International Journal of Pure and Applied Mathematics
Volume 59 No. 2 2010, 123-132

THE SCHWARZIAN DERIVATIVE: SOME DISCUSSIONS AND OBSERVATIONS

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Abstract: We study the Schwarzian derivative for a number of examples and particularly some extreme values of the norm of this derivative.

AMS Subject Classification: 30C45
Key Words: Schwarzian derivative, norm, univalent functions

1. Introduction

To decide whether or not a function is univalent has traditionally been an interesting and many times hard problem. In line of this the Schwarzian derivative has been an important object of study much due to Nehari's Theorem [5] and Kraus' Theorem [4], giving one sufficient and one necessary condition for univalence, but leaving a gap between them. The purpose of the present article is to study these two results more closely with special aim to investigate what happens when the norm of the Schwarzian derivative is near the special values of Nehari and Kraus. The discussions are mainly known, but in a new setting with a number of illustrations to exemplify the theory.

Received: December 11, 2009

2. Some Preliminaries

Definition 1. (Univalent Functions) A function f analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ is said to be univalent in $\mathbb{D}$ if it does not take the same value twice, i.e.

$$z_1 \neq z_2 \Rightarrow f(z_1) \neq f(z_2), \quad z_1, z_2 \in \mathbb{D}.$$

The theory of univalent functions is largely concerned with the family S of functions f analytic and univalent in $\mathbb{D}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, thus having the form $f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots, \quad z \in \mathbb{D}$. The most known example of a function in S is the Koebe function

$$k(z) = \frac{z}{(1-z)^2}$$

and its rotations. If the image $f(\mathbb{D})$ is starlike with respect to the origin, the function is called starlike and the class of such functions is denoted S^*.

Definition 2. (The Schwarzian Derivative) Let f be an analytic, locally univalent function in $\mathbb{D}$. Then the Schwarzian Derivative is defined by

$$S_f(z) = \left[\frac{f''(z)}{f'(z)}\right]' - \frac{1}{2}\left[\frac{f''(z)}{f'(z)}\right]^2.$$

The norm of the Schwarzian derivative is defined as

$$\|S_f\|_2 = \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 \left|S_f(z)\right|.$$

Remark 3. To motivate the study of the Schwarzian derivative, we note that

— The Schwarzian derivative is important mainly because of its invariance under linear fractional transformation.

— If f is a linear fractional transformation then $\|S_f\|_2 = 1$, so in some sense $\|S_f\|_2$ measures how far away a function is from a linear fractional transformation.

— Also the pre-Schwarzian derivative defined as the logarithmic derivative of f' has been studied in many papers, see e.g [1], [2], [6].

The norm of the Schwarzian derivative is the key in the following classical results.

Theorem 4. (Kraus-Nehari) *If a function f is univalent in $\mathbb{D}$, then*

$$\|S_f\|_2 \leq 6.$$

If a function f is analytic in $\mathbb{D}$ and

$$\|S_f\|_2 \leq 2,$$

then f is univalent in $\mathbb{D}$.

Both constants 2 and 6 are the best possible, leaving a gap between them in which the norm of the Schwarzian derivative is a number between 2 and 6. Analytic functions with these norms may be univalent or not, but the Schwarzian derivative cannot be used to decide this.

We will investigate the border line cases of the norm more closely by asking:

— Which functions in S are such that $\|S_f\|_2 = 6$?

— Which functions in S are such that $\|S_f\|_2 = 2$?

Partial answers illustrated by a number of examples will be offered.

3. The Boundary Case $\|S_f\|_2 = 6$

In earlier papers an alternative method of frequency analysis has been derived, see [7], [8]. In this method we construct from observations of a signal, certain univalent functions called slit mapping functions. It was then natural to start the investigation by studying the Schwarzian derivative for these slit mapping functions. They are defined as follows:

Definition 5. (Slit Mapping Functions) Let Sl denote the subfamily of S^* consisting of functions of the form

$$f(z) = \frac{z}{\prod_{j=1}^{I}(1 - 2z\cos\omega_j + z^2)^{r_j}}, \quad \text{where} \quad \sum_{j=1}^{I} r_j = 1.$$

A function $f \in Sl$ maps $\mathbb{D}$ onto the complement of $2I$ radial slits, i.e. $f(\mathbb{D})^c$ consists of $2I$ radial slits, two of which are on the positive and negative real axis. f has an analytic extension to the closed unit disk $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$ except for the zeros of the denominator.

We choose one particular mapping (see Figure 1) to illustrate what happens to the norm of the Schwarzian derivative:

Example 6. (A Slit Mapping Function) Let $I = 3$ with $\omega_1 = \frac{\pi}{6}$, $\omega_2 = \frac{\pi}{2}$,

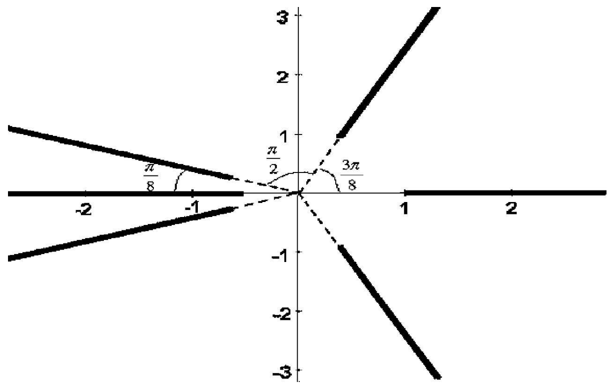


Figure 1: Slits given by a function in Sl

$\omega_3 = \frac{3\pi}{4}$ and $\alpha_1 = \frac{\sqrt{3}}{4}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{1}{4}$. Then

$$f(e^{i\theta}) = \frac{1}{2(\cos\theta - \frac{\sqrt{3}}{2})^{\frac{3}{8}}(\cos\theta)^{\frac{1}{2}}(\cos\theta + \frac{\sqrt{2}}{2})^{\frac{1}{8}}}. \tag{1}$$

This function maps $\mathbb{D}$ onto the complement of the slits given in Figure 1.

When we have derived the Schwarzian derivative of function (1), we substitute the polar representation $z = r(\cos\theta + i\sin\theta)$ into the expression $(1 - |z|^2)^2 S_f(z)$. Then we choose a proper number of radii; $r = 0.01, 0.1, 0.2, ..., 0.9, 0.96$ to illustrate the norm of the Schwarzian derivative. The reason for choosing more than one radius is that the norm is not satisfying the maximum principle. *Maple* produces the curves in Figure 2.

The curves indicate that $\|S_f\|_2 = 6$. Actually, explicit computations show that $\lim_{r\to 1}(1 - \left|re^{i\theta}\right|^2)^2 S_f(re^{i\theta}) = -6$, so the maximum norm is obtained as r tends to 1.

From properties of the Koebe function it follows that all slit functions in S lead to the same result as in the previous example. The simplest example is a single slit mapping

$$f(z) = \frac{z(1+az)}{(1-z)^2}; \quad a \in \left\{ w : \left| w + \frac{1}{2} \right| = \frac{1}{2} \right\}. \tag{2}$$

But also other functions have this property, as shown in the following example.

Example 7. (A Polynomial) Let p be the function

$$p(z) = z + \frac{1}{2}z^2,$$

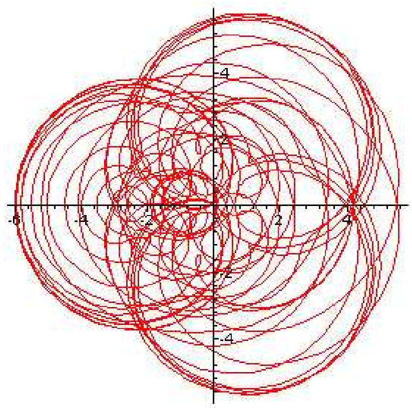


Figure 2: The norm of the Schwarzian derivative of a slit mapping function

representing a univalent function with extremal properties for second degree polynomials: The coefficient 1/2 is the largest value possible if the polynomial is to be univalent. It maps $\mathbb{D}$ onto the interior of a cardioid. The norm of the Schwarzian derivative of $p(z)$ is obtained from

$$S_p(z)(1-|z|^2)^2 = -\frac{3}{2}\frac{(1-|z|^2)^2}{(1+z)^2}. \tag{3}$$

With the same procedure as in Example 6, the graphs are like in Figure 3. The figure indicates that the boundary value -6 is obtained as $r \to 1$, which also can be seen directly from the expression (3).

With these examples at hand it is nearby to ask: What do the functions have in common that brings them to have this 6-property? What is their common characteristic? These questions lead to the following result:

Theorem 8. *Let $f \in S$, and assume that f is analytic in the point $e^{i\alpha}$. Then*

$$\lim_{r\to 1}(1-\left|re^{i\alpha}\right|^2)^2 S_f(re^{i\alpha}) = \begin{cases} 0 & \text{if } \ f'(e^{i\alpha}) \neq 0, \\ -6e^{-2i\alpha} & \text{if } \ f'(e^{i\alpha}) = 0 \ \text{ and } \ f''(e^{i\alpha}) \neq 0. \end{cases} \tag{4}$$

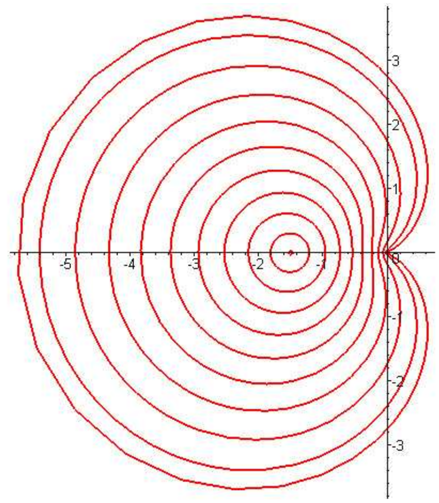


Figure 3: The norm of the Schwarzian derivative of a polynomial

Proof. The proof is straight forward: $S_f(z)$ may be rewritten as

$$S_f(z) = \frac{1}{(f'(z))^2}\left[f'''(z)f'(z) - \frac{3}{2}(f''(z))^2\right].$$

We substitute $z = re^{i\alpha}$ to obtain the expression

$$(1-\left|re^{i\alpha}\right|^2)^2 S_f(re^{i\alpha}) = \frac{(1-r^2)^2}{(f'(re^{i\alpha}))^2}\left[f'''(re^{i\alpha})f'(re^{i\alpha}) - \frac{3}{2}(f''(re^{i\alpha}))^2\right].$$

From this it is obvious that if $f'(e^{i\alpha}) \neq 0$, then the limit as $r \to 1$ of $(1-r^2)^2 S_f(re^{i\alpha})$ is 0.

If $f'(e^{i\alpha}) = 0$ we get an 0/0-form, and since the limit is radial we use L'Hopitals rule on the expression to obtain the given result in (4). In this case we have to use that $f''(e^{i\alpha}) \neq 0$. □

It can be proved that $f' = 0$ is necessary for the Schwarzian derivative to have norm 6.

4. The Boundary Case $\|S_f\|_2 = 2$

Nehari's Theorem says that if we have an analytic function and know that $\|S_f\|_2 \leq 2$, then the function is univalent. The constant 2 is best possible, and can not be replaced by any larger number. This is proved by Hille [3] by using the function

$$f(z) = \left(\frac{1-z}{1+z}\right)^{\alpha}, \quad \alpha \in \mathbb{C},$$

which is univalent if and only if the exponent $\alpha = a + ib$ satisfies $a^2 + b^2 \leq 2\,|a|$. We have e.g. that the choice $\alpha = ib$ produces a non-univalent function f with $\|S_f\|_2 \leq 2(1+b^2)(1-|z|^2)^{-2}$.

We restrict ourselves to investigate functions that we know in advance are univalent, transform them and ask: Is it possible to find a univalent function with Schwarzian norm 2 that becomes non-univalent with a small variation leading to norm greater than 2? We include 2 examples.

Example 9. (A Log Function 1) Let f be the function

$$f(z) = \frac{1}{2c}\ln\left(\frac{1+cz}{1-cz}\right), \; 0 < |c| \leq 1. \tag{5}$$

For $c = 1$ this function maps $\mathbb{D}$ onto the horizontal strip between $-\pi/4$ and $\pi/4$. The Schwarzian derivative of (5) is given in the expression

$$S_f(z)(1-|z|^2)^2 = \frac{2c^2(1-|z|^2)^2}{(-1+cz)^2(1+cz)^2}.$$

It is not difficult to show directly that $\|S_f\|_2 < 2$ when $|c| < 1$ and $\|S_f\|_2 = 2$ when $|c| = 1$.

Example 10. (A Log Function 2) Let f be the function

$$f_c(z) = \ln(1+z) + cz^2, \quad c \in \mathbb{C}.$$

Because of both logarithmic and non-logarithmic terms, the function involves lots of discussions with not too nice expressions. The Schwarzian derivative is

$$S_f(z) = -\frac{1}{2}\frac{-1+12(-c+c^2+4c^2z+6c^2z^2+4c^2z^3+c^2z^4)-32cz-20cz^2}{(1+2cz+2cz^2)^2(1+z)^2}.$$

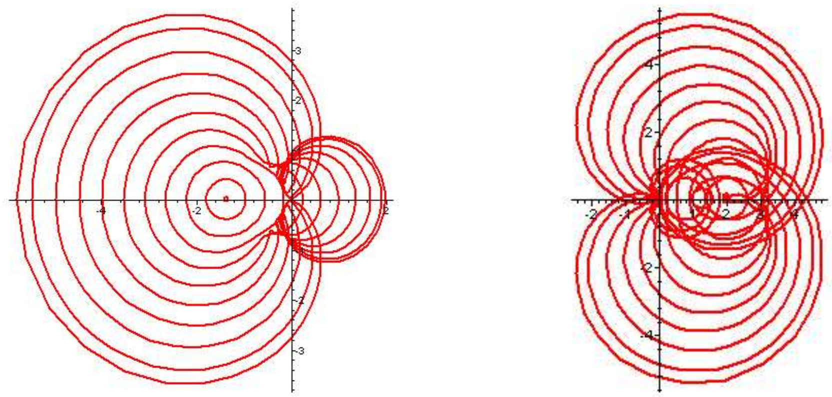
Let $|z| < 1$, $|w| < 1$. Then a necessary and sufficient condition for univalence is

$$f_c \in S \Longleftrightarrow -c \notin V := \left\{\omega_v : \omega_v = \frac{\ln(1+z)-\ln(1+w)}{z^2-w^2}\right\}.$$

A necessary and sufficient condition for local univalence is

$$f_c'(z) \neq 0 \Longleftrightarrow -c \notin U := \left\{\omega_u : \omega_u = \frac{1}{2z(1+z)}\right\},$$

where $U \subseteq V$. But these results do not give a “picture” of the set that c may be chosen from in order to give a univalent function. To get such a picture, we concentrate on the denominator term $1+2cz+2cz^2$ in $S_f(z)$. Zeros of this term are $z = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-\frac{2}{c}}$ and if we require $\|S_f\|_2 \leq 6$ and choose the same



(a) $c = -\frac{1}{4}$. (b) $c = \frac{1}{2}$.

Figure 4: The norm of the Schwarzian derivative for extremal cases of a Log function

values of r as in Example 6, exact intervals on the real and imaginary axis may be given:

c real: It is easy to see that $c \in \left[-\frac{1}{4}, \frac{1}{2}\right] \Rightarrow \|S_f\|_2 \leq 6$; giving candidates for univalence. Figures 4(a) and 4(b) illustrate what happens to the norm of the Schwarzian derivative for the extremal cases $c = -\frac{1}{4}$ and $c = \frac{1}{2}$.

To be sure that the function is univalent, we require $\|S_f\|_2 \leq 2$. Numerically this is the case for values of c approximately in the interval $[-0.197, 0.255]$.

c imaginary: It is easy to see that $c \in \left[-\frac{\sqrt{3}}{6}i, \frac{\sqrt{3}}{6}i\right] \Longrightarrow \|S_f\|_2 \leq 6$; giving candidates for univalence. Figure 5 illustrate what happens in the extremal case $c = -\frac{\sqrt{3}}{6}i$. For the positive imaginary axis we obtain a similar result due to symmetry about the real axis. In this case if we require $\|S_f\|_2 \leq 2$, numerically we find that c approximately must be chosen from the interval $[-0.228i, 0.228i]$.

Other values of c: The expressions become more complicated, but may be solved numerically. The domain of permitted values of c are given by the innermost domain of Figure 6. When c is chosen in this area, we obtain a function f_c which has the possibility to be univalent (i.e. giving $\|S_f\|_2 \leq 6$). The domain of values of c producing functions f_c that we *know* are univalent by the requirement $\|S_f\|_2 \leq 2$, is considerably smaller.

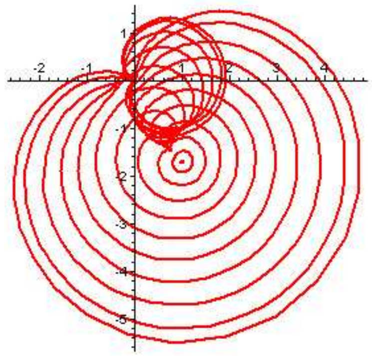


Figure 5: $c = -\frac{\sqrt{3}}{6}i$

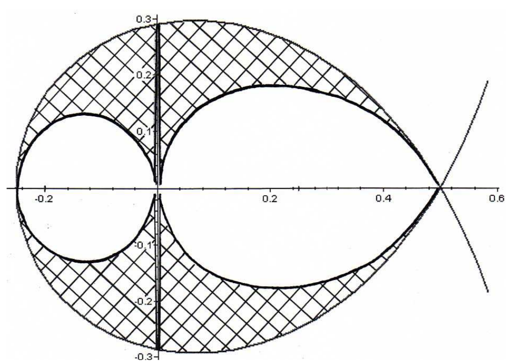


Figure 6: Other values of c

In Figure 6 we have included also another border line. This is obtained as we know that the derivative of each function has to be different from 0. Not all of the border line is included, only the part that produce bounds for our values of c.

Put together we obtain an interesting figure: Outside the outer border are non-permitted values of c. Inside the border we first meet an area of c-values for which f_c is locally univalent but not univalent. This is except for the two segments of the imaginary axis. The region is shredded in Figure 6. Inside the "Schwarzian border" bounding the innermost domain, we have values of c for which $\|S_f\|_2 \leq 6$.

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