) we obtain the formula (14). □

In particular if $\frac{n}{2} - l \le k < \frac{n}{2}$,

$$L^l \left\{\delta^{(k-1)}(P_+)\right\} = 0 \tag{18}$$

if n is even. Therefore $\delta^{(k-1)}(P_+)$ is homogeneous solution of the ultrahyperbolic operator iterated l time:

$$L^l = \left\{\frac{\partial^2}{\partial x_1^2} + \dots \frac{\partial^2}{\partial x_\mu^2} - \frac{\partial^2}{\partial x_{\mu+1}^2} - \dots - \frac{\partial^2}{\partial x_{\mu+\nu}^2}\right\}^l. \tag{19}$$

By putting $k = s+1, l = 1$ in (18) we have that $\delta^{(s)}(P_+)$ is homogeneous solution of the ultrahyperbolic operator if $s = \frac{n-4}{2}$.

The formula (14) not depend of signature of μ and ν where $\mu+\nu=n$ is the dimension of the space. By putting $k=\frac{n}{2}-r$, with $\frac{n}{2}-r\geq 0$, for n even, we have

$$L^l\left\{\delta^{(\frac{n}{2}-r-1)}(P_+)\right\}=\frac{2^{2l}\Gamma(r)}{\Gamma(r-l)}\delta^{(\frac{n}{2}-r-1+l)}(P_+) \tag{20}$$

if $l<r<\frac{n}{2}+l-1$.

The formula (20) appears in [5], formula (19), under condition μ and ν odd.

On the other hand, from [5], formula (19), we have

$$L^l\left\{\delta^{(\frac{n}{2}-r-1)}(P_+)\right\}=2^{2r-1}(-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}\Gamma(r)(-1)^{n+1}L^{l-r}\left\{\delta(x)\right\} \tag{21}$$

if $r\leq l,\mu$ and ν are both odd. In particular by putting $l=r$ in (21) we have

$$L^l\left\{\delta^{(\frac{n}{2}-r-1)}(P_+)\right\}=(-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}2^{2r-1}\Gamma(r)(-1)^{n+1}\delta(x) \tag{22}$$

if μ and ν are both odd and $\frac{n}{2}-r\geq 0$. Formula (22) means that

$$E_{n,r,\mu,\nu}=\frac{1}{(-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}2^{2r-1}\Gamma(r)(-1)^{n+1}}\delta^{(\frac{n}{2}-r-1)}(P_+) \tag{23}$$

is elemntal solution of the ultrahyperbolic operation defined by (13) iterated $r-$times, under the conditions μ and ν are both odd and $\frac{n}{2}-r\geq 0$.

Lemma 2. *Let $\delta^{(k-1)}(P_+)$ be the distributional defined by (4) and L^j the ultrahyperbolic operator iterated j times defined by (13), then the following formula is valid*

$$L^l\left\{\delta^{(k-1)}(P_+)\right\}=a_{l,k,n}\delta^{(k+l-1)}(P_+) \tag{24}$$

if $k\geq\frac{n}{2}$, where

$$a_{l,k,n}=(-1)^l\frac{2^{2l}(k+l-\frac{n}{2})!}{(k-\frac{n}{2})!}. \tag{25}$$

Proof. In order to obtain the formula (24), we need the following formulae:

$$\delta^{(k-1)}(P_+)=b_{k,\nu,n}L^{k-\frac{n}{2}}\left\{\delta\right\} \tag{26}$$

(see [3], formula (45)) if μ and ν are both even and $k\geq\frac{n}{2}$, and

$$\delta^{(k-1)}(P_+)=d_{k,\nu,n}L^{k-\frac{n}{2}}\left\{\delta\right\} \tag{27}$$

(see [3], formula (48)) if μ and ν are both odd and $k\geq\frac{n}{2}$, where

$$b_{k,\nu,n}=\frac{(-1)^k(-1)^{\frac{\nu}{2}}\pi^{\frac{n}{2}}}{4^{k-\frac{n}{2}}(k-\frac{n}{2})!} \tag{28}$$

and

$$d_{k,\nu,n} = \frac{(-1)^k (-1)^{\frac{\nu+1}{2}} \pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}} (k-\frac{n}{2})!} \left[\Psi(\frac{\mu}{2}) - \Psi(\frac{n}{2}) \right]. \tag{29}$$

The formulae (27) and (29) are in the sense of regularization of $\delta^{(k-1)}(P_+)$ (see [7], p. 49). In (29) for integral and half-integral values of the argument, $\Psi(x)$ is given by

$$\Psi(k) = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k-1}, \quad k = 2, 3, \ldots \tag{30}$$

and

$$\Psi(k + \frac{1}{2}) = -\gamma - 2\ln 2 + 2\left(1 + \frac{1}{3} + \cdots + \frac{1}{2k-1}\right) \tag{31}$$

$k = 1, 2, 3, \ldots$ where γ is the Euler's constant. Now taking into account the property

$$L^j\{\delta\} * L^l\{\delta\} = L^{j+l}\{\delta\} \tag{32}$$

(see [2], formula (5.10)), where L^j is defined by (13), from (26), (27), (30) and (31), we have

$$\begin{aligned} L^l\delta * \delta^{(k-1)}(P_+) &= L^l\delta * b_{k,\nu,n} L^{k-\frac{n}{2}}\{\delta\} \\ &= b_{k,\nu,n} L^{l+k-\frac{n}{2}}\{\delta\} = \frac{b_{k,\nu,n}}{b_{k+l,\nu,n}} \delta^{(k+l-1)}(P_+) \\ &= (-1)^l \frac{2^{2l}(k+l-\frac{n}{2})!}{(k-\frac{n}{2})!} \delta^{(k+l-1)}(P_+), \end{aligned} \tag{33}$$

if μ and ν are both even and $k \geq \frac{n}{2}$ and

$$\begin{aligned} L^l\delta * \delta^{(k-1)}(P_+) &= L^l\delta * c_{k,\nu,n} L^{k-\frac{n}{2}}\{\delta\} \\ &= c_{k,\nu,n} L^{l+k-\frac{n}{2}}\{\delta\} = \frac{c_{k,\nu,n}}{c_{k+l,\nu,n}} \delta^{(k+l-1)}(P_+) \\ &= (-1)^l \frac{2^{2l}(k+l-\frac{n}{2})!}{(k-\frac{n}{2})!} \delta^{(k+l-1)}(P_+), \end{aligned} \tag{34}$$

if μ and ν are both odd and $k \geq \frac{n}{2}$. From (33) and (34) we obtain the formulae (24) and (25). □

3. $L^l\left\{\delta^{(k-1)}(P-c^2)\right\}$

Using the Leray formula (see [9], p. 102) the distribution $\delta^{(k-1)}(P(x)-c^2)$ can be defined by the following form

$$\left\langle \delta^{(k-1)}(P(x)-c^2),\varphi\right\rangle = \left\langle \delta^{(k-1)}(t),\Phi(t)\right\rangle = (-1)^{k-1}\Phi^{(k-1)}(0) \tag{35}$$

(see [8], p. 189), where $P(x)$ is defined by (1), c is a constant,

$$\Phi(t) = \int\limits_{P-c^2=t} \varphi w_{P(x)-c^2}(x,dx) \tag{36}$$

(see [8], p. 189), w_u is an $(n-1)$-dimensional form on u, defined as follows:

$$du\Lambda dw = dx_1\Lambda dx_2\Lambda\ldots\Lambda dx_n \tag{37}$$

and the manifold $u(x)=s$ has the orientation such that $w(x,dx)>0$.

On the other hand, from [4], formula (49), we have the following formula

$$\delta^{(k-1)}(P(x)-c^2) = \sum_{\nu=o}^{\infty}\frac{(-c^2)^\nu}{\nu!}\delta^{(k-1)}(P_+). \tag{38}$$

Lemma 3. *Let $\delta^{(k-1)}(P(x)-c^2)$ be the distribution defined by (35) and L^l the ultrahyperbolic operator defined by (19), then the following formula is valid*

$$L^l\left\{\delta^{(k-1)}(P-c^2)\right\} = \sum_{j=o}^{\infty}\frac{(-c^2)^j}{j!}\frac{2^{2j}\Gamma(\frac{n}{2}-k-j)}{\Gamma(\frac{n}{2}-(k+j)-l)}\delta^{(k+l+j-1)}(P_+) \tag{39}$$

if $k+j<\frac{n}{2}$.

Proof. The formula (39) is consequence of the formulae (14) and (31). □

We observe that if $\frac{n}{2}-(k+j)-l\leq 0$ and $k+j<\frac{n}{2}$, then

$$L^l\left\{\delta^{(k-1)}(P-c^2)\right\} = 0. \tag{40}$$

Therefore $\delta^{(k-1)}(P-c^2)$ is homogeneous solution of the operator

$$\left\{\sum_{i=1}^{\mu}\frac{\partial^2}{\partial x_i^2} - \sum_{i=\mu+1}^{\mu+\nu}\frac{\partial^2}{\partial x_i^2}\right\}^j \text{ if } \frac{n}{2}-(k+j)-l\leq 0 \text{ and } k+j<\frac{n}{2}.$$

Therefore, using (40) the formula (39) can be rewritten in the following form

$$L^l\left\{\delta^{(k-1)}(P-c^2)\right\} = \sum_{j=o}^{\frac{n}{2}-k-l-1}\frac{(-c^2)^j}{j!}\frac{2^{2j}\Gamma(\frac{n}{2}-k-j)}{\Gamma(\frac{n}{2}-(k+j)-l)}\delta^{(k+l+j-1)}(P_+) \tag{41}$$

if $\frac{n}{2} - k - l - 1 \geq 0$.

We observe that by putting $c^2 = 0$ in (40) and (41) we obtain (14) and (18)respectively.

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