International Journal of Pure and Applied Mathematics
Volume 58 No. 4 2010, 425-434

ANALYTICAL SOLUTION TO A HIGHLY NONLINEAR EARTH-SATELLITE PITCH ATTITUDE LIBRATION EQUATION

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Abstract: We derive the dynamical equations that characterizes the pitch librations of an earth-satellite. To obtain analytical solution of the resulting highly nonlinear equation, we adopt a scheme that successively augments the nonlinearity level of the equation by adding nonlinear terms. We compare our results with available numerical solution.

AMS Subject Classification: 70F05, 85-02
Key Words: analytical solution, nonlinear, earth-satellite, libration, dynamics

1. Introduction

An earth-satellite is basically controlled by gravitational force of the earth. Even so, because of the presence of several other forces such as atmospheric

Received: January 5, 2010

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drag, solar radiation pressure, and solar and lunar gravitational fields [2], the path of a satellite is, in general, a complicated three dimensional curve [4]. Thus, studying the dynamics of the motion of a satellite requires the solution of highly nonlinear equations.

The dynamics of several types of earth-satellite systems have been studied by researchers. Of note is the recent work by Misra [8] who studied the dynamics and control of tethered satellite systems. Working with Wong, Misra [11] extended this to the analysis of multi-tethered spacecrafts. They derived and analyzed the equations of motion governing such systems using the Langrangian approach. In addition, they considered spiral motion at the end masses. Kumar and coworkers [7], and Patel, working with colleagues [9], conducted studies on the stabilization of satillite motion using solar rings. They appealed to Euler moment equations to derive complex nonlinear governing equations. Kumar then proposed and analyzed the use of solar rings in the stabilization of satellite attitude [6]. Work on the main satellite libration has been limited.

The differential equations describing orbiting earth-satellite pitch attitude librations are usually nonlinear [8] and [10]. Such nonlinear equations usually cannot be solved analytically. Typically, approximate solutions are obtained by linearizing about a fixed point. Consequently, solutions obtained in this way may lead to the satellite drifting from the intended trajectory [9].

We present the equations governing the motion of an orbiting earth-satellite pitch attitude librations. These equations turn out to be highly nonlinear. We obtain an analytical solution to the main equation by successively augmenting the nonlinearity level of the system by adding nonlinear terms. Our analytical solution is compared with an available numerical solution.

2. Derivation of Pitch Equation in Terms of Anomaly

The attitude librations of an earth-satellite is typically defined in terms of the roll, pitch and yaw (Euler angles) [5] and [6]. In order to derive the governing equation of the system we proceed as follows: First we make the following assumptions:

1. Earth is spherically symmetrical and is of uniform density and thus it is treated as a point mass.

2. The satellite orbits high enough above the earth, as such, drag force is negligible.

3. There is no maneuvering or significant change in path, hence thrust force

is ignored.

4. Other forces such as those due to solar radiation and electromagnetic fields are negligible compared to the earth's gravity.

5. Since the satellite is relatively close to the earth, the gravitational attraction of the sun and other third bodies are ignored.

We proceed to give the list symbols and parameters used in the model.

Symbol	***Description***
r	Orbital radius
μ	Gravitational parameters
ϕ	Roll attitude angle
ψ	Yaw attitude angle
θ	Pitch attitude angle
ω_c	Orbital rate corresponding to circular orbit at perigee
e	Eccentricity of the orbit
φ	True anolomaly of the satellite
σ	Inertial ratio

Next we consider the satellite as a rigid non-spinning body with no internal momentum and no passive or active control subsystem. The earth is considered as a point mass. The earth and orbiting satellite are thus considered as a two body system. Consequently, the attitude dynamics can be described by a system of equations of a rigid body dynamics. Using Newtonian universal law of gravitation and ignoring the departure of the motion from Newton's law due to relativity effects [4], the governing equations can be derived as:

$$\begin{aligned} I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z &= M_x, \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z &= M_y, \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y &= M_z, \end{aligned} \tag{1}$$

where I_x, I_y, I_z are the non-zero components of the moment of inertia tensor of the body, and M_x, M_y, M_z are components of the external torque vector acting on the body. In the case of an orbiting satellite, the external torque attributed to gravity can be expressed as:

$$\begin{aligned} M_x &= -\frac{3\mu}{r_c^3}(I_y - I_z)\cos^2\theta\cos\phi\sin\phi, \\ M_y &= -\frac{3\mu}{r_c^3}(I_x - I_z)\sin\theta\cos\phi\cos\phi, \\ M_z &= -\frac{3\mu}{r_c^3}(I_y - I_x)\sin\theta\cos\phi\sin\phi. \end{aligned} \tag{2}$$

The attitude equations are then obtained by substituting (2) in (1).

Next we substitute for the angular velocities and their derivatives. This leads to

$$\begin{aligned} I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z &= -\frac{3\mu}{r_c^3}(I_y - I_z)\cos^2\theta\cos\phi\sin\phi, \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z &= -\frac{3\mu}{r_c^3}(I_x - I_z)\sin\theta\cos\theta\cos\phi, \\ I_z + (I_y - I_x)\omega_x\omega_y &= -\frac{3\mu}{r_c^3}(I_y - I_x)\sin\theta\cos\theta\sin\phi. \end{aligned} \tag{3}$$

We note that satellite motions (librations) usually include a combination of relatively small rotational motions compared to the main orbital motion [10]. Consequently, the dynamics of the physical system controlling the motion are left intact when terms representing such small motions are neglected. This enables us to neglect terms of degree greater than or equal to two which are of lower order in terms of accuracy. Thus in terms of the Euler angles of the satellite the dynamic equations become:

$$\begin{aligned} I_x\ddot{\phi} - I_x\dot{\omega}_0\sin\Psi - \omega_0(I_z - I_y + I_x) - (I_z - I_y)(\omega_0^2 + \frac{3\mu}{r_c^3})\sin\theta &= 0, \\ I_y\ddot{\theta} + \frac{3\mu}{r_c^3}(I_x - I_z)\sin\theta &= I_y\dot{\omega}_0, \\ I_z\dot{\psi} + I_z\dot{\omega}_0 + \omega(I_z - I_y + I_x)\dot{\phi} - \omega_0^2(I_x - I_y)\sin\psi &= 0. \end{aligned} \tag{4}$$

The second equation in system (3) represents the uncoupled pitch motion. This can be rearranged as:

$$\ddot{\theta} + \frac{3\mu(I_x - I_z)}{r_c^3 I_y}\sin 2\theta = \dot{\omega}_0. \tag{5}$$

Expressing the radius r_c in terms of the orbital eccentricity e, radius at periapsis r_p, and the true anomaly φ we have:

$$r_c = \frac{r_p(1+e)}{1 + e\cos\varphi}. \tag{6}$$

Hence,

$$\frac{\mu}{r_c^3} = \frac{\mu}{\mu r_p^3}(\frac{1 + e\cos\varphi}{1+e})^3 = \omega_c^2(\frac{1 + e\cos\varphi}{1+e})^3,$$

where $\omega_c^2 = \frac{\mu}{r_p^3}$, with ω_c being the orbital angular rate for a circular orbit of radius r_p. The semi-latus rectum p is related to r_p (the radius at periapsis) by $p = r_p(1+e)$. Setting $h^2 = \mu p$ we have:

$$\omega_0 = \phi = \frac{h}{r_c^2} = \omega_c\frac{(1 + e\cos\varphi)^2}{(1+e)^{3/2}}. \tag{7}$$

Thus

$$\dot{\omega} = -2\omega_c^2 e \sin\varphi\Big(\frac{1+e\cos\varphi}{1+e}\Big)^3,$$

where the dot represents derivative with respect to time. Substituting for $\dot{\varphi}$ in (5), we obtain the pitch equation for eccentric orbit in terms of the true anomaly φ as:

$$\ddot{\theta} + \frac{3}{2}\omega_c^2(\frac{1+e\cos\varphi}{1+e})^3\sigma\sin 2\theta = -2\omega_c^2 e\sin\varphi(\frac{1+e\cos\varphi}{1+e})^3, \tag{8}$$

where:

θ is the pitch attitude angle;

ω_c is the orbital rate corresponding to a circular orbit;

e is the eccentricity of the orbit;

φ is the true anomaly of the satellite.

3. Analytical Solution of the Nonlinear Equation

From (8), we now make the substitutions:

$$\begin{aligned} x_1 &= \theta, && (9)\\ \dot{x_1} &= \dot{\theta} = x_2, && (10)\\ \alpha_1 &= \frac{\omega_c^2}{2}\Big(\frac{1+e\cos\varphi}{1+e}\Big)^3, && (11)\\ \alpha_2 &= e\sin\varphi. && (12) \end{aligned}$$

By eliminating time t and by appropriate substitutions from (9)-(12) we can represent the governing equations by a single vector equation as:

$$x(\varphi) = F(x,\varphi), \tag{13}$$

where

$$x(\varphi) = \begin{bmatrix} x_1(\varphi) \\ x_2(\varphi) \end{bmatrix}$$

and

$$F(x,\varphi) = \begin{bmatrix} \frac{k_1}{(1+e\cos\varphi)^2} \\ \frac{3\sigma k_2}{2}(1+e\cos\varphi)(-2\sin 2x_1 - \frac{4e}{3\sigma}\sin\varphi \end{bmatrix}.$$

We express the nonlinear equation $\dot{x} = F(x,\varphi)$ in the form

$$\dot{x} = Ax + G(x,\varphi), \tag{14}$$

where

$$G(x,\varphi) = F(x,\varphi) - Ax = \begin{bmatrix} 0 \\ z(x,\varphi) \end{bmatrix}$$

with $Z(x,\varphi) = 3\sigma k_2(1 + e\cos\varphi)(x_1 - \frac{1}{2}\sin 2x_1 - \frac{2e}{3\sigma}\sin\varphi)$. Thus $\dot{x}(\varphi) = Ax + G(\varphi)$, where

$$G(\varphi) = \begin{bmatrix} 0 \\ -4ek_2(1 + e\cos\varphi)\sin\varphi \end{bmatrix} = \begin{bmatrix} 0 \\ -4ek_2(\sin\varphi + \frac{e}{2}\sin 2\varphi) \end{bmatrix}.$$

The general solution to equation (14) is given by

$$x(\varphi) = \Phi(\varphi)x_0 + \Phi(\varphi)\int_0^{\varphi} \Phi(s)^{-1}G(s)ds. \tag{15}$$

We now proceed to evaluate the integral

$$J = \int_0^{\varphi} \Phi(s)^{-1}G(s)ds. \tag{16}$$

By linearization (see, for example, [7]) the fundamental matrix $\Phi(\varphi)$ can be expressed as

$$\Phi(\varphi) = Q(\varphi)e^{\varphi R}, \tag{17}$$

where

$$\Lambda(\varphi) = e^{\mu\varphi}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

and

$$e^{\varphi R} = P\Lambda(\varphi)P^{-1}, P = \begin{bmatrix} \cos\omega\varphi & \sin\omega\varphi \\ -\sin\omega\varphi & \cos\omega\varphi \end{bmatrix}.$$

Here, $u + i\omega$ are the eigenvalues of R and

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \pm i \begin{bmatrix} v_1 \\ V_2 \end{bmatrix}$$

are the corresponding eigenvectors. In addition,

$$\Phi(\varphi) = Q(\varphi)P_x\Lambda(\varphi)P_x^{-1} = Q(\varphi)M(\varphi),$$

where $Q(\varphi) = P_0 + P_1\cos\varphi + q_1\sin\varphi$ and $M(\varphi) = P_x\Lambda(\varphi)P_x^{-1}\varphi$ is harmonic with frequency of ω and $Q(\varphi)$ is periodic with period of 2π. Since $\Phi(\varphi)$ is a product of $Q(\varphi)$ and $M(\varphi)$, it is harmonic with multiple frequencies of $\omega - 1$, ω and $\omega + 1$. Thus $\Phi(\varphi)$ can be expressed in terms of trigonometric functions as:

$$\begin{aligned} \Phi(\varphi) &= e^{\mu\varphi}\big(f_1\cos\omega\varphi + f_2\sin\omega\varphi + f_3\cos(\omega-1)\varphi + f_4\sin(\omega-1)\varphi \\ &+ f_5\cos(\omega+1)\varphi + f_6\sin(\omega+1)\varphi\big), \end{aligned} \tag{18}$$

where $f_1, ..., f_6$ are real matrices and are given by:

$$\begin{aligned}
f_1 &= p_0, \\
f_2 &= \begin{bmatrix} -(p_{011}k_1 + p_{012}k_3) & (p_{011}k_2 + (p_{012}k_1) \\ -(p_{021}k_1 + p_{022}k_3) & (p_{021}k_2 + p_{022}k_1) \end{bmatrix}, \\
f_3 &= p_1, \\
f_4 &= \begin{bmatrix} -(p_{111}k_1 + p_{112}k_3) & (p_{111}k_2 + p_{112}k_1) \\ -(p_{121}k_1 + p_{122}k_3) & (p_{121}k_2 + p_{122}k_1) \end{bmatrix}, \\
f_5 &= f_3, \\
f_6 &= f_4.
\end{aligned}$$

The fundamental matrix $\Phi(s)$ is endowed with special properties which enables us to find the its inverse easily as:

$$\begin{aligned}
\Phi^{-1}(s) &= e^{\mu s}\big(g_1 \cos\omega s + g_2 \sin\omega s + g_3 \cos(\omega-1)s + g_4 \sin(\omega-1)s \\
&+ g_5 \cos(\omega+1)s + g_6 \sin(\omega+1)s\big). \qquad (19)
\end{aligned}$$

Here,

$$\begin{aligned}
g_1 &= \begin{bmatrix} p_{022} & -p_{012} \\ -p_{021} & p_{011} \end{bmatrix}, \\
g_2 &= \begin{bmatrix} (p_{021}k_2 + p_{022}k_1) & -(p_{011}k_2 + p_{012}k_1) \\ (p_{021}k_1 + p_{022}k_3) & -(p_{011}k_1 + p_{012}k_3) \end{bmatrix}, \\
g_3 &= \begin{bmatrix} p_{122} & -p_{112} \\ -p_{121} & p_{111} \end{bmatrix}, \\
g_4 &= \begin{bmatrix} (p_{121}k_2 + p_{122}k_1) & -(p_{111}k_2 + p_{112}k_1) \\ (p_{121}k_1 + p_{122}k_3) & -(p_{111}k_1 + p_{112}k_3) \end{bmatrix}, \\
g_5 &= g_3, \\
g_6 &= g_4,
\end{aligned}$$

and

$$G(s) = \begin{bmatrix} 0 \\ \dfrac{-2e\omega_c}{(1+e)^{\frac{3}{2}}}\Big(\sin s + \dfrac{e}{2}\sin 2s\Big) \end{bmatrix}. \qquad (20)$$

The integrand in (16) can now be expressed in the form:

$$\begin{aligned}
\Phi^{-1}(s)G(s) &= e^{\mu s}\big(J_1 \cos(\omega-3)s + J_2 \sin(\omega-3)s + (J_3 \cos(\omega-2)s + J_4 \sin(\omega-2)s) \\
&+ (J_5 \cos(\omega-1)s + J_6 \sin(\omega-1)s) + (J_7 \cos\omega s + J_8 \sin\omega s) + (J_9 \cos(\omega+1)s \\
&+ J_{10} \sin(\omega+1)s) + (J_{11} \cos(\omega+2)s + J_{12} \sin(\omega+2)s) + (J_{13} \cos(\omega+3)s \\
&+ J_{14} \sin(\omega+3)))s. \qquad (21)
\end{aligned}$$

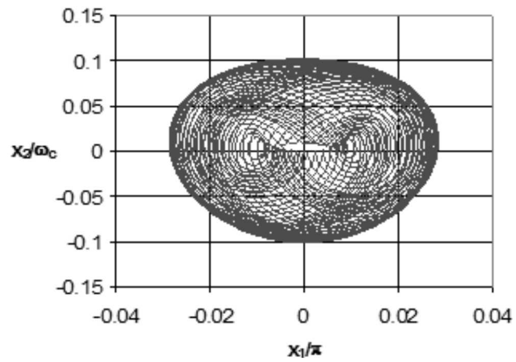

Figure 1: Phase portrait of the nonlinear response

The J's are real vectors each of which has a factor of $\frac{e^2\omega_c}{(1+e)^{3/2}}$, with the components extracted from the matrices $g_1, ..., g_6$ above.

Expressing the integrand in (16) in the form (17) now reduces the problem of solving the highly nonlinear equation to a routine term by term integration. The analytical solution for typical parameter values and the corresponding numerical solution are displayed in the form of phase-plane plots.

4. Results

To assess the validity of the method of solution to the nonlinear equation, we choose typical parameter values of the model and compare our results with published data. Our result is displayed in the form of phase-plane plot shown in Figure 1.

Figure 2 represents the phase portrait of the equation as presented in [1] obtained by means of numerical integration. Comparison of Figures 1 and 2 shows remarkable agreement both quantitatively and qualitatively.

The corresponding anomaly history plot is displayed in Figure 3.

5. Conclusion

The solution to the nonlinear equations describing the motion of a satellite represents a major challenge. We presented an analytical method of solving a

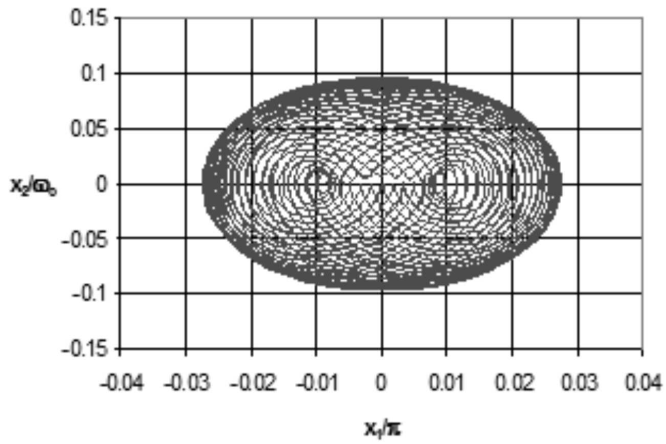

Figure 2: Phase-plane portrait from numerical solution of the nonlinear response

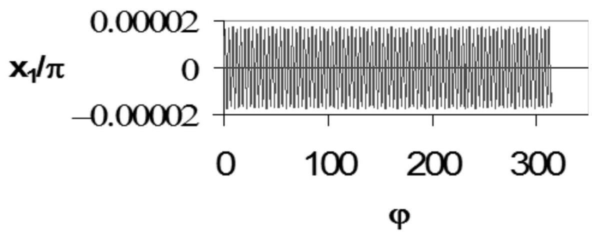

Figure 3: Anomaly history of the nonlinear response

highly nonlinear system of equations governing an earth satellite pitch librations. Comparison of our results with published data obtained through numerical computations shows close agreement. We believe that using our method, a general theoretical basis for solving similar nonlinear systems could be established.

Acknowledgments

This paper is dedicated to the memory of the late Professor Anthony K. Amos, who introduced the first author to the topic. The first author is also indebted to the Department of Aerospace Engineering, Pennsylvania State University, University Park, PA, USA, for welcoming him to undertake his PhD dissertation. The core of this paper is based on the dissertation.

The third author made valuable contributions before passing away in 2006.

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