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CURVES $X \subset \mathbb{P}^n$ WITH 2-DIMENSIONAL SETS WITH X-RANK TWO

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Abstract: Here we look at integral and non-degenerate curves $X \subset \mathbb{P}^n$ such that the subset of $\mathbb{P}^n$ formed by the points with X-rank n is two-dimensional, giving an example in positive characteristic.

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Fix an integral and non-degenerate curve $X \subseteq \mathbb{P}^n$ defined over an algebraically closed field $\mathbb{K}$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \ \ \rangle$ denote the linear span. Hence $r_X(P) = 1$ if and only if $P \in X$. Since X is non-degenerate, the X-rank is defined and $r_X(P) \leq n+1$ for all $P \in \mathbb{P}^n$. For any integer $k \geq 1$ set $E_X(k) := \{P \in \mathbb{P}^n : r_X(P) = k\}$ and $F_X(k) := \{P \in \mathbb{P}^n : r_X(P) \geq k\}$. If $\mathrm{char}(\mathbb{K}) = 0$, then $F_X(n+1) = \emptyset$ (see [9], Proposition 5.1). In positive characteristic we have $F_X(n+2) = \emptyset$ and if $E_X(n+1) \neq \emptyset$, then X is a strange curve and $E_X(n+1)$ is a unique point: the strange point of X (see [3], Theorem 2). Very few strange curves have $E_X(n+1) \neq \emptyset$. In [3] we also gave a sufficient condition for the non-emptiness of $E_X(n+1)$. If X is a rational normal curve and $\mathrm{char}(\mathbb{K}) = 0$, then $E_X(n) = TX \backslash X$ is 2-dimensional by a theorem of Comas and Seiguer (see [7] or [9], Theorem 4.1). Here we ask the following question.

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Question 1. Assume $\mathrm{char}(\mathbb{K}) = 0$. Is it true that $\dim(E_X(n)) = 2$ if and only if X is a rational normal curve?

For an affermative answer when $n = 3$ and X has only planar singularities, see Proposition 1. In positive characteristic this question has a negative answer. We gave a counterexample in [2], Example 2, when $n = 3$. However, $n = 3$ is a very special case, because there are so many space curves X with $E_X(3) \neq \emptyset$ (see [10]). Here we recycle the examples in [3] to show that Question 1 is not true in positive characteristic for arbitrary $n \geq 3$. In arbitrary characteristic it is easy to check that $\dim(E_X(n)) \leq 2$ for an arbitrary curve X.

We recall that the integral curve $X \subset \mathbb{P}^n$, 4 is said to be *strange* if there is $O \in \mathbb{P}^n$ such that $O \in T_Q X$ for all $Q \in X_{reg}$ (or, equivalently, for a general $Q \in X$) (see [5]). If X is strange, a point as above is called a *strange point* of X. If X is not a line, and X is strange, then $p := \mathrm{char}(\mathbb{K}) > 0$ and the strange point of X is unique. We recall the following definition (see [3], Definition 1).

Definition 1. Let $X \subset \mathbb{P}^n$, $n \geq 2$, be a non-degenerate strange curve and let O be its strange point. Let $\ell_O : \mathbb{P}^n \backslash \{O\} \to \mathbb{P}^{n-1}$ be the linear projection from O and $T \subset \mathbb{P}^{n-1}$ the closure of $\ell_O(Y \backslash \{O\})$. Thus T is non-degenerate and

$$\deg(X) = p^e s \cdot \deg(T) + \mu, \tag{1}$$

where μ is the multiplicity of X at O, while s and p^e are the separable and the inseparable degree of $\ell_O|X$, respectively (see [5], Theorem 2.3). Now assume $n \geq 3$, $\mu = 0$ (i.e. $O \notin X$) and $s = 1$. We say that X is *flat* or *flat with respect to its strange point* O if for any $S \subset X$ such that $\sharp(S) \leq n$ we have $\dim(\langle S \rangle) = \dim(\langle \ell_O(S) \rangle)$.

Here there is our example (see below for the proof that it works).

Example 1. Fix an integer $n \geq 3$ and a non-degenerate flat strange curve $X \subset \mathbb{P}^n$. Let O be the strange point of X and $H \subset \mathbb{P}^n$ any hyperplane such that $O \notin H$. See the linear projection ℓ_O from O as a morphism $\ell_O : \mathbb{P}^n \backslash \{O\} \to H$. Since X is flat, $O \notin X$. Set $T := \ell_O(X)$ and let $C(O,T) \subset \mathbb{P}^n$ the cone with vertex O and base T. Then $E_X(n) \cap C(0,T) = C(0,T) \backslash (\{O\} \cup X)$ and hence $\dim(E_X(n)) = 2$

We recall that there are two types of flat strange curves: the one such that $T := \ell_O(X)$ is the rational normal curve of $\mathbb{P}^{n-1}$ and the other ones (see [3]). All the flat strange curves with a rational normal curve as $\ell_O(X)$ are explicitely described by a weighted homogeneous equation in a Hirzebruch surface F_{n-1}. This construction was done in a more general setting in [3], which generalized the description of all strange plane curves given by V. Bayer and A. Hefez (see [5]). We explicit write down it in the case giving flat strange curves with rational normal curve as $\ell_O(X)$ (see Example 2).

Proof of Example 1. Take X as in Example 1. By [3], Theorem 2, we have $E_X(n+1) = F_X(n+1) = \{O\}$. Obviously $r_X(P) = 1$ for all $P \in X$. We have $r_X(P) \le n$ for all $P \in \mathbb{P}^n \backslash \{O\}$ (see [3], Theorem 2). Fix $P \in C(0,T) \backslash (\{O\} \cup X)$. Since $r_X(P) \le n$ for all $P \in \mathbb{P}^n \backslash \{O\}$, it is sufficient to prove $r_X(P) \ge n$. Assume $r_X(P) \le n-1$. Fix $S \subset X$ computing $r_X(P)$, i.e. $P \in \langle S \rangle$ and $\sharp(S) = r_X(P)$. Since X is flat, there is a unique $A \in X$ such that $\ell_O(A) = \ell_O(P)$. First assume $A \notin \langle S \rangle$. Hence $S \cup \{A\}$ is linearly independent and $\sharp(S \cup \{A\}) = \sharp(S) + 1 \le n$. Since $\{A, P\} \in \langle S \cup \{A\} \rangle$, the linear space $\langle S \cup \{A\} \rangle$ contains the line $\langle \{P, A\} \rangle$. Since $O \in \langle \{P, A\} \rangle$, we get $O \in \langle S \cup \{A\} \rangle$. Thus $r_X(O) \le n$, contradicting [3], Theorem 2.

Now assume $A \in \langle S \rangle$. Since $P \in \langle S \rangle$, the linear space $\langle S \rangle$ contains the line $\langle \{P, A\} \rangle$. Since $O \in \langle \{P, A\} \rangle$, we get $O \in \langle S \rangle$. Thus $r_X(O) \le n-1$, contradicting [3], Theorem 2. □

Example 2. Fix $O \in \mathbb{P}^n$, $n \ge 2$, and a hyperplane H of $\mathbb{P}^n$ such that $O \notin H$. Fix homogeneous coordinates $z_0, \dots, z_n$ of $\mathbb{P}^n$ such that $H = \{z_0 = 0\}$ and $O = (1; 0; \dots; 0)$. We use $z_1, \dots, z_n$ as homogeneous coordinates of H. Let $T \subseteq H$ be an integral, non-degenerate subvariety and $C(O,T) \subseteq \mathbb{P}^n$ the cone with vertex O and base T. Since T is non-degenerate, we may identify $H^0(H, \mathcal{O}_H(1))$ with an n-dimensional linear subspace V of $H^0(T, \mathcal{O}_T(1))$. Since T is non-degenerate and $O \notin H$, $C(O,T)$ is non-degenerate. Hence we may identify $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ with an $(n+1)$-dimensional linear subspace V_O of $H^0(C(O,T), \mathcal{O}_{C(0,1)}(1))$. Hence we may use $z_0, \dots, z_n$ as a basis of V_O, $z_1, \dots, z_n$ as a basis of V and write $z_0 | T \equiv 0$. We fix integers $e > 0$, $s \ge 1$ and $\mu \ge 0$ (p^e is the degree of inseparability of the projection from the strange point O, s is the degree of separability of the projection from the strange point O and μ is the multiplicity of the strange variety at its strange point O). Take a new variable w and give it weight p^e, while impose that $z_1, \dots, z_n$ have weight 1. Let $V(p^e, s, \mu)$ be the set of all $f \in \mathbb{K}[z_1, \dots, z_n, w]$ with $f = \sum_{i=0}^{s} f_s w^i$, with each $f_i \in \mathbb{K}[z_1, \dots, z_n]$ homogeneous of degree $\mu + p^{e(s-i)}$. Hence each element of $V(p^e, s, \mu)$ has total weight $\mu + p^e s$. Let $v : \Pi \to \mathbb{P}^n$ be the blowing-up of O. Let $D(O,T) \subseteq \Pi$ be the strict transform of $C(O,T)$. Set $u := v | D(O,T)$. Since $O \notin H$, we may see ℓ_O as a morphism $\ell_O : \mathbb{P}^n \backslash \{O\} \to H$. It is well-known that this morphism extends to a morphism $\alpha : \Pi \to H$ which is a $\mathbb{P}^1$-bundle, i.e. $\Pi \cong \mathbb{P}(\mathcal{O}_H(1) \oplus \mathcal{O}_H)$ (see [12], §I.4, or [8], p. 373). Restrict this construction to T. Since $O \notin H$ and $T \subseteq H$, we have $v^{-1}(T) \cong T$. Hence

$\alpha|v^{-1}(T)$ induces a projection $\pi : D(O,T) \to T$ which makes $D(O,T)$ a $\mathbb{P}^1$-bundle on T: the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_T(1) \oplus \mathcal{O}_T)$. The cone $C(O,T)$ is obtained from $D(O,T)$ contracting a section E of π. The scheme $D(O,T)\backslash E$ is the total space of the line bundle $\mathcal{O}_T(1)$. The variable w is the variable for the fibers of $D(O,T)\backslash E \to T$. For each $f = \sum_{i=0}^{s} f_s w^i \in V(p^e, s, \mu)$ its zero locus $Z_f \subseteq C(O,T)$ is obtained by taking the intersection of the cone $C(O,T)$ with the hypersurface $V_f := \{\sum_{i=0}^{s} f_s z_0^{p^{ei}} = 0\}$. The restriction to Z_f of the linear projection from O has separable degree s and inseparable degree p^e and O is a strange point of Z_f. Notice that $\deg(V_f) = \mu + sp^s$. This is the reason to assign weight p^e to w: when we need to use f to obtain a hypersurface V_f we set $w = z_0^{p^e}$. This construction gives all strange varieties with prescribed μ , s, p^e and T if $\mathrm{Pic}(D(O,T)$ is generated by the exceptional divisor $v^{-1}(O) \cap D(O,T)$ and $v^*(\mathcal{O}_{\mathbb{P}^n}(1))|D(O,T)$. This is the case when T is a rational normal curve, which we now consider.

Now assume that T is a rational normal curve. We claim that $D(O,T)$ is isomorphic to the Hirzebruch surface F_{n-1}. We have $\mathrm{Pic}(F_{n-1}) \cong \mathbb{Z}^{\oplus 2}$ and we take as a basis of $\mathrm{Pic}(F_{n-1})$ a fiber F of the ruling β of F_{n-1} and the section h with negative self-intersection of β (see [8], p. 523, or [4], p. 141). Hence $F \cong h \cong \mathbb{P}^1$, $h^2 = 1-n$, $h \cdot F = 1$ and $F^2 = 0$. Set $H := h + (n-1)F$. Notice that $H \cdot h = 0$, $H \cdot F = 1$ and $H^2 = n-1$. It is well-known that H is a spanned line bundle and $h^0(F_{n-1}, H) = n+1$. Let $\psi : F_{n-1} \to \mathbb{P}^n$ be the morphism induced by ψ. Since $H^2 > 0$, $\psi(F_{n-1})$ is a surface. Since $H \cdot h = 0$, ψ contracts h to a point. Set $O := \psi(h)$. Since $H \cdot f = 1$, each fiber of β is mapped by ψ isomorphically onto a line. Since $F \cdot h = 1$, each of these lines contains O. Thus $\psi(F_{n-1})$ is a surface cone with vertex containing O. Since $H^2 = n-1$, to check that $\psi(F_{n-1})$ is a cone over a rational normal curve it is sufficient to check that ψ has degree 1. We just quote again [8], p. 523, or its quotation [4], p. 141).

Proposition 1. *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Let* $X \subset \mathbb{P}^3$ *be an integral and non-degenerate curve with only planar singularities. The following conditions are equivalent:*

(i) $\dim(E_X(3)) \geq 2$*;*

(ii) $\dim(E_X(3)) = 2$*;*

(iii) X *is a rational normal curve.*

Proof. Since $\mathbb{P}^3$ is the secant varieties of X, a non-empty open subset of $\mathbb{P}^3$ has X-rank 2. Hence (i) and (ii) are equivalent in arbitrary characteristic. The quoted theorem of Comas and Seiguer gives $E_X(3) = TX\backslash X$ and hence

$\dim(E_X(3)) = 2$ if X is a rational normal curve. Now assume that X is not a rational normal curve, i.e. assume $d := \deg(X) \ge 4$.

(a) Here we check $r_X(P) = 2$ for a general $Q \in X$ and a general $P \in T_QX$. Fix a general $Q \in X$ and a general $P \in T_QX$. Let $\ell_P : \mathbb{P}^3 \backslash \{P\} \to \mathbb{P}^2$ be the linear projection from P. Set $v_P := \ell_P|X$. If v_P is not injective, then $r_X(P) = 2$. Hence from now on we assume the injectivity of v_P. If $\deg(v_P) \ge 2$, then $r_X(P) = 2$ (here we use that $\mathrm{char}(\mathbb{K}) = 0$ and hence any purely inseparable finite morphism $X \to \mathbb{P}^2$ has degree 1). Hence we may assume $\deg(v_P) = 1$. Thus $Y := v_P(X)$ is a degree d plane curve birational to X. If $\sharp((X \cap T_QX)_{red}) \ge 2$, then $r_X(P) \le 2$ for all $P \in T_QX$. Hence we may assume $(X \cap T_QX)_{red} = \{Q\}$. Since $\mathrm{char}(\mathbb{K}) = 0$ and Q is general, T_QX has order of contact 2 with X at Q and the osculating plane of X has order of contact 3 with X at Q. Hence $v_P(Q)$ is an ordinary cusp of the plane curve Y (see [10], middle of p. 97). Since X has only planar singularities and P is a general point of the tangent developable TX of X, $P \notin T_AX$ for all $A \in T_AX$. Hence v_P is unramified at each point of $\mathrm{Sing}(X)$. Since $\mathrm{char}(\mathbb{K}) = 0$, X is not a strange curve. Hence two general secant lines are disjoint. Since Q is general and P is general in T_QX, we get that T_QX is the unique tangent line of X_{reg} containing P. Thus Y is a degree d plane curve such that $p_a(Y) = p_a(X) + 1$. We have $p_a(Y) = (d-1)(d-2)/2$. Castelnuovo's upper bound for the arithmetic genus of space curves gives $d = 3$, contradiction.

(b) Step (a) gives $\dim((E_X(3) \cap (\cup_{Q \in X_{reg}} T_QX))) \le 1$. Hence the finiteness of $\mathrm{Sing}(X)$ shows that to conclude the proof it is sufficient to prove $\dim(T_AX \cap E_X(3)) \le 1$ for all $A \in \mathrm{Sing}(X)$. Fix $A \in \mathrm{Sing}(X)$. By assumption T_AX is a plane. Every secant line of X intersects T_AX. Assume that in this way we do not cover a non-empty open subset of T_AX. Since $\mathrm{char}(\mathbb{K}) = 0$, X is not a strange curve. Hence we cover in this way finitely many points and finitely many (at least one) quasi-projective curves. Call T the closure of one of these curve. Since X is not strange, a dimensional count gives that for a general $Q \in T$ there are infinitely many secant lines to X passing through A. Hence the linear projection from A does not inducs a birational morphism of X. Since $\dim(T) = \dim(X)$, we got a contradiction (see [6]). $\square$

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