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ON THE VARIATIONAL PROBLEMS IN A MULTI-CONNECTED DOMAIN ASSOCIATED WITH THE THEORY OF LIQUID CRYSTALS

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Abstract: We consider the asymptotic behavior of minimizers of the (modified) Landau-de Gennes functional with the Dirichlet boundary condition as the elastic coefficient tends to large. We show that the some of results in Pan [6] hold even if the domain is multi-connected.

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1. Introduction

In this paper, we consider the variational problems of the modified Landau-de Gennes model associated with the theory of liquid crystals. We treat the modified energy functional

$$\mathcal{E}_0[\psi,\boldsymbol{n}] = \int_\Omega \{|\nabla_{q\boldsymbol{n}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + K_1|\operatorname{div}\boldsymbol{n}|^2 + K_2|\operatorname{curl}\boldsymbol{n}|^2\}dx, \quad (1.1)$$

where ψ is a complex-valued function called order parameter, $\boldsymbol{n}$ is a unit vector field called director field, q is a real number called wave number, κ is a positive

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constant called the Ginzburg-Landau parameter of liquid crystal, K_1 and K_2 are positive material constants called elastic coefficients, and Ω is the domain in $\mathbb{R}^3$ occupied by the liquid crystals. We used the notation $\nabla_{q\boldsymbol{n}}\psi = \nabla\psi - iq\boldsymbol{n}\psi$, $i = \sqrt{-1}$.

We shall examine the asymptotic behavior of minimizers of (1.1) under the Dirichlet boundary condition for the director field as $K_1 \to \infty$ or $K_2 \to \infty$. Here we note that if $\boldsymbol{n}$ is a critical point of the (modified) Oseen-Frank functional:

$$\mathcal{F}_0[\boldsymbol{n}] = \int_\Omega \{K_1|\operatorname{div}\boldsymbol{n}|^2 + K_2|\operatorname{curl}\boldsymbol{n}|^2\}dx, \tag{1.2}$$

the functional $\mathcal{E}_0$ has a critical point $(0, \boldsymbol{n})$ which corresponds with a nematic phase. We are interested in non-trivial minimizers $(\psi, \boldsymbol{n})$ with $\psi \neq 0$ which corresponds with the phase transition from a nematic phase to a smectic phase. Pan [6, Section 5] resolved the asymptotic behavior of minimizers of $\mathcal{E}_0$ under the Dirichlet boundary condition for the director field as $K_1 \to \infty$ or $K_2 \to \infty$ in the case where Ω is a bounded, simply-connected domain in $\mathbb{R}^3$ with a smooth boundary.

In the present paper, we show that the similar results hold even if Ω is a bounded, multi-connected domain in $\mathbb{R}^3$ with a smooth boundary.

In order to treat the case where Ω is multi-connected, throughout this paper we assume that Ω satisfies the following conditions.

(H.1) Ω is a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega = \Gamma$ which is a manifold of calss C^r $(r \geq 2)$ of dimension 2, and Ω is locally situated on one side of Γ. Moreover, Γ has a finite number of connected components $\Gamma_1, \Gamma_2, \dots, \Gamma_{m+1}$ where $m \geq 0$ and Γ_{m+1} denoting the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

(H.2) There exist n manifolds $\Sigma_1, \Sigma_2, \dots, \Sigma_n$ $(n \geq 0)$ of dimension 2 of class C^r such that $\Sigma_i \cap \Sigma_j = \emptyset$ $(i \neq j)$ and they are not tangential to Γ such that $\dot{\Omega} = \Omega \setminus \Sigma$ where $\Sigma = \cup_{j=1}^n \Sigma_i$ is simply connected and Lipschitzian.

The number n is called the first Betti number which equals the number of handles of Ω and m is called the second Betti number which is equals the number of holes. We say that Ω is simply connected if $n = 0$, and that Ω has no holes if $m = 0$. If we define the spaces

$$\begin{aligned}
\mathbb{H}_1(\Omega) &= \{\boldsymbol{u} \in L^2(\Omega, \mathbb{R}^3)\,;\operatorname{curl}\boldsymbol{u} = 0, \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega, \boldsymbol{u}\cdot\boldsymbol{\nu}|_\Gamma = 0\},\\
\mathbb{H}_2(\Omega) &= \{\boldsymbol{u} \in L^2(\Omega, \mathbb{R}^3)\,;\operatorname{curl}\boldsymbol{u} = 0, \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega, \boldsymbol{u}\times\boldsymbol{\nu}|_\Gamma = 0\},
\end{aligned}$$

it is well known that

$$\dim \mathbb{H}_1(\Omega) = n \quad \text{and} \quad \dim \mathbb{H}_2(\Omega) = m.$$

Moreover, $\mathbb{H}_i(\Omega)$ $(i = 1, 2)$ is a closed subspace of $L^2(\Omega, \mathbb{R}^3)$, and $\mathbb{H}_i(\Omega) \subset H^1(\Omega, \mathbb{R}^3)$ for $i = 1, 2$. Furthermore, if Γ is of class $C^{r,\theta}$ with $r \geq 2$ and $0 < \theta < 1$,

$$\mathbb{H}_i(\Omega) \subset C^{r-1,\theta}(\overline{\Omega}, \mathbb{R}^3) \quad (i = 1, 2).$$

For these facts, see Dautray and Lions [3]. Let $\mathbb{P}_i$ be the orthogonal projection from $L^2(\Omega, \mathbb{R}^3)$ onto $\mathbb{H}_i(\Omega)^\perp$, where $\mathbb{H}_i(\Omega)^\perp$ denotes the orthogonal complement of $\mathbb{H}_i(\Omega)$ in $L^2(\Omega, \mathbb{R}^3)$.

Let $\boldsymbol{n}_0$ be a given smooth unit vector field on Γ. Then we define the spaces

$$\begin{aligned} H^1(\Omega, \mathbb{S}^2) &= \{\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3); |\boldsymbol{u}| = 1 \text{ a.e. in } \Omega\}, \\ H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0) &= \{\boldsymbol{u} \in H^1(\Omega, \mathbb{S}^2); \boldsymbol{u} = \boldsymbol{n}_0 \text{ on } \Gamma\}, \\ \mathbb{H}(\Omega, \boldsymbol{n}_0) &= H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0). \end{aligned}$$

Here $H^1(\Omega, \mathbb{R}^3)$ and $H^1(\Omega, \mathbb{C})$ denote the usual Sobolev spaces of real vector valued functions and complex-valued functions, respectively. We consider the following variational problem:

$$C_d(K_1, K_2, \kappa, q) = \inf_{(\psi, \boldsymbol{n}) \in \mathbb{H}(\Omega, \boldsymbol{n}_0)} \mathcal{E}_0[\psi, \boldsymbol{n}]. \tag{1.3}$$

We note that for any $K_1, K_2, \kappa > 0$ and any real number q, the minimizers of (1.3) exist (see Proposition 2.1 in the next section, and see [6] for simply connected domain Ω). Also we define

$$N(K_1, K_2, \boldsymbol{n}_0) = \inf_{\boldsymbol{u} \in H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0)} \mathcal{F}_0[\boldsymbol{u}]. \tag{1.4}$$

According to Hardt, Kinderlehrer and Lin [5], if $\boldsymbol{n}_0$ is a smooth vector field on Γ, then $H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0) \neq \emptyset$, and $N(K_1, K_2, \boldsymbol{n}_0)$ is achieved in $H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0)$. Define

$$\mathbb{G}(\boldsymbol{n}_0) = \{\boldsymbol{u} = \nabla\phi + \boldsymbol{h}^1; \phi \in H^2(\Omega), \boldsymbol{h}^1 \in \mathbb{H}_1(\Omega), \\ |\boldsymbol{u}| = 1 \text{ a.e. in } \Omega, \boldsymbol{u} = \boldsymbol{n}_0 \text{ on } \Gamma\}.$$

We note that in the case where Ω is simply connected, since $\mathbb{H}_1(\Omega) = \{0\}$, the definition of $\mathbb{G}(\boldsymbol{n}_0)$ is identical to

$$\{\nabla\phi; \phi \in H^2(\Omega), |\nabla\phi| = 1 \text{ a.e. in } \Omega, \nabla\phi = \boldsymbol{n}_0 \text{ on } \Gamma\}$$

defined in [6]. Thanks to [3, p. 235, (1.76)], for any integer $k \geq 0$,

$$\{\boldsymbol{u} \in H^k(\Omega, \mathbb{R}^3); \operatorname{curl} \boldsymbol{u} = 0 \text{ in } \Omega\} = \nabla H^{k+1}(\Omega) \oplus \mathbb{H}_1(\Omega), \tag{1.5}$$

where $\oplus$ means the orthogonal direct sum in $L^2(\Omega, \mathbb{R}^3)$. Now it is easily verified that

$$\mathbb{G}(\boldsymbol{n}_0) = \{\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3); \operatorname{curl} \boldsymbol{u} = 0 \text{ in } \Omega, |\boldsymbol{u}| = 1 \text{ a.e. in } \Omega, \\ \boldsymbol{u} = \boldsymbol{n}_0 \text{ on } \Gamma\}. \tag{1.6}$$

If $\mathbb{G}(\boldsymbol{n}_0) \neq \emptyset$, define

$$G(\boldsymbol{n}_0) = \inf_{\boldsymbol{u}\in\mathbb{G}(\boldsymbol{n}_0)} \int_\Omega |\operatorname{div} \boldsymbol{u}|^2 dx. \tag{1.7}$$

On the other hand, define

$$\mathbb{R}(\boldsymbol{n}_0) = \{\boldsymbol{u} = \operatorname{curl} \boldsymbol{v} + \boldsymbol{h}^2; \boldsymbol{v} \in H^2(\Omega, \mathbb{R}^3), \boldsymbol{h}^2 \in \mathbb{H}_2(\Omega), |\boldsymbol{u}| = 1 \text{ a.e. in } \Omega, \boldsymbol{u} = \boldsymbol{n}_0 \text{ on } \Gamma\}.$$

Also according to [3, p. 235, (1.76)], for any integer $k \geq 0$,

$$\{\boldsymbol{u} \in H^k(\Omega, \mathbb{R}^3); \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\} = \operatorname{curl} H^{k+1}(\Omega, \mathbb{R}^3) \oplus \mathbb{H}_2(\Omega). \tag{1.8}$$

Thus it is also verified that

$$\mathbb{R}(\boldsymbol{n}_0) = \{\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3); \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega, |\boldsymbol{u}| = 1 \text{ a.e. in } \Omega,\ \boldsymbol{u} = \boldsymbol{n}_0 \text{ on } \Gamma\} \tag{1.9}$$

and if $\mathbb{R}(\boldsymbol{n}_0) \neq \emptyset$, define

$$R(\boldsymbol{n}_0) = \inf_{\boldsymbol{u}\in\mathbb{R}(\boldsymbol{n}_0)} \int_\Omega |\operatorname{curl} \boldsymbol{u}|^2 dx, \tag{1.10}$$

$$d(K_1, \kappa, q, \boldsymbol{n}_0) = \inf_{(\phi,\boldsymbol{u})\in H^1(\Omega,\mathbb{C})\times\mathbb{R}(\boldsymbol{n}_0)} \int_\Omega \{|\nabla_{q\boldsymbol{u}}\phi|^2 - \kappa^2|\phi|^2 + \frac{\kappa^2}{2}|\phi|^4 + K_2|\operatorname{curl} \boldsymbol{u}|^2\} dx. \tag{1.11}$$

There are some articles which are relevant to these minimizing problems, see Aramaki [1], Bate and Pan [2], Pan [6], [7], [8], and Pan and Qi [9].

We are in a position to state the main theorem (cf. [6, Theorem 5.4]).

Theorem 1.1. *(1) Let $\mathbb{G}(\boldsymbol{n}_0) \neq \emptyset$. Then the following holds.*

(1.i) If $K_1 G(\boldsymbol{n}_0) - N(K_1, K_2, \boldsymbol{n}_0) < \frac{\kappa^2}{2}|\Omega|$, the minimizers of $C_d(K_1, K_2, \kappa, q)$ are non-trivial.

(1.ii) Let $(\psi_j, \boldsymbol{u}_j) \in \mathbb{H}(\Omega, \boldsymbol{n}_0)$ be a minimizer of $C_d(K_1, K_2^{(j)}, \kappa, q)$ for fixed $K_1, \kappa > 0$, $q \in \mathbb{R}$ and $K_2 = K_2^{(j)} > 0$ where $K_2^{(j)} \to \infty$ as $j \to \infty$. Then we see that

$$\lim_{j\to\infty} \mathcal{E}_0[\psi_j, \boldsymbol{u}_j] = K_1 G(\boldsymbol{n}_0) - \frac{\kappa^2}{2}|\Omega|.$$

Moreover, there exists subsequence $(\psi_{j_l}, \boldsymbol{u}_{j_l})$ and $\phi_0 \in H^2(\Omega)$ such that $\nabla\phi_0$ is a minimizer of $G(\boldsymbol{n}_0)$ and

$$(\psi_{j_l}, \boldsymbol{u}_{j_l}) \to (ce^{iq\phi_0}, \nabla\phi_0)$$

with a constant c, $|c| = 1$, strongly in $\mathbb{H}(\Omega, \boldsymbol{n}_0)$ as $l \to \infty$.

(2) Let $\mathbb{R}(\boldsymbol{n}_0) \neq \emptyset$. Then the following holds.

(2.i) If $d(K_1, \kappa, q, \boldsymbol{n}_0) < N(K_1, K_2, \boldsymbol{n}_0)$*, the minimizers of* $C_d(K_1, K_2, \kappa, q)$ *are non-trivial.*

(2.ii) Let $(\psi_j, \boldsymbol{u}_j) \in \mathbb{H}(\Omega, \boldsymbol{n}_0)$ *be a minimizer of* $C_d(K_1^{(j)}, K_2, \kappa, q)$ *for fixed* $K_2, \kappa > 0$*,* $q \in \mathbb{R}$ *and* $K_1 = K_1^{(j)} > 0$ *where* $K_1^{(j)} \to \infty$ *as* $j \to \infty$*. Then we see that*

$$\lim_{j\to\infty} \mathcal{E}_0[\psi_j, \boldsymbol{u}_j] = d(K_2, \kappa, q, \boldsymbol{n}_0).$$

Moreover, there exists subsequence $(\psi_{j_l}, \boldsymbol{u}_{j_l})$*,* $(\psi_0, \boldsymbol{u}_0) \in H^1(\Omega, \mathbb{C}) \times \mathbb{R}(\boldsymbol{n}_0)$ *which is a minimizer of* $d(K_2, \kappa, q, \boldsymbol{n}_0)$ *and*

$$(\psi_{j_l}, \boldsymbol{u}_{j_l}) \to (\psi_0, \boldsymbol{u}_0)$$

strongly in $\mathbb{H}(\Omega, \boldsymbol{n}_0)$ *as* $l \to \infty$*.*

Remark 1.2. The results are almost identical with [6, Theorem 5.4]. However, since we treat the case where the domain Ω is multi-connected, we have to modify its proof in the case where Ω is multi-connected.

The plan of this paper is as follows. In Section 2, we give some basic estimates and the existence of minimizers of $\mathcal{E}_0$. Section 3 is devoted to the asymptotic behavior as K_1 or (and) K_2 tends to infinity. Finally in Section 4, we give a proof of Theorem 1.1.

2. Preliminaries

In this section, we give some properties which are needed for the proof of Theorem 1.1. First, we state the regularity and some estimates which are well known (cf. Dautray and Lions [3] and also Temam [10]).

Let Ω is bounded domain in $\mathbb{R}^3$ with C^{k+2} boundary $\Gamma = \partial\Omega$ $(k \geq 0)$. Then we see that

$$H^{k+1}(\Omega, \mathbb{R}^3) = \{\boldsymbol{u} \in L^2(\Omega, \mathbb{R}^3)\,;\operatorname{curl}\boldsymbol{u} \in H^k(\Omega, \mathbb{R}^3),\\ \operatorname{div}\boldsymbol{u} \in H^k(\Omega), \boldsymbol{u}\cdot\boldsymbol{\nu}|_\Gamma \in H^{k+1/2}(\Gamma)\} \quad (2.1)$$

or

$$H^{k+1}(\Omega, \mathbb{R}^3) = \{\boldsymbol{u} \in L^2(\Omega, \mathbb{R}^3)\,;\operatorname{curl}\boldsymbol{u} \in H^k(\Omega, \mathbb{R}^3),\\ \operatorname{div}\boldsymbol{u} \in H^k(\Omega), \boldsymbol{u}\times\boldsymbol{\nu}|_\Gamma \in H^{k+1/2}(\Gamma, \mathbb{R}^3)\}. \quad (2.2)$$

Here $\cdot|_\Gamma$ means the trace operator to Γ. Moreover, the following estimate holds: There exists a constant $C > 0$ such that

$$\|\boldsymbol{u}\|_{H^{k+1}(\Omega,\mathbb{R}^3)} \le C\{\|\boldsymbol{u}\|_{L^2(\Omega,\mathbb{R}^3)} + \|\mathrm{curl}\,\boldsymbol{u}\|_{H^k(\Omega,\mathbb{R}^3)} + \|\mathrm{div}\,\boldsymbol{u}\|_{H^k(\Omega)} + \|\boldsymbol{u}\cdot\boldsymbol{\nu}\|_{H^{k+1/2}(\Gamma)}\} \quad (2.3)$$

or

$$\|\boldsymbol{u}\|_{H^{k+1}(\Omega,\mathbb{R}^3)} \le C\{\|\boldsymbol{u}\|_{L^2(\Omega,\mathbb{R}^3)} + \|\mathrm{curl}\,\boldsymbol{u}\|_{H^k(\Omega,\mathbb{R}^3)} + \|\mathrm{div}\,\boldsymbol{u}\|_{H^k(\Omega)} + \|\boldsymbol{u}\times\boldsymbol{\nu}\|_{H^{k+1/2}(\Gamma,\mathbb{R}^3)}\} \quad (2.4)$$

for any $\boldsymbol{u} \in H^{k+1}(\Omega,\mathbb{R}^3)$, respectively. Of course, the right hand sides of (2.3) and (2.4) are estimated by $H^{k+1}(\Omega,\mathbb{R}^3)$ norm of $\boldsymbol{u}$.

Here we note that if Ω is simply connected, we can omit the term $\|\boldsymbol{u}\|_{L^2(\Omega,\mathbb{R}^3)}$ in the right hand side of (2.3), and if Ω has no holes, we can omit the term $\|\boldsymbol{u}\|_{L^2(\Omega,\mathbb{R}^3)}$ in the right hand side of (2.4) (cf. Aramaki [1]).

Now we prove the existence of minimizers of $C_d(K_1,K_2,\kappa,q)$.

Proposition 2.1. *For any $K_1,K_2,\kappa > 0$ and any $q \in \mathbb{R}$, the minimizers $(\psi,\boldsymbol{u})$ of the variational problem (1.3) in $\mathbb{H}(\Omega,\boldsymbol{n}_0) = H^1(\Omega,\mathbb{C})\times H^1(\Omega,\mathbb{S}^2,\boldsymbol{n}_0)$ exist, and they satisfy the following Euler equations*

$$\begin{cases} -\nabla_{q\boldsymbol{u}}^2\psi = \kappa^2(1-|\psi|^2)\psi, & \\ -K_1\nabla(\mathrm{div}\,\boldsymbol{u}) + K_2\mathrm{curl}^2\boldsymbol{u} - q\Im(\overline{\psi}\nabla_{q\boldsymbol{u}}\psi) = \lambda(x)\boldsymbol{u}, & \text{on } \Omega, \\ (\nabla_{q\boldsymbol{u}}\psi)\cdot\boldsymbol{\nu} = 0, \quad \boldsymbol{u} = \boldsymbol{n}_0 & \text{on } \Gamma, \end{cases} \quad (2.5)$$

where $\lambda(x)$ is the Lagrangean multiplier which depends on x. Moreover, every minimizing sequence is precompact in $\mathbb{H}(\Omega,\boldsymbol{n}_0)$.

Proof. First we note that $C_d(K_1,K_2,\kappa,q)$ is finite. In fact, if we choose $\boldsymbol{u}_0 \in H^1(\Omega,\mathbb{S}^2,\boldsymbol{n}_0)$ and take $(0,\boldsymbol{u}_0)$ as a test field of $\mathcal{E}_0$, we easily see that $C_d(K_1,K_2,\kappa,q)$ is finite.

Let $\{(\psi_j,\boldsymbol{u}_j)\} \subset \mathbb{H}(\Omega,\boldsymbol{n}_0)$ be a minimizing sequence of $C_d(K_1,K_2,\kappa,q)$. Then there exists a constant $C > 0$ such that

$$\int_\Omega \{|\nabla_{q\boldsymbol{u}_j}\psi_j|^2 + \frac{\kappa^2}{2}(1-|\psi_j|^2)^2 + K_1|\mathrm{div}\,\boldsymbol{u}_j|^2 + K_2|\mathrm{curl}\,\boldsymbol{u}_j|^2\}dx \le C_d(K_1,K_2,\kappa,q) + \frac{\kappa^2}{2}|\Omega| + o(1) \le C. \quad (2.6)$$

Since $|\boldsymbol{u}_j| = 1$ a.e. in Ω, $\boldsymbol{u}_j\cdot\boldsymbol{\nu} = \boldsymbol{n}_0\cdot\boldsymbol{\nu}$ on Γ, it follows from (2.3) that $\{\boldsymbol{u}_j\}$ is bounded in $H^1(\Omega,\mathbb{R}^3)$. Thus passing to a subsequence, we may assume that $\boldsymbol{u}_j \to \boldsymbol{u}_0$ weakly in $H^1(\Omega,\mathbb{R}^3)$ and strongly in $L^4(\Omega,\mathbb{R}^3)$. Since $\boldsymbol{u}_0$ satisfies that $|\boldsymbol{u}_0| = 1$ a.e. in Ω and $\boldsymbol{u}_0 = \boldsymbol{n}_0$ on Γ, we see that $\boldsymbol{u}_0 \in H^1(\Omega,\mathbb{S}^2,\boldsymbol{n}_0)$. We also have

$$\int_\Omega |\mathrm{div}\,\boldsymbol{u}_0|^2 dx \le \liminf_{j\to\infty}\int_\Omega |\mathrm{div}\,\boldsymbol{u}_j|^2 dx, \quad (2.7)$$

$$\int_\Omega |\mathrm{curl}\, \boldsymbol{u}_0|^2 dx \leq \liminf_{j\to\infty} \int_\Omega |\mathrm{curl}\, \boldsymbol{u}_j|^2 dx.$$

Next, we estimate ψ_j. By (2.6), $\{|\nabla_{q\boldsymbol{u}_j}\psi_j|\}$ is bounded in $L^2(\Omega)$ and $\{|\psi_j|\}$ is bounded in $L^4(\Omega)$. Since $\{|\boldsymbol{u}_j\psi_j|\}$ is bounded in $L^2(\Omega)$, $\{\nabla\psi_j = \nabla_{q\boldsymbol{u}_j}\psi_j + iq\boldsymbol{u}_j\psi_j\}$ is bounded in $L^2(\Omega, \mathbb{C}^3)$. Thus $\{\psi_j\}$ is bounded in $H^1(\Omega, \mathbb{C})$. Passing to a subsequence, we may assume that $\psi_j \to \psi_0$ weakly in $H^1(\Omega, \mathbb{C})$ and strongly in $L^4(\Omega, \mathbb{C})$. Since $\boldsymbol{u}_j\psi_j \to \boldsymbol{u}_0\psi_0$ strongly in $L^2(\Omega, \mathbb{C}^3)$, we see that $\nabla_{q\boldsymbol{u}_j}\psi_j \to \nabla_{q\boldsymbol{u}_0}\psi_0$ weakly in $L^2(\Omega, \mathbb{C}^3)$. Therefore, we have

$$\int_\Omega |\nabla_{q\boldsymbol{u}_0}\psi_0|^2 dx \leq \liminf_{j\to\infty} \int_\Omega |\nabla_{q\boldsymbol{u}_j}\psi_j| dx. \tag{2.8}$$

Since $\psi_j \to \psi_0$ strongly in $L^4(\Omega, \mathbb{C})$,

$$\int_\Omega (1 - |\psi_0|^2)^2 dx = \lim_{j\to\infty} \int_\Omega (1 - |\psi_j|^2)^2 dx. \tag{2.9}$$

Hence, we have

$$\mathcal{E}_0[\psi_0, \boldsymbol{u}_0] \leq \liminf_{j\to\infty} \mathcal{E}_0[\psi_j, \boldsymbol{u}_j] = C_d(K_1, K_2, \kappa, q).$$

Since $(\psi_0, \boldsymbol{u}_0) \in \mathbb{H}(\Omega, \boldsymbol{n}_0)$, $(\psi_0, \boldsymbol{u}_0)$ is a minimizer. Finally we show that $(\psi_j, \boldsymbol{u}_j)$ converges to $(\psi_0, \boldsymbol{u}_0)$ strongly in $\mathbb{H}(\Omega, \boldsymbol{n}_0)$. Since it follows from (2.6) that

$$\limsup_{j\to\infty} \mathcal{E}_0[\psi_j, \boldsymbol{u}_j] \leq C_d(K_1, K_2, \kappa, q) = \mathcal{E}_0[\psi_0, \boldsymbol{u}_0] \leq \liminf_{j\to\infty} \mathcal{E}_0[\psi_j, \boldsymbol{u}_j].$$

Thus $\lim_{j\to\infty} \mathcal{E}_0[\psi_j, \boldsymbol{u}_j] = \mathcal{E}_0[\psi_0, \boldsymbol{u}_0]$. By (2.6), (2.7), (2.8) and (2.9), we have

$$\begin{aligned}
\lim_{j\to\infty} \int_\Omega |\nabla_{q\boldsymbol{u}_j}\psi_j|^2 dx &= \int_\Omega |\nabla_{q\boldsymbol{u}_0}\psi_0|^2 dx, \\
\lim_{j\to\infty} \int_\Omega |\mathrm{div}\, \boldsymbol{u}_j|^2 dx &= \int_\Omega |\mathrm{div}\, \boldsymbol{u}_0|^2 dx, \\
\lim_{j\to\infty} \int_\Omega |\mathrm{curl}\, \boldsymbol{u}_j|^2 dx &= \int_\Omega |\mathrm{curl}\, \boldsymbol{u}_0|^2 dx.
\end{aligned}$$

Since

$$\begin{aligned}
\nabla_{q\boldsymbol{u}_j}\psi_j &\to \nabla_{q\boldsymbol{u}_0}\psi_0 \quad \text{weakly in } L^2(\Omega, \mathbb{C}^3), \\
\mathrm{div}\, \boldsymbol{u}_j &\to \mathrm{div}\, \boldsymbol{u}_0 \quad \text{weakly in } L^2(\Omega), \\
\mathrm{curl}\, \boldsymbol{u}_j &\to \mathrm{curl}\, \boldsymbol{u}_0 \quad \text{weakly in } L^2(\Omega, \mathbb{R}^3),
\end{aligned}$$

it follows that

$$\begin{aligned}
\nabla_{q\boldsymbol{u}_j}\psi_j &\to \nabla_{q\boldsymbol{u}_0}\psi_0 \quad \text{strongly in } L^2(\Omega, \mathbb{C}^3), \\
\mathrm{div}\, \boldsymbol{u}_j &\to \mathrm{div}\, \boldsymbol{u}_0 \quad \text{strongly in } L^2(\Omega), \\
\mathrm{curl}\, \boldsymbol{u}_j &\to \mathrm{curl}\, \boldsymbol{u}_0 \quad \text{strongly in } L^2(\Omega, \mathbb{R}^3).
\end{aligned}$$

Since $\boldsymbol{u}_j \to \boldsymbol{u}_0$ strongly in $L^2(\Omega, \mathbb{R}^3)$ and $\boldsymbol{u}_j \cdot \boldsymbol{\nu} = \boldsymbol{n}_0 \cdot \boldsymbol{\nu} = \boldsymbol{u}_0 \cdot \boldsymbol{\nu}$ on Γ, applying the estimate (2.3), we see that $\boldsymbol{u}_j \to \boldsymbol{u}_0$ strongly in $H^1(\Omega, \mathbb{R}^3)$. Since $\boldsymbol{u}_j\psi_j \to \boldsymbol{u}_0\psi_0$ and $\nabla_{q\boldsymbol{u}_j}\psi_j \to \nabla_{q\boldsymbol{u}_0}\psi_0$ strongly in $L^2(\Omega, \mathbb{C}^3)$, we see that

$$\nabla\psi_j = \nabla_{q\boldsymbol{u}_j}\psi_j + iq\boldsymbol{u}_j\psi_j \to \nabla_{q\boldsymbol{u}_0}\psi_0 + iq\boldsymbol{u}_0\psi_0 = \nabla\psi_0$$

strongly in $L^2(\Omega, \mathbb{C}^3)$. Thus $\psi_j \to \psi_0$ strongly in $H^1(\Omega, \mathbb{C})$. That is to say $(\psi_j, \boldsymbol{u}_j) \to (\psi_0, \boldsymbol{u}_0)$ strongly in $\mathbb{H}(\Omega, \boldsymbol{n}_0)$.

We consider the Euler-Lagrange equations. Let $(\psi, \boldsymbol{u})$ be a minimizer of (1.3). Then if we use the Lagrange multiplier $\lambda(x)$ and consider a equation

$$\frac{d}{dt}\left[\mathcal{E}_0[\psi, \boldsymbol{u} + t\boldsymbol{v}] - \int_\Omega \lambda(x)(|\boldsymbol{u} + t\boldsymbol{v}|^2 - 1)dx\right]_{t=0} = 0$$

for $\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^3)$, we can see that for any $\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^3)$,

$$\int_\Omega \{K_1(\operatorname{div}\boldsymbol{u})(\operatorname{div}\boldsymbol{v}) + K_2(\operatorname{curl}\boldsymbol{u})\cdot(\operatorname{curl}\boldsymbol{v}) - q\Im[\overline{\psi}\nabla_{q\boldsymbol{u}}\psi]\cdot\boldsymbol{v} - \lambda(x)\boldsymbol{u}\cdot\boldsymbol{v}\}dx = 0. \quad (2.10)$$

So we get the second equation of the Euler equations (2.5). We can seek for the unknown function λ, if we choose $\boldsymbol{v} = \eta\boldsymbol{u}$ for any $C_0^\infty(\Omega)$ function η in (2.10) (cf. [5]). It is easy to see that the other equations of (2.5) hold. This completes the proof of Proposition 2.1. □

3. Asymptotic Behavior for Large K_1 or K_2

In this section, we examine the asymptotic behavior of minimizers of $\mathcal{E}_0$ as K_1 or K_2 tends to infinity.

If K_1 and K_2 tend to infinity at the same speed (corresponding with the nematic-smectic C transition), we give the following result.

Proposition 3.1. *Let $(\psi_j, \boldsymbol{u}_j) \in \mathbb{H}(\Omega, \boldsymbol{n}_0)$ be a minimizer of $\mathcal{E}_0$ for $\kappa > 0$, $K_1 = K_1^{(j)}, K_2 = K_2^{(j)}$ where*

$$K_1^{(j)} \to \infty, \quad K_2^{(j)} \to \infty, \quad \frac{K_1^{(j)}}{K_2^{(j)}} \to a > 0 \quad \text{as} \quad j \to \infty.$$

Then there exist a subsequence $\{(\psi_{j_l}, \boldsymbol{u}_{j_l})\}$, $(\psi_0, \boldsymbol{u}_0) \in \mathbb{H}(\Omega, \boldsymbol{u}_0)$, where $\boldsymbol{u}_0$ is a minimizer of the functional

$$\int_\Omega \{a|\operatorname{div}\boldsymbol{u}|^2 + |\operatorname{curl}\boldsymbol{u}|^2\}dx$$

on $H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0)$, and ψ_0 is a minimizer of the functional

$$\mathcal{G}_{q\boldsymbol{u}_0}[\phi] = \int_\Omega \{\nabla_{q\boldsymbol{u}_0}\phi|^2 - \kappa^2|\phi|^2 + \frac{\kappa^2}{2}|\phi|^4\}dx$$

on $H^1(\Omega, \mathbb{R})$.

Proof. Since [6] does not give the proof, we shall give it.

Define

$$N(a, \boldsymbol{n}_0) = \inf_{\boldsymbol{u} \in H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0)} \int_\Omega \{a|\operatorname{div} \boldsymbol{u}|^2 + |\operatorname{curl} \boldsymbol{u}|^2\} dx,$$

and let $\widehat{\boldsymbol{u}}_0 \in H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0)$ be a minimizer of $N(a, \boldsymbol{n}_0)$. Then we have

$$\begin{aligned} C_d(K_1^{(j)}, K_2^{(j)}, \kappa, q) &= \int_\Omega \{|\nabla_{q\boldsymbol{u}_j} \psi_j|^2 + \frac{\kappa^2}{2}(1 - |\psi_j|^2)^2 \\ &\quad + K_1^{(j)} |\operatorname{div} \boldsymbol{u}_j|^2 + K_2^{(j)} |\operatorname{curl} \boldsymbol{u}_j|^2\} dx - \frac{\kappa^2}{2}|\Omega| \\ &\leq \mathcal{E}_0[0, \widehat{\boldsymbol{u}}_0] \\ &= \int_\Omega \{K_1^{(j)} |\operatorname{div} \widehat{\boldsymbol{u}}_0|^2 + K_2^{(j)} |\operatorname{curl} \widehat{\boldsymbol{u}}_0|^2\} dx. \end{aligned} \tag{3.1}$$

From this inequality we have

$$\begin{aligned} \frac{K_1^{(j)}}{K_2^{(j)}} \int_\Omega |\operatorname{div} \boldsymbol{u}_j|^2 dx + \int_\Omega |\operatorname{curl} \boldsymbol{u}_j|^2 dx &\leq \frac{K_1^{(j)}}{K_2^{(j)}} \int_\Omega |\operatorname{div} \widehat{\boldsymbol{u}}_0|^2 dx \\ &\quad + \int_\Omega |\operatorname{curl} \widehat{\boldsymbol{u}}_0|^2 dx + \frac{\kappa^2}{2K_2^{(j)}}|\Omega|. \end{aligned}$$

Thus $\{\operatorname{curl} \boldsymbol{u}_j\}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$, $\{\operatorname{div} \boldsymbol{u}_j\}$ is bounded in $L^2(\Omega)$, $|\boldsymbol{u}_j| = 1$ a.e. in Ω and $\boldsymbol{u}_j \cdot \boldsymbol{\nu} = \boldsymbol{n}_0 \cdot \boldsymbol{\nu}$ on Γ. Therefore, from (2.3), $\{\boldsymbol{u}_j\}$ is bounded in $H^1(\Omega, \mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{u}_j \to \boldsymbol{u}_0$ weakly in $H^1(\Omega, \mathbb{R}^3)$, strongly in $L^2(\Omega, \mathbb{R}^3)$. Thus we have $\boldsymbol{u}_0 \in H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0)$, and we see that

$$\begin{aligned} N(a, \boldsymbol{n}_0) &\leq a \int_\Omega |\operatorname{div} \boldsymbol{u}_0|^2 dx + \int_\Omega |\operatorname{curl} \boldsymbol{u}_0|^2 dx \\ &\leq \liminf_{j\to\infty} \left\{ \frac{K_1^{(j)}}{K_2^{(j)}} \int_\Omega |\operatorname{div} \boldsymbol{u}_j|^2 dx + \int_\Omega |\operatorname{curl} \boldsymbol{u}_j|^2 dx \right\} \\ &\leq \limsup_{j\to\infty} \left\{ \frac{K_1^{(j)}}{K_2^{(j)}} \int_\Omega |\operatorname{div} \boldsymbol{u}_j|^2 dx + \int_\Omega |\operatorname{curl} \boldsymbol{u}_j|^2 dx \right\} \\ &\leq \limsup_{j\to\infty} \Big\{ \frac{K_1^{(j)}}{K_2^{(j)}} \int_\Omega |\operatorname{div} \boldsymbol{u}_j|^2 dx + \int_\Omega |\operatorname{curl} \boldsymbol{u}_j|^2 dx \\ &\qquad + \frac{\kappa^2}{2K_2^{(j)}}|\Omega| \Big\} \end{aligned}$$

$$\leq \quad a\int_\Omega |\mathrm{div}\,\widehat{\boldsymbol{u}}_0|^2dx + \int_\Omega |\mathrm{curl}\,\widehat{\boldsymbol{u}}_0|^2dx = N(a,\boldsymbol{n}_0).$$

Thus $\boldsymbol{u}_0$ is a minimizer of $N(a,\boldsymbol{n}_0)$. Moreover, we get

$$\lim_{j\to\infty}\int_\Omega |\mathrm{div}\,\boldsymbol{u}_j|^2dx = \int_\Omega |\mathrm{div}\,\boldsymbol{u}_0|^2dx,$$

$$\lim_{j\to\infty}\int_\Omega |\mathrm{curl}\,\boldsymbol{u}_j|^2dx = \int_\Omega |\mathrm{curl}\,\boldsymbol{u}_0|^2dx.$$

Therefore, we see that $\mathrm{curl}\,\boldsymbol{u}_j \to \mathrm{curl}\,\boldsymbol{u}_0$ strongly in $L^2(\Omega,\mathbb{R}^3)$, $\mathrm{div}\,\boldsymbol{u}_j \to \mathrm{div}\,\boldsymbol{u}_0$ strongly in $L^2(\Omega)$. Since $\boldsymbol{u}_j \to \boldsymbol{u}_0$ strongly in $L^2(\Omega,\mathbb{R}^3)$ and $\boldsymbol{u}_j\cdot\boldsymbol{\nu} = \boldsymbol{n}_0\cdot\boldsymbol{\nu} = \boldsymbol{u}_0\cdot\boldsymbol{\nu}$ on Γ, it follows from from (2.3) that $\boldsymbol{u}_j \to \boldsymbol{u}_0$ strongly in $H^1(\Omega,\mathbb{R}^3)$. Since $\boldsymbol{u}_0$ is clearly a minimizer of $N(K_1^{(j)}/K_2^{(j)},\boldsymbol{n}_0)$, we have

$$\int_\Omega \{K_1^{(j)}|\mathrm{div}\,\boldsymbol{u}_0|^2 + K_2^{(j)}|\mathrm{curl}\,\boldsymbol{u}_0|^2\}dx$$
$$\leq \int_\Omega \{K_1^{(j)}|\mathrm{div}\,\boldsymbol{u}_j|^2 + K_2^{(j)}|\mathrm{curl}\,\boldsymbol{u}_j|^2\}dx.$$

Therefore, from (3.1), we get

$$\int_\Omega \{|\nabla_{q\boldsymbol{u}_j}\psi_j|^2 + \frac{\kappa^2}{2}(1-|\psi_j|^2)^2\}dx \leq C.$$

Thus $\{\nabla_{q\boldsymbol{u}_j}\psi_j\}$ is bounded in $L^2(\Omega,\mathbb{C}^3)$ and $\{\psi_j\}$ is bounded in $L^4(\Omega,\mathbb{C})$. Since $\{q\boldsymbol{u}_j\psi_j\}$ is bounded in $L^2(\Omega,\mathbb{C}^3)$, we see that $\nabla\psi_j$ is bounded in $L^2(\Omega,\mathbb{C}^3)$, so $\{\psi_j\}$ is bounded in $H^1(\Omega,\mathbb{C})$. Passing to a subsequence, we may assume that $\psi_j \to \psi_0$ weakly in $H^1(\Omega,\mathbb{C})$ and strongly in $L^4(\Omega,\mathbb{R}^3)$. Hence we have

$$\int_\Omega \{\nabla_{q\boldsymbol{u}_0}\psi_0|^2 + \frac{\kappa^2}{2}(1-|\psi_0|^2)^2\}dx$$
$$\leq \liminf_{j\to\infty}\int_\Omega \{\nabla_{q\boldsymbol{u}_j}\psi_j|^2 + \frac{\kappa^2}{2}(1-|\psi_j|^2)^2\}dx. \quad (3.2)$$

Hence we have

$$\begin{aligned}\mathcal{E}_0[\psi_j,\boldsymbol{u}_j] &= \inf_{(\psi,\boldsymbol{u})\in\mathbb{H}(\Omega,\boldsymbol{n}_0)}\int_\Omega \{|\nabla_{q\boldsymbol{u}}\psi|^2 + \frac{\kappa^2}{2}(1-|\psi|^2)^2 + K_1^{(j)}|\mathrm{div}\,\boldsymbol{u}|^2 \\ &\quad + K_2^{(j)}|\mathrm{curl}\,\boldsymbol{u}|^2\}dx - \frac{\kappa^2}{2}|\Omega| \\ &\leq \inf_{\psi\in H^1(\Omega,\mathbb{C})}\int_\Omega \{|\nabla_{q\boldsymbol{u}_0}\psi|^2 + \frac{\kappa^2}{2}(1-|\psi|^2)^2 + K_1^{(j)}|\mathrm{div}\,\boldsymbol{u}_0|^2 \\ &\quad + K_2^{(j)}|\mathrm{curl}\,\boldsymbol{u}_0|^2\}dx - \frac{\kappa^2}{2}|\Omega|.\end{aligned}$$

Therefore, we get

$$\int_\Omega \{|\nabla_{q\boldsymbol{u}_j}\psi_j|^2 + \frac{\kappa^2}{2}(1-|\psi_j|^2)^2 + K_1^{(j)}|\operatorname{div}\boldsymbol{u}_0|^2 + K_2^{(j)}|\operatorname{curl}\boldsymbol{u}_0|^2 dx \tag{3.3}$$
$$\le \inf_{\psi\in H^1(\Omega,\mathbb{C})}\int_\Omega \{|\nabla_{q\boldsymbol{u}_0}\psi|^2 + \frac{\kappa^2}{2}(1-|\psi|^2)^2 + K_1^{(j)}|\operatorname{div}\boldsymbol{u}_0|^2$$
$$+ K_2^{(j)}|\operatorname{curl}\boldsymbol{u}_0|^2\}dx.$$

Since

$$\begin{aligned}\int_\Omega |\nabla_{q\boldsymbol{u}_j}\psi_j|^2 dx &= \int_\Omega |\nabla_{q\boldsymbol{u}_0}\psi_j + iq(\boldsymbol{u}_0-\boldsymbol{u}_j)\psi_j|^2 dx \\ &\le \int_\Omega |\nabla_{q\boldsymbol{u}_0}\psi_j|^2 dx + C\|\boldsymbol{u}_0-\boldsymbol{u}_j\|^2_{L^4(\Omega,\mathbb{R}^3)}.\end{aligned}$$

Hence $\{\psi_j\}$ is a minimizing sequence of $\mathcal{G}_{q\boldsymbol{u}_0}$. Thus by (2.14), we see that ψ_0 is a minimizer of $\mathcal{G}_{q\boldsymbol{u}_0}$. We also have $\nabla\psi_j \to \nabla\psi_0$ strongly in $L^2(\Omega,\mathbb{R}^3)$, so $\psi_j \to \psi_0$ strongly in $H^1(\Omega,\mathbb{C})$. □

From now on, we consider the cases where K_1 is fixed and $K_2 \to \infty$ or K_2 is fixed and $K_1 \to \infty$.

First we shall show that if $\mathbb{G}(\boldsymbol{n}_0) \ne \emptyset$, $G(\boldsymbol{n}_0)$ is achieved in $\mathbb{G}(\boldsymbol{n}_0)$ and the asymptotic behaviors of $N(K_1,K_2,\boldsymbol{n}_0)$ and minimizers of $N(K_1,K_2,\boldsymbol{n}_0)$ as $K_2 \to \infty$ where $N(K_1,K_2,\boldsymbol{n}_0)$ is defined by (1.3).

Proposition 3.2. (cf. [6]) *Assume that $\mathbb{G}(\boldsymbol{n}_0) \ne \emptyset$. Then we have the following.*

(i) $G(\boldsymbol{n}_0)$ is achieved in $\mathbb{G}(\boldsymbol{n}_0)$.

(ii) $\lim_{K_2\to\infty} N(K_1,K_2,\boldsymbol{n}_0) = K_1 G(\boldsymbol{n}_0)$.

(iii) Let $\boldsymbol{u}_j$ be a minimizer of $N(K_1,K_2,\boldsymbol{n}_0)$ with $K_2 = K_2^{(j)}$ where $K_2^{(j)} \to \infty$ as $j \to \infty$. Then there exist a subsequence $\{\boldsymbol{u}_{j_l}\}$, $\phi \in H^2(\Omega)$ and $\boldsymbol{h}^1 \in \mathbb{H}_1(\Omega)$ such that $\boldsymbol{u} := \nabla\phi + \boldsymbol{h}^1$ is a minimizer of $G(\boldsymbol{n}_0)$, and $\boldsymbol{u}_{j_l} \to \boldsymbol{u}$ strongly in $H^1(\Omega,\mathbb{R}^3)$ as $l \to \infty$.

Proof. (i) Let $\{\boldsymbol{u}_j\} \subset \mathbb{G}(\boldsymbol{n}_0)$ be a minimizing sequence of $G(\boldsymbol{n}_0)$, i.e.,

$$\lim_{j\to\infty}\|\operatorname{div}\boldsymbol{u}_j\|^2_{L^2(\Omega)} = G(\boldsymbol{n}_0).$$

First we claim that $\{\boldsymbol{u}_j\}$ is bounded in $H^1(\Omega,\mathbb{R}^3)$. In fact, since $\boldsymbol{u}_j \in \mathbb{G}(\boldsymbol{n}_0)$, $\|\boldsymbol{u}_j\|_{L^2(\Omega,\mathbb{R}^3)} = \sqrt{|\Omega|}$ and $\boldsymbol{u}_j = \boldsymbol{n}_0$ on Γ and $\operatorname{curl}\boldsymbol{u}_j = 0$ in Ω. Thus from the estimate (2.3), we see that there exist constants C, C' such that

$$\begin{aligned}\|\boldsymbol{u}_j\|_{H^1(\Omega,\mathbb{R}^3)} &\le C\{\|\boldsymbol{u}_j\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{div}\boldsymbol{u}_j\|_{L^2(\Omega)} + \|\operatorname{curl}\boldsymbol{u}_j\|_{L^2(\Omega,\mathbb{R}^3)} \\ &\quad + \|\boldsymbol{u}_j\cdot\boldsymbol{\nu}\|_{H^{1/2}(\Gamma)}\}\end{aligned}$$

$$\leq \quad C\{\sqrt{|\Omega|}+G(\boldsymbol{n}_0)+1+\|\boldsymbol{n}_0\cdot\boldsymbol{\nu}\|_{H^{1/2}(\Gamma)}\}\leq C'$$

for large j.

Passing to a subsequence, we may assume that $\boldsymbol{u}_j\to\boldsymbol{u}_0$ weakly in $H^1(\Omega,\mathbb{R}^3)$, strongly $L^2(\Omega,\mathbb{R}^3)$ and $\boldsymbol{u}_j\to\boldsymbol{u}_0$ a.e. in Ω. So we have $|\boldsymbol{u}_0|=1$ a.e. in Ω, $\boldsymbol{u}_0=\boldsymbol{n}_0$ on Γ and $\operatorname{curl}\boldsymbol{u}_0=0$ in Ω. Since $\operatorname{div}\boldsymbol{u}_j\to\operatorname{div}\boldsymbol{u}_0$ weakly in $L^2(\Omega)$, we see that

$$\int_\Omega|\operatorname{div}\boldsymbol{u}_0|^2dx\leq\liminf_{j\to\infty}\int_\Omega|\operatorname{div}\boldsymbol{u}_j|^2dx=G(\boldsymbol{n}_0).$$

Thus $\boldsymbol{u}_0$ is a minimizer of $G(\boldsymbol{n}_0)$ in $\mathbb{G}(\boldsymbol{n}_0)$.

(ii) and (iii). Let $\widehat{\boldsymbol{u}}_0$ be a minimizer of $G(\boldsymbol{n}_0)$. Taking $\widehat{\boldsymbol{u}}_0$ as a test field of $N(K_1,K_2,\boldsymbol{n}_0)$, we see that

$$N(K_1,K_2,\boldsymbol{n}_0)\leq\int_\Omega K_1|\operatorname{div}\widehat{\boldsymbol{u}}_0|^2dx=K_1G(\boldsymbol{n}_0). \tag{3.4}$$

Let $\{K_2^{(j)}\}$ be any sequence such that $K_2^{(j)}\to\infty$ as $j\to\infty$ and let $\boldsymbol{u}_j\in H^1(\Omega,\mathbb{S}^2,\boldsymbol{n}_0)$ be a minimizer of $N(K_1,K_2,\boldsymbol{n}_0)$ with $K_2=K_2^{(j)}$. By (3.4), we have

$$\int_\Omega\{K_1|\operatorname{div}\boldsymbol{u}_j|^2+K_2^{(j)}|\operatorname{curl}\boldsymbol{u}_j|^2\}\leq K_1G(\boldsymbol{n}_0). \tag{3.5}$$

Therefore,

$$\begin{aligned}&\int_\Omega|\operatorname{div}\boldsymbol{u}_j|^2dx\leq G(\boldsymbol{n}_0),\\&\int_\Omega|\operatorname{curl}\boldsymbol{u}_j|^2dx\leq\frac{K_1}{K_2^{(j)}}G(\boldsymbol{n}_0)\to0\quad\text{as } j\to\infty.\end{aligned} \tag{3.6}$$

By the estimate (2.3), we have

$$\begin{aligned}\|\boldsymbol{u}_j\|^2_{H^1(\Omega,\mathbb{R}^3)}\quad&\leq\quad C\{\|\boldsymbol{u}_j\|^2_{L^2(\Omega,\mathbb{R}^3)}+\|\operatorname{div}\boldsymbol{u}_j\|^2_{L^2(\Omega)}+\|\operatorname{curl}\boldsymbol{u}_j\|^2_{L^2(\Omega,\mathbb{R}^3)}\\&\qquad+\|\boldsymbol{u}_j\cdot\boldsymbol{\nu}\|^2_{H^{1/2}(\Gamma)}\}\\&\leq\quad C\Big\{|\Omega|+G(\boldsymbol{n}_0)+\frac{K_1}{K_2^{(j)}}G(\boldsymbol{n}_0)+\|\boldsymbol{n}_0\cdot\boldsymbol{\nu}\|^2_{H^{1/2}}\Big\}.\end{aligned}$$

Thus we see that $\{\boldsymbol{u}_j\}$ is bounded in $H^1(\Omega,\mathbb{R}^3)$. Passing to a subseqence, we may assume that $\boldsymbol{u}_j\to\boldsymbol{u}$ weakly in $H^1(\Omega,\mathbb{R}^3)$, strongly in $L^2(\Omega,\mathbb{R}^3)$ and moreover, $\boldsymbol{u}_j\to\boldsymbol{u}$ a.e. in Ω. Therefore, we have $|\boldsymbol{u}|=1$ a.e. in Ω and $\boldsymbol{u}=\boldsymbol{n}_0$ on Γ, so $\boldsymbol{u}\in H^1(\Omega,\mathbb{S}^2,\boldsymbol{n}_0)$. Since $\operatorname{curl}\boldsymbol{u}_j\to\operatorname{curl}\boldsymbol{u}$ weakly in $L^2(\Omega,\mathbb{R}^3)$ and $\operatorname{curl}\boldsymbol{u}_j\to0$ strongly in $L^2(\Omega,\mathbb{R}^3)$ from (3.6), we have

$$\operatorname{curl}\boldsymbol{u}=0\quad\text{in }\Omega. \tag{3.7}$$

Since $\operatorname{div} \boldsymbol{u}_j \to \operatorname{div} \boldsymbol{u}$ weakly in $L^2(\Omega)$, we get, from (3.6),

$$\int_\Omega |\operatorname{div} \boldsymbol{u}|^2 dx \le \liminf_{j\to\infty} \int_\Omega |\operatorname{div} \boldsymbol{u}_j|^2 dx \le G(\boldsymbol{n}_0).$$

Thus $\boldsymbol{u}$ is a minimizer of $G(\boldsymbol{n}_0)$. Since if we again use (3.6), we see that

$$\limsup_{j\to\infty} \int_\Omega |\operatorname{div} \boldsymbol{u}_j| dx \le G(\boldsymbol{n}_0) \le \int_\Omega |\operatorname{div} \boldsymbol{u}|^2 dx \le \liminf_{j\to\infty} \int_\Omega |\operatorname{div} \boldsymbol{u}_j|^2 dx,$$

so we have

$$\lim_{j\to\infty} \int_\Omega |\operatorname{div} \boldsymbol{u}_j|^2 dx = \int_\Omega |\operatorname{div} \boldsymbol{u}|^2 dx.$$

Since $\operatorname{div} \boldsymbol{u}_j \to \operatorname{div} \boldsymbol{u}$ weakly in $L^2(\Omega)$, $\operatorname{div} \boldsymbol{u}_j \to \operatorname{div} \boldsymbol{u}$ strongly in $L^2(\Omega)$. Applying (2.3) to $\boldsymbol{u}_j - \boldsymbol{u}$, we see that $\boldsymbol{u}_j \to \boldsymbol{u}$ strongly in $H^1(\Omega, \mathbb{R}^3)$. By (3.4) and (3.5), we have

$$K_1 G(\boldsymbol{n}_0) \ge N(K_1, K_2^{(j)}, \boldsymbol{n}_0) \ge K_1 \int_\Omega |\operatorname{div} \boldsymbol{u}_j|^2 dx$$
$$\to K_1 \int_\Omega |\operatorname{div} \boldsymbol{u}|^2 dx = K_1 G(\boldsymbol{n}_0)$$

as $j \to \infty$. Thus we get

$$\lim_{j\to\infty} N(K_1, K_2^{(j)}, \boldsymbol{n}_0) = K_1 G(\boldsymbol{n}_0).$$

Hence (ii) and (iii) were proved. □

Remark 3.3. Since $\operatorname{curl} \boldsymbol{u} = 0$ in Ω for any $\boldsymbol{u} \in \mathbb{G}(\boldsymbol{n}_0)$, it follows from (1.5) that we can write $\boldsymbol{u} = \nabla\phi + \boldsymbol{h}^1$ where $\phi \in H^2(\Omega)$ and $\boldsymbol{h}^1 \in \mathbb{H}_1(\Omega)$. Thus in a particular case where Ω is simply connected, since $\mathbb{H}_1(\Omega) = \{0\}$, Proposition 3.2 is almost identical with [6, Lemma 5.2].

Proposition 3.4. (cf. [6]) *Assume that $\mathbb{R}(\boldsymbol{n}_0) \ne \emptyset$. Then we have the following.*

(i) $R(\boldsymbol{n}_0)$ is achieved in $\mathbb{R}(\boldsymbol{n}_0)$.

(ii) $\lim_{K_1\to\infty} N(K_1, K_2, \boldsymbol{n}_0) = K_2 R(\boldsymbol{n}_0)$.

(iii) Let $\boldsymbol{u}_j$ be a minimizer of $N(K_1, K_2, \boldsymbol{n}_0)$ with $K_1 = K_1^{(j)}$ where $K_1^{(j)} \to \infty$ as $j \to \infty$. Then there exist a subsequence $\{\boldsymbol{u}_{j_l}\}$ and $\boldsymbol{u}_0 \in \mathbb{R}(\boldsymbol{n}_0)$ where $\boldsymbol{u}_0$ is a minimizer of $R(\boldsymbol{n}_0)$ such that $\boldsymbol{u}_{j_l} \to \boldsymbol{u}_0$ strongly in $H^1(\Omega, \mathbb{R}^3)$ as $l \to \infty$.

Proof. (i) Let $\{\boldsymbol{u}_j\} \subset \mathbb{R}(\boldsymbol{n}_0)$ be a minimizing sequence of $R(\boldsymbol{n}_0)$, i.e.,

$$\lim_{j\to\infty} \|\operatorname{curl} \boldsymbol{u}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} = R(\boldsymbol{n}_0).$$

We claim that $\{\boldsymbol{u}_j\}$ is bounded in $H^1(\Omega, \mathbb{R}^3)$.

In fact, since $\|\boldsymbol{u}_j\|_{L^2(\Omega,\mathbb{R}^3)} = \sqrt{|\Omega|}, \boldsymbol{u}_j = \boldsymbol{n}_0$ on Γ and $\operatorname{div}\boldsymbol{u}_j = 0$ in Ω, applying the estimate (2.4) with $k=0$ to $\boldsymbol{u}_j$,

$$\begin{aligned}\|\boldsymbol{u}_j\|_{H^1(\Omega,\mathbb{R}^3)} &\leq C\{\|\boldsymbol{u}_j\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{div}\boldsymbol{u}_j\|_{L^2(\Omega)} + \|\operatorname{curl}\boldsymbol{u}_j\|_{L^2(\Omega,\mathbb{R}^3)} \\ &\qquad + \|\boldsymbol{u}_j \times \boldsymbol{\nu}\|_{H^{1/2}(\Gamma,\mathbb{R}^3)}\} \\ &\leq C\{\sqrt{|\Omega|} + R(\boldsymbol{n}_0) + 1 + \|\boldsymbol{n}_0 \times \boldsymbol{\nu}\|_{H^{1/2}(\Gamma,\mathbb{R}^3)}\} \leq C'\end{aligned}$$

for large j.

Therefore, passing to a subsequence, we may assume that $\boldsymbol{u}_j \to \boldsymbol{u}_0$ weakly in $H^1(\Omega,\mathbb{R}^3)$, strongly in $L^2(\Omega,\mathbb{R}^3)$ and $\boldsymbol{u}_j \to \boldsymbol{u}_0$ a.e. in Ω. Thus we have $|\boldsymbol{u}_0| = 1$ a.e. in Ω and $\boldsymbol{u}_0 = \boldsymbol{n}_0$ on Γ. Since $0 = \operatorname{div}\boldsymbol{u}_j \to \operatorname{div}\boldsymbol{u}_0$ weakly in $L^2(\Omega)$, we have $\operatorname{div}\boldsymbol{u}_0 = 0$ in Ω. Thus $\boldsymbol{u}_0 \in \mathbb{R}(\boldsymbol{n}_0)$ and since $\operatorname{curl}\boldsymbol{u}_j \to \operatorname{curl}\boldsymbol{u}_0$ weakly in $L^2(\Omega,\mathbb{R}^3)$, we have

$$\int_\Omega |\operatorname{curl}\boldsymbol{u}_0|^2 dx \leq \liminf_{j\to\infty} \int_\Omega |\operatorname{curl}\boldsymbol{u}_j|^2 dx = R(\boldsymbol{n}_0).$$

Hence $\boldsymbol{u}_0$ is a minimizer of $R(\boldsymbol{n}_0)$.

Next, we shall prove (ii) and (iii).

Let $\widehat{\boldsymbol{u}}_0$ be a minimizer of $R(\boldsymbol{n}_0)$ and taking $\widehat{\boldsymbol{u}}_0$ as a test field of $N(K_1,K_2,\boldsymbol{n}_0)$, we see that

$$\begin{aligned}N(K_1,K_2,\boldsymbol{n}_0) &\leq \int_\Omega \{K_1|\operatorname{div}\widehat{\boldsymbol{u}}_0|^2 + K_2|\operatorname{curl}\widehat{\boldsymbol{u}}_0|^2\}dx \\ &= K_2\int_\Omega |\operatorname{curl}\widehat{\boldsymbol{u}}_0|^2 dx = K_2 R(\boldsymbol{n}_0).\end{aligned} \tag{3.8}$$

Let $\{K_1^{(j)}\}$ be any sequence such that $K_1^{(j)} \to \infty$ as $j \to \infty$ and let $\boldsymbol{u}_j \in H^1(\Omega,\mathbb{S}^2,\boldsymbol{n}_0)$ be a minimizer of $N(K_1,K_2,\boldsymbol{n}_0)$ for $K_1 = K_1^{(j)}$. By (3.8), we have

$$\int_\Omega \{K_1^{(j)}|\operatorname{div}\boldsymbol{u}_j|^2 + K_2|\operatorname{curl}\boldsymbol{u}_j|^2\}dx \leq K_2 R(\boldsymbol{n}_0), \tag{3.9}$$

therefore,

$$\begin{aligned}\int_\Omega |\operatorname{div}\boldsymbol{u}_j|^2 dx &\leq \frac{K_2}{K_1^{(j)}} R(\boldsymbol{n}_0) \to 0, \\ \int_\Omega |\operatorname{curl}\boldsymbol{u}_j|^2 dx &\leq R(\boldsymbol{n}_0).\end{aligned} \tag{3.10}$$

Applying the estimate (2.4) with $k=0$ to $\boldsymbol{u}_j$, we see that

$$\begin{aligned}\|\boldsymbol{u}_j\|^2_{H^1(\Omega,\mathbb{R}^3)} \leq C\{&\|\boldsymbol{u}_j\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{div}\boldsymbol{u}_j\|^2_{L^2(\Omega)} \\ &+ \|\operatorname{curl}\boldsymbol{u}_j\|_{L^2(\Omega,\mathbb{R}^3)} + \|\boldsymbol{u}_j \times \boldsymbol{\nu}\|_{H^{1/2}(\Gamma,\mathbb{R}^3)}\}\end{aligned}$$

$$\leq C\{|\Omega| + \frac{K_2}{K_1^{(j)}} R(\boldsymbol{n}_0) + R(\boldsymbol{n}_0) + \|\boldsymbol{n}_0 \times \boldsymbol{\nu}\|_{H^{1/2}(\Gamma,\mathbb{R}^3)}\}.$$

Thus $\{\boldsymbol{u}_j\}$ is bounded in $H^1(\Omega,\mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{u}_j \to \boldsymbol{u}_0$ weakly in $H^1(\Omega,\mathbb{R}^3)$, strongly in $L^2(\Omega,\mathbb{R}^3)$, and moreover, $\boldsymbol{u}_j \to \boldsymbol{u}_0$ a.e. in Ω. Thus we see that $|\boldsymbol{u}_0| = 1$ a.e. in Ω, $\boldsymbol{u}_0 = \boldsymbol{n}_0$ on Γ, so $\boldsymbol{u}_0 \in H^1(\Omega,\mathbb{S}^2,\boldsymbol{n}_0)$. Since $\operatorname{div}\boldsymbol{u}_j \to \operatorname{div}\boldsymbol{u}_0$ weakly in $L^2(\Omega)$ and from (3.10) $\operatorname{div}\boldsymbol{u}_j \to 0$ strongly in $L^2(\Omega)$, we see that $\operatorname{div}\boldsymbol{u}_0 = 0$ in Ω. From (3.10), we also have

$$\int_\Omega |\operatorname{curl}\boldsymbol{u}_0|^2 dx \leq \liminf_{j\to\infty} \int_\Omega |\operatorname{curl}\boldsymbol{u}_j|^2 dx \leq R(\boldsymbol{n}_0).$$

Thus it follows that $\boldsymbol{u}_0$ is a minimizer of $R(\boldsymbol{n}_0)$ and so from (3.10),

$$\limsup_{j\to\infty} \int_\Omega |\operatorname{curl}\boldsymbol{u}_j|^2 dx \leq R(\boldsymbol{n}_0) = \int_\Omega |\operatorname{curl}\boldsymbol{u}_0|^2 dx \leq \liminf_{j\to\infty} \int_\Omega |\operatorname{curl}\boldsymbol{u}_j|^2 dx.$$

Hence

$$\lim_{j\to\infty} \int_\Omega |\operatorname{curl}\boldsymbol{u}_j|^2 dx = \int_\Omega |\operatorname{curl}\boldsymbol{u}_0|^2 dx.$$

From this equality and the fact that $\operatorname{curl}\boldsymbol{u}_j \to \operatorname{curl}\boldsymbol{u}_0$ weakly in $L^2(\Omega,\mathbb{R}^3)$, we see that $\operatorname{curl}(\boldsymbol{u}_j - \boldsymbol{u}_0) \to 0$ strongly in $L^2(\Omega,\mathbb{R}^3)$. Since $\operatorname{div}(\boldsymbol{u}_j - \boldsymbol{u}_0) = \operatorname{div}\boldsymbol{u}_j \to 0$ strongly in $L^2(\Omega)$, $\boldsymbol{u}_j - \boldsymbol{u}_0 \to 0$ strongly in $L^2(\Omega,\mathbb{R}^3)$. Moreover, since $(\boldsymbol{u}_j - \boldsymbol{u}_0) \times \boldsymbol{\nu} = 0$ on Γ, it follows from the estimate (2.4) that $\boldsymbol{u}_j \to \boldsymbol{u}_0$ strongly in $H^1(\Omega,\mathbb{R}^3)$. From (3.9),

$$K_2 R(\boldsymbol{n}_0) \geq N(K_1^{(j)}, K_2, \boldsymbol{n}_0) \geq K_2 \int_\Omega |\operatorname{curl}\boldsymbol{u}_j|^2 dx$$
$$\to K_2 \int_\Omega |\operatorname{curl}\boldsymbol{u}_0|^2 dx = K_2 R(\boldsymbol{n}_0)$$

as $j \to \infty$. Therefore, we get

$$\lim_{j\to\infty} N(K_1^{(j)}, K_2, \boldsymbol{n}_0) = K_2 R(\boldsymbol{n}_0). \qquad \square$$

Remark 3.5. From (1.8), we can write $\boldsymbol{u}_0$ in Proposition 3.4 (iii) into the form $\boldsymbol{u}_0 = \operatorname{curl}\boldsymbol{v}_0 + \boldsymbol{h}_0^2$ where $\boldsymbol{v}_0 \in H^2(\Omega,\mathbb{R}^3)$ and $\boldsymbol{h}_0^2 \in \mathbb{H}_2(\Omega)$. Thus in a particular case where Ω has no holes, we can write $\boldsymbol{u}_0 = \operatorname{curl}\boldsymbol{v}_0$ where $\boldsymbol{v}_0 \in H^2(\Omega,\mathbb{R}^3)$.

4. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. First, we prove (1). Assume that $\mathbb{G}(\boldsymbol{n}_0) \neq \emptyset$ and let $\widehat{\boldsymbol{u}}_0$ be a minimizer of $G(\boldsymbol{n}_0)$. Then from (1.5), we can write

$\widehat{\boldsymbol{u}}_0 = \nabla\widehat{\phi}_0 + \widehat{\boldsymbol{h}}_0^1$ where $\widehat{\phi}_0 \in H^2(\Omega)$ and $\widehat{\boldsymbol{h}}_0^1 \in \mathbb{H}_1(\Omega)$. Taking $\psi = e^{iq\widehat{\phi}_0}, \boldsymbol{u} = \nabla\widehat{\phi}_0$ as a test field of $\mathcal{E}_0$, since $\operatorname{div}\boldsymbol{u} = \Delta\widehat{\phi}_0 = \operatorname{div}\widehat{\boldsymbol{u}}_0$ and $\operatorname{curl}\boldsymbol{u} = 0$, we have

$$C_d(K_1, K_2, \kappa, q) \le \mathcal{E}_0[\psi, \boldsymbol{u}] = -\frac{\kappa^2}{2}|\Omega| + \int_\Omega K_1|\operatorname{div}\widehat{\boldsymbol{u}}_0|^2 dx \qquad (4.1)$$
$$= K_1 G(\boldsymbol{n}_0) - \frac{\kappa^2}{2}|\Omega|.$$

On the other hand, if $\boldsymbol{u}$ is a minimizer of $N(K_1, K_2, \boldsymbol{n}_0)$, we have

$$\mathcal{E}_0[0, \boldsymbol{u}] = \int_\Omega \{K_1|\operatorname{div}\boldsymbol{u}|^2 + K_2|\operatorname{curl}\boldsymbol{u}|^2\}dx = N(K_1, K_2, \boldsymbol{n}_0). \qquad (4.2)$$

Therefore if $K_1 G(\boldsymbol{n}_0) - N(K_1, K_2, \boldsymbol{n}_0) < \frac{\kappa^2}{2}|\Omega|$, we get

$$C_d(K_1, K_2, \kappa, q) \le K_1 G(\boldsymbol{n}_0) - \frac{\kappa^2}{2}|\Omega| < N(K_2, K_2, \boldsymbol{n}_0) = \mathcal{E}[0, \boldsymbol{u}].$$

This means that the minimizers of $C_d(K_1, K_2, \kappa, q)$ are non-trivial.

Let $(\psi_j, \boldsymbol{u}_j) \in \mathbb{H}(\Omega, \boldsymbol{n}_0) = H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0)$ be a minimizer of $C_d(K_1, K_2^{(j)}, \kappa, q)$ where $K_2^{(j)} \to \infty$ as $j \to \infty$. By (4.1),

$$C_d(K_1, K_2^{(j)}, \kappa, q) = \mathcal{E}_0[\psi_j, \boldsymbol{u}_j] \le K_1 G(\boldsymbol{n}_0) - \frac{\kappa^2}{2}|\Omega|.$$

That is to say,

$$\int_\Omega \{|\nabla_{q\boldsymbol{u}_j}\psi_j|^2 + \frac{\kappa^2}{2}(1 - |\psi_j|^2)^2$$
$$+ K_1|\operatorname{div}\boldsymbol{u}_j|^2 + K_2^{(j)}|\operatorname{curl}\boldsymbol{u}_j|^2\}dx \le K_1 G(\boldsymbol{n}_0). \quad (4.3)$$

It follows that $\operatorname{curl}\boldsymbol{u}_j \to 0$ strongly in $L^2(\Omega, \mathbb{R}^3)$, $\{\operatorname{div}\boldsymbol{u}_j\}$ is bounded in $L^2(\Omega)$. Since $|\boldsymbol{u}_j| = 1$ a.e. in Ω and $\boldsymbol{u}_j = \boldsymbol{n}_0$ on Γ, it follows from the estimate (2.3), we see that $\{\boldsymbol{u}_j\}$ is bounded in $H^1(\Omega, \mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{u}_j \to \boldsymbol{u}_0$ weakly in $H^1(\Omega, \mathbb{R}^3)$, strongly in $L^2(\Omega, \mathbb{R}^3)$ and $\boldsymbol{u}_j \to \boldsymbol{u}_0$ a.e. in Ω. Hence we see that $|\boldsymbol{u}_0| = 1$ a.e. in Ω and $\boldsymbol{u}_0 = \boldsymbol{n}_0$ on Γ. Moreover, since $\operatorname{curl}\boldsymbol{u}_j \to 0$ strongly in $L^2(\Omega, \mathbb{R}^3)$ and $\operatorname{curl}\boldsymbol{u}_j \to \operatorname{curl}\boldsymbol{u}_0$ weakly in $L^2(\Omega, \mathbb{R}^3)$, we have $\operatorname{curl}\boldsymbol{u}_0 = 0$ in Ω. Thus $\boldsymbol{u}_0 \in \mathbb{G}(\boldsymbol{n}_0)$ and so from (4.3),

$$G(\boldsymbol{n}_0) \le \int_\Omega |\operatorname{div}\boldsymbol{u}_0|^2 dx \le \liminf_{j\to\infty} \int_\Omega |\operatorname{div}\boldsymbol{u}_j|^2 dx \le G(\boldsymbol{n}_0). \qquad (4.4)$$

Thus $\boldsymbol{u}_0$ is a minimizer of $G(\boldsymbol{n}_0)$.

On the other hand, it follows from (4.3) that $\{|\nabla_{q\boldsymbol{u}_j}\psi_j|\}$ is bounded in $L^2(\Omega)$ and $\{\psi_j\}$ is bounded in $L^4(\Omega, \mathbb{C})$. Since $\{q\boldsymbol{u}_j\psi_j\}$ is bounded in $L^2(\Omega, \mathbb{C}^3)$, we see that $\{\nabla\psi_j = \nabla_{q\boldsymbol{u}_j}\psi_j + iq\boldsymbol{u}_j\psi_j\}$ is bounded in $L^2(\Omega, \mathbb{C}^3)$. Thus $\{\psi_j\}$ is bounded in $H^1(\Omega, \mathbb{C})$. Passing to a subsequence, we may assume that $\psi_j \to \psi_0$

weakly in $H^1(\Omega,\mathbb{C})$ and strongly in $L^4(\Omega,\mathbb{C})$. Using (4.3), we have

$$\begin{aligned}\int_\Omega \{|\nabla_{q\boldsymbol{u}_0}\psi_0|^2+\frac{\kappa^2}{2}(1-|\psi_0|^2)^2\}dx+K_1G(\boldsymbol{n}_0)\\ \le \liminf_{j\to\infty}\mathcal{E}_0[\psi_j,\boldsymbol{u}_j]+\frac{\kappa^2}{2}|\Omega|\\ \le \limsup_{j\to\infty}\mathcal{E}_0[\psi_j,\boldsymbol{u}_j]+\frac{\kappa^2}{2}|\Omega|\\ \le K_1G(\boldsymbol{n}_0).\end{aligned}$$

Thus we have

$$\begin{aligned}&\lim_{j\to\infty}\mathcal{E}_0[\psi_j,\boldsymbol{u}_j]=K_1G(\boldsymbol{n}_0)-\frac{\kappa^2}{2}|\Omega|, \qquad (4.5)\\ &\lim_{j\to\infty}\int_\Omega\{|\nabla_{q\boldsymbol{u}_j}\psi_j|^2+\frac{\kappa^2}{2}(1-|\psi_j|^2)^2\}dx\\ &\quad=\int_\Omega\{|\nabla_{q\boldsymbol{u}_0}\psi_0|^2+\frac{\kappa^2}{2}(1-|\psi_0|^2)^2\}dx=0,\\ &\lim_{j\to\infty}\int_\Omega|\operatorname{div}\boldsymbol{u}_j|^2dx=\int_\Omega|\operatorname{div}\boldsymbol{u}_0|^2dx=G(\boldsymbol{n}_0),\\ &\lim_{j\to\infty}K_2^{(j)}\int_\Omega|\operatorname{curl}\boldsymbol{u}_j|^2dx=0.\end{aligned}$$

Since $\psi_j\to\psi_0$ strongly in $L^4(\Omega,\mathbb{C})$, it follows from the second equality of (4.5) that

$$\lim_{j\to\infty}\int_\Omega|\nabla\psi_j|^2dx=\int_\Omega|\nabla\psi_0|^2dx.$$

Since $\nabla\psi_j\to\nabla\psi_0$ weakly in $L^2(\Omega,\mathbb{C}^3)$, we see that $\nabla\psi_j\to\nabla\psi_0$ strongly in $L^2(\Omega,\mathbb{C}^3)$, so $\psi_j\to\psi_0$ strongly in $H^1(\Omega,\mathbb{C})$. From the second equality of (4.5), since $|\psi_0|=1$ a.e. in Ω and $\nabla\psi_0-iq\boldsymbol{u}_0\psi_0=0$ in Ω, we can write $\psi_0=e^{1q\varphi_0}$. On the other hand, from (1.5), we can write $\boldsymbol{u}_0=\nabla\phi_0+\boldsymbol{h}_0^1$ where $\phi_0\in H^2(\Omega),\boldsymbol{h}_0^1\in\mathbb{H}_1(\Omega)$. Thus

$$0=\nabla\psi_0-iq\boldsymbol{u}_0\psi_0=iq(\nabla(\varphi_0-\phi_0)-\boldsymbol{h}_0^1)e^{iq\varphi_0},$$

and so $\nabla(\varphi_0-\phi_0)=\boldsymbol{h}_0^1$. Since $\nabla H^1(\Omega)$ is orthogonal to $\mathbb{H}_1(\Omega)$ by (1.5), we see that $\boldsymbol{h}_0^1=0,\nabla(\varphi_0-\phi_0)=0$ in Ω. Hence we can write $\varphi_0-\phi_0$ is a constant. So we may take $\varphi_0=\phi_0+c_1$ where c_1 is a constant. Since $\operatorname{div}\boldsymbol{u}_j\to\operatorname{div}\boldsymbol{u}_0$ weakly in $L^2(\Omega)$, from the third equality of (4.5), we see that $\operatorname{div}\boldsymbol{u}_j\to\operatorname{div}\boldsymbol{u}_0$ strongly in $L^2(\Omega)$. From the last equality of (4.5), $\operatorname{curl}\boldsymbol{u}_j\to\operatorname{curl}\boldsymbol{u}_0=0$ strongly in $L^2(\Omega)$. Moreover, since $\boldsymbol{u}_j\to\boldsymbol{u}_0$ strongly in $L^2(\Omega,\mathbb{R}^3)$ and $\boldsymbol{u}_j\cdot\boldsymbol{\nu}=\boldsymbol{n}_0\cdot\boldsymbol{\nu}=\boldsymbol{u}_0\cdot\boldsymbol{\nu}$ on Γ, it follows from (2.3) that $\boldsymbol{u}_j\to\boldsymbol{u}_0=\nabla\phi_0$ strongly in $H^1(\Omega,\mathbb{R}^3)$. Thus

$(\psi_j, \boldsymbol{u}_j) \to (ce^{iq\phi_0}, \nabla\phi_0)$ with $|c| = 1$, strongly in $\mathbb{H}(\Omega, \boldsymbol{n}_0)$.

(2) We assume that $\mathbb{R}(\boldsymbol{n}_0) \neq \emptyset$. Let $(\widehat{\phi}, \widehat{\boldsymbol{u}}) \in H^1(\Omega, \mathbb{C}) \times \mathbb{R}(\boldsymbol{n}_0)$ be a minimizer of $d(K_2, \kappa, q, \boldsymbol{n}_0)$ which is defined in (1.11). If we take $(\widehat{\phi}, \widehat{\boldsymbol{u}})$ as a test field of $\mathcal{E}_0$, we see that

$$C_d(K_1, K_2, \kappa, q) \leq d(K_2, \kappa, q, \boldsymbol{n}_0). \tag{4.6}$$

If $d(K_2, \kappa, q, \boldsymbol{n}_0) < N(K_1, K_2, \boldsymbol{n}_0)$, then

$$C_d(K_1, K_2, \kappa, q) < N(K_1, K_2, \boldsymbol{n}_0) = \mathcal{E}_0[0, \boldsymbol{u}]$$

for any minimizer $\boldsymbol{u}$ of $N(K_1, K_2, \boldsymbol{n}_0)$. Therefore, in this case, the minimizers of $C_d(K_1, K_2, \kappa, q)$ are non-trivial.

(2ii) Let $(\psi_j, \boldsymbol{u}_j) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0)$ be a minimizer of $C_d(K_1^{(j)}, K_2, \kappa, q)$ where $K_1^{(j)} \to \infty$ as $j \to \infty$. Then, from (4.6)

$$\begin{aligned}\int_\Omega \{|\nabla_{q\boldsymbol{u}_j}\psi_j|^2 + \frac{\kappa^2}{2}(1 - |\psi|^2)^2 + K_1^{(j)}|\operatorname{div}\boldsymbol{u}_j|^2 + K_2|\operatorname{curl}\boldsymbol{u}_j|^2\}dx \\ = C_d(K_1, K_2, \kappa, q) + \frac{\kappa^2}{2}|\Omega| \leq d(K_2, \kappa, q, \boldsymbol{n}_0) + \frac{\kappa^2}{2}|\Omega|.\end{aligned}$$

As in (1), we see that $\{\psi_j\}$ is bounded in $H^1(\Omega, \mathbb{C})$. Since $\{\operatorname{curl}\boldsymbol{u}_j\}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$, $\operatorname{div}\boldsymbol{u}_j \to 0$ strongly in $L^2(\Omega)$, $|\boldsymbol{u}_j| = 1$ a.e. in Ω and $\boldsymbol{u}_j = \boldsymbol{n}_0$ on Γ, it follows from (2.4) that $\{\boldsymbol{u}_j\}$ is bounded in $H^1(\Omega, \mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{u}_j \to \boldsymbol{u}_0$ weakly in $H^1(\Omega, \mathbb{R}^3)$, strongly in $L^2(\Omega, \mathbb{R}^3)$, $\psi_j \to \psi_0$ weakly in $H^1(\Omega, \mathbb{C})$, strongly in $L^4(\Omega, \mathbb{C})$. Thus we can that $|\boldsymbol{u}_0| = 1$ a.e. in Ω, $\boldsymbol{u}_0 = \boldsymbol{n}_0$ on Γ and $\operatorname{div}\boldsymbol{u}_0 = 0$ in Ω. By (1.8), we can write $\boldsymbol{u}_0 = \operatorname{curl}\boldsymbol{v}_0 + \boldsymbol{h}_0^2$ where $\boldsymbol{v}_0 \in H^2(\Omega, \mathbb{R}^3), \boldsymbol{h}_0^2 \in \mathbb{H}_2(\Omega)$. Thus $\boldsymbol{u}_0 \in \mathbb{R}(\boldsymbol{n}_0)$. From (4.6), we have

$$\begin{aligned} d(K_2, \kappa, q, \boldsymbol{n}_0) &\leq \int_\Omega \{|\nabla_{q\boldsymbol{u}_0}\psi_0|^2 - \kappa^2|\psi_0|^2 + \frac{\kappa^2}{2}|\psi_0|^4 \\ &\qquad + K_2|\operatorname{curl}\boldsymbol{u}_0|^2\}dx \\ &\leq \liminf_{j\to\infty} \mathcal{E}_0[\psi_j, \boldsymbol{u}_j] \leq \limsup_{j\to\infty} \mathcal{E}_0[\psi_j, \boldsymbol{u}_j] \\ &\leq d(K_2, \kappa, q, \boldsymbol{n}_0). \end{aligned}$$

Therefore, we obtain

$$\lim_{j\to\infty} \mathcal{E}_0[\psi_j, \boldsymbol{u}_j] = d(K_2, \kappa, q, \boldsymbol{n}_0), \tag{4.7}$$

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega \{|\nabla_{q\boldsymbol{u}_j}\psi_0|^2 - \kappa^2|\psi_j|^2 + \frac{\kappa^2}{2}|\psi_j|^4 + K_1^{(j)}|\operatorname{div}\boldsymbol{u}_j|^2 + K_2|\operatorname{curl}\boldsymbol{u}_j|^2\}dx \\ = \int_\Omega \{|\nabla_{q\boldsymbol{u}_0}\psi_0|^2 - \kappa^2|\psi_0|^2 + \frac{\kappa^2}{2}|\psi_0|^4 + K_2|\operatorname{curl}\boldsymbol{u}_0|^2\}dx\end{aligned}$$

$$= d(K_2, \kappa, q, \boldsymbol{n}_0),$$

$$\lim_{j\to\infty} K_1^{(j)} \int_\Omega |\operatorname{div} \boldsymbol{u}_j|^2 dx = 0.$$

Thus $(\psi_0, \boldsymbol{u}_0)$ achieves $d(K_2, \kappa, q, \boldsymbol{n}_0)$. Since

$$\liminf_{j\to\infty} \int_\Omega \{|\nabla_{q\boldsymbol{u}_j}\psi_j|^2 - \kappa^2|\psi_j|^2 + \frac{\kappa^2}{2}|\psi_j|^4\}dx$$
$$\geq \int_\Omega \{|\nabla_{q\boldsymbol{u}_0}\psi_0|^2 - \kappa^2|\psi_0|^2 + \frac{\kappa^2}{2}|\psi_0|^4\}dx,$$
$$\liminf_{j\to\infty} \int_\Omega |\operatorname{curl} \boldsymbol{u}_j|^2 dx \geq \int_\Omega |\operatorname{curl} \boldsymbol{u}_0|^2 dx,$$

we have

$$\int_\Omega \{|\nabla_{q\boldsymbol{u}_0}\psi_0|^2 - \kappa^2|\psi_0|^2 + \frac{\kappa^2}{2}|\psi_0|^4 + K_2|\operatorname{curl} \boldsymbol{u}_0|^2\}dx$$
$$\leq \liminf_{j\to\infty} \int_\Omega \{|\nabla_{q\boldsymbol{u}_j}\psi_j|^2 - \kappa^2|\psi_j|^2 + \frac{\kappa^2}{2}|\psi_j|^4\}dx$$
$$+ \liminf_{j\to\infty} \int_\Omega K_2|\operatorname{curl} \boldsymbol{u}_j|^2\}dx,$$
$$\lim_{j\to\infty} \int_\Omega \{|\nabla_{q\boldsymbol{u}_j}\psi_j|^2 - \kappa^2|\psi_j|^2 + \frac{\kappa^2}{2}|\psi_j|^4 + K_2|\operatorname{curl} \boldsymbol{u}_j|^2\}dx$$
$$= d(K_2, \kappa, q, \boldsymbol{n}_0).$$

Thus we have

$$\lim_{j\to\infty} \int_\Omega \{|\nabla_{q\boldsymbol{u}_j}\psi_j|^2 - \kappa^2|\psi_j|^2 + \frac{\kappa^2}{2}|\psi_j|^4\}dx \tag{4.8}$$
$$= \int_\Omega \{|\nabla_{q\boldsymbol{u}_0}\psi_0|^2 - \kappa^2|\psi_0|^2 + \frac{\kappa^2}{2}|\psi_0|^4\}dx,$$
$$\lim_{j\to\infty} \int_\Omega |\operatorname{curl} \boldsymbol{u}_j|^2 dx = \int_\Omega |\operatorname{curl} \boldsymbol{u}_0|^2 dx.$$

Since $\operatorname{div} \boldsymbol{u}_j \to \operatorname{div} \boldsymbol{u}_0$ weakly in $L^2(\Omega)$ and $\operatorname{curl} \boldsymbol{u}_j \to \operatorname{curl} \boldsymbol{u}_0$ weakly in $L^2(\Omega, \mathbb{R}^3)$, taking the third equation of (4.7) into consideration, we see that $\boldsymbol{u}_j \to \boldsymbol{u}_0$ strongly in $H^1(\Omega, \mathbb{R}^3)$. Moreover, since $\nabla_{q\boldsymbol{u}_j}\psi_j \to \nabla_{q\boldsymbol{u}_0}\psi_0$ strongly in $L^2(\Omega, \mathbb{C})$, we see that $\psi_j \to \psi_0$ strongly in $H^1(\Omega, \mathbb{C})$. Thus $(\psi_j, \boldsymbol{u}_j) \to (\psi_0, \boldsymbol{u}_0)$ strongly in $\mathbb{H}(\Omega, \boldsymbol{n}_0)$. This completes the proof of Theorem 1.1.

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