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MIXED E-, E*-CONVEX FUNCTIONS AND THEIR CORRESPONDING MIXED HESSIAN MATRICES

Emre Tokgöz
[1]Department of Computer Science
University of Oklahoma
Norman, OK, 73019, USA
e-mail: Emre.Tokgoz-1@ou.edu

Abstract: Many minimization problems include functions with integer variables or a combination of integer and real variables. Tokgöz [9] defined mixed T and T* convex functions by using L and L$^\natural$ convex function definitions and introduced their corresponding mixed Hessian matrices for quadratic L and L$^\natural$ convex functions. In this paper, similar to the extension of L and L$^\natural$ convex functions to the mixed T and T*-convex functions, the domain of M and M$^\natural$ convex functions are extended to include real variables in the domain. Mixed Hessian matrices are defined for quadratic mixed E and E* convex functions with properties similar to those of the Hessian matrix for real variables.

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1. Introduction

Some of the definitions of discrete and integer convex functions and their introducers are as follows: Integrally convex functions by Favati and Tardella [7], discretely convex functions by Miller [1], strong discrete convex functions by Yüceer [11], M-convex and L-convex functions by Murota [4] and [5], L$^\natural$-convex functions by Fujishige and Murota [8], M$^\natural$-convex functions by Murota and Shioura [6], and D-convex functions by Ui [10]. It is shown by Ui [10]

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that the definition of a D-convex function has special cases as M, L, M$^\natural$, L$^\natural$, integrally and discretely convex functions. One important result is that these unified forms of convex functions have the property that every local minimum is a global minimum and vice versa.

Integer and discrete convex functions can also be characterized by properties of their Hessian matrices. Hessian matrices are defined for strong discrete convex functions by Yüceer [11], for L-convex functions by Moriguchi and Murota [9], and for M-convex functions by Hirai and Murota [3]. The discrete Hessian matrix is symmetric in a local neighborhood and vanishes when the function is affine. It is also known that the Hessian matrix of a discrete function corresponds to the coefficient matrix of the function that it represents if the function is a second degree polynomial with respect to its integer variables [8].

Tokgöz [2] constructed mixed Hessian matrices for mixed T and T* convex functions by using the matrices constructed by Moriguchi and Murota [8] which represent second degree L and L$^\natural$-convex polynomial functions. In this paper, similar to the mixed Hessian matrices constructed by Tokgöz [2], the mixed Hessian matrices will be constructed to represent functions that are defined on a combination of integer and real spaces which satisfy M and M$^\natural$-convexity properties of second degree polynomials with respect to the integer variables and convexity with respect to the real variables.

Let $\Im_\aleph$ denote the integer vector space with coordinates indexed by the elements of $\aleph$ with $\aleph \neq \emptyset$. In the case when the number of elements of $\aleph$ is n, i.e. $|\aleph| = n$, we have the corresponding n-dimensional integer vector space $\Im|_\mathbb{Z} = \mathbb{Z}^n$. Let $\mathbb{R}^m$ denote the m-dimensional Euclidean space.

A function $\varphi : \mathbb{R}^m \to \mathbb{R}$ is convex if

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

for $\forall\ x, y \in \mathbb{R}^m$ and $0 \leq \lambda \leq 1$. One way of verifying the convexity of a C^2 function is to show that the corresponding Hessian matrix at each point is positive definite. If the inequality is strict then the Hessian matrix is positive semi-definite.

Tokgöz [2] defined the first difference of a discrete function $\Omega : \Im|_\mathbb{Z} \to \mathbb{R}$ as follows:

$$\nabla_j(\Omega(\psi,\xi)) = \Omega(\psi + e_j, \xi) - \Omega(\psi, \xi),$$

where e_j is a unit vector and the second difference is defined by

$$\nabla_{ij}(\Omega(\psi,\xi)) = \Omega(\psi + e_i + e_j, \xi) - \Omega(\psi + e_i, \xi) - \Omega(\psi + e_j, \xi) + \Omega(\psi, \xi).$$

All definitions and applications below are based on the local neighborhood of a point.

2. Mixed E−Convex Functions

One nice property of M-convex functions is that they can be extended to convex functions in real variables (Murota, [5]). In fact, it is the key point of this paper to construct Hessian matrices to represent functions that are defined on a combination of integer and real spaces which satisfy the M-convexity and M$^\natural$-convexity properties of second degree polynomials with respect to the integer variables and convexity with respect to the real variables.

Given a vector $u = \{u(v) : v \in \aleph\} \in \Im|_{\mathbb{Z}}$, the positive support of u is defined by

$$support_+(u) = \{v \in \aleph : u(v) > 0\}$$

and the negative support of u is defined by

$$support_-(u) = \{v \in \aleph : u(v) < 0\}.$$

Denote by the characteristic vector of $u \in \Im|_{\mathbb{Z}}$ by $\chi_u = \chi_{\{u\}}$ where the characteristic vector of a subset $U \subset \aleph$ is defined to be $\chi_U(u) = 1$ if $u \in U$ and $\chi_U(u) = 0$ otherwise. A convex function $g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ satisfies the inqualities

$$\begin{aligned} g(x) + g(y) &\geq g((1-\gamma)x + \gamma y) + g(\gamma x + (1-\gamma) y), \\ g(x) + g(y) &\geq g(x - \gamma(x-y)) + g(y + \gamma(x-y)), \end{aligned}$$

which is easy to see from the definition of a real variable convex function. Keeping this in mind, the definition of an M-convex function is given as follows (Hirai and Murota, [3]):

Definition 2.1. Let $\rho : \Im|_{\mathbb{Z}} \to \mathbb{R}$ be a function with non-empty domain $\left(\text{i.e. } D_{\Im|_{\mathbb{Z}}}\rho \neq \emptyset\right)$. ρ is said to be M-convex if for $\alpha, \beta \in D_{\Im|_{\mathbb{Z}}}\rho$ with $u \in support_+(\alpha - \beta)$, there exists $v \in support_-(\alpha - \beta)$ such that

$$\rho(\alpha) + \rho(\beta) \geq \rho(\alpha - \chi_u + \chi_v) + \rho(y + \chi_u - \chi_v).$$

Note that the Laplacian of ρ in this case is

$$\nabla_{uv}\rho(\alpha) = \rho(\alpha + \chi_v - \chi_u) - \rho(\alpha).$$

Using M-convexity definition, a mixed E-convex function is defined as follows:

Definition 2.2. A function $\phi : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$ is mixed E-convex if ϕ is M-convex with respect to the integer variables and C^2 strictly convex with respect to the real variables.

Throughout this paper let $\phi : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$, $\phi|_{\Im|_{\mathbb{Z}}} = \phi_1$, $\phi|_{\mathbb{R}^m} = \phi_2$, the real variable extension of ϕ_1 to be $\overline{\phi}_1$, the real variable extension of ϕ to be $\overline{\phi} : \mathbb{R}^{n+m} \to \mathbb{R}$ unless stated otherwise.

We introduce the mixed Hessian matrix H for a mixed E-convex function as follows: $n \times n$ upper left discrete submatrix portion of H will be assumed to be corresponding to the quadratic M-convex function portion ϕ_1. The lower $m \times m$ submatrix of H is the $m \times m$ Hessian matrix of ϕ associated with the real variables of ϕ. The upper right and lower left submatrices consist of the first differences of first differential and the first differential of first difference of the given mixed function. Thus S has the following form:

$$H = \begin{bmatrix} \left[\nabla_{ij}\phi(\psi,\xi)\right]_{n\times n} & \left[\frac{\partial}{\partial\xi_k}(\nabla_j(\phi(\psi,\xi)))\right]_{n\times m} \\ \left[\nabla_j(\frac{\partial}{\partial\xi_k}\phi(\psi,\xi))\right]_{m\times n} & \left[\frac{\partial^2}{\partial\xi_k\partial\xi_t}\phi(\psi,\xi)\right]_{m\times m} \end{bmatrix}$$
$$= \begin{bmatrix} X_{n\times n} & Y_{n\times m} \\ Z_{m\times n} & W_{m\times m} \end{bmatrix} \quad (2.1)$$

where $\psi \in \Im|_{\mathbb{Z}}, \xi \in \mathbb{R}^m, 1 \le i,\, j \le n$ and $1 \le k,\, t \le m$. In the matrix S, ∇_i and ∇_{ij} denote the first and second differences with respect to the discrete variables ψ_k of ϕ and, $\frac{\partial}{\partial\xi_k}$ and $\frac{\partial^2}{\partial\xi_k\partial\xi_t}$ denote the first and second differentials of ϕ with respect to the continuous variables ξ_i, respectively.

When we consider Hessian matrix for an M-convex function, for each $\vartheta \in \Im|_{\mathbb{Z}}$, we consider the neighborhood

$$\aleph(\vartheta) = \{\vartheta + \chi_u + \chi_v : u, v \in \aleph\}.$$

Considering $\phi_1 : \Im|_{\mathbb{Z}} \to \mathbb{R}$ and $\vartheta \in \aleph(\vartheta)$, the discrete Hessian matrix components are defined as follows:

$$X_{ij}(\phi_1;\vartheta) = \phi_1(\vartheta + \chi_u + \chi_v) - \frac{1}{2}(\phi_1(\vartheta + 2\chi_u) + \phi_1(\vartheta + 2\chi_v)).$$

Hirai and Murota [3] defined a local quadratic extension $\overline{\phi}_1$ of ϕ_1 with respect to the real variables as follows:

$$\overline{\phi}_1(x;\vartheta) = \frac{1}{2}\sum_{i\in\aleph}\phi_1(\vartheta + \chi_u + \chi_v)(x-\vartheta)_u + \frac{1}{2}\sum_{i\in\aleph}X_{ij}(\phi_1;\vartheta)(x-\vartheta)_u(x-\vartheta)_v.$$

Lemma 2.1. *The mixed Hessian matrix H defined for a mixed E-convex function as above is well defined.*

Proof. As mentioned in the beginning of this section, one nice property of M-convex functions is that they can be extended to real variable convex functions. Extending ϕ_1 to $\overline{\phi}_1$ gives a convex function in $\mathbb{R}^n$. Hence we have a function that has $(n+m)$ real variables with the Hessian matrix

$$\overline{H} = \left[\frac{\partial^2}{\partial\xi_k \partial\xi_t}\phi(\psi,\xi)\right]_{(n+m)\times(n+m)}$$

which is well defined.

Another way of proving this lemma is as follows: Let $\overline{\phi}$ be a real convex function with $(n+m)$ real variables and has the Hessian matrix $\overline{H}$ where we assume that the restriction of n variables of $\overline{\phi}_1$ to $\Im|_{\mathbb{Z}}$ gives the M-convex function as above. In this case, it is enough to discretize the differentials of $\overline{\phi}$ for the n variables that corresponds to the M-convex portion of $\overline{\phi}$. This gives us the mixed Hessian matrix H as above. □

Many cases are possible, depending on the conditions placed on the upper right and lower left submatrices of H in a neighborhood of a point. In the simplest simple case we have the following lemma.

Lemma 2.3. *Let ϕ be a mixed E-convex function that is linear for both the discrete and the real variables. Then the mixed Hessian matrix H above is the zero matrix if the degree of the polynomial is one. If the degree of the polynomial is not one then the corresponding matrix has the form of* (2.1) *with at least one of X, W, Y or Z non-zero.*

Proof. If the degree of the given polynomial is one, the first differences of ϕ are constants hence the second differences and first differentials of the first differences of ϕ are zero. Similarly, the first differentials of ϕ are constant hence the second differentials and first differences of the first differentials are zero. This proves that all the entries of the matrix H are zero. In the simplest case, if ϕ has only integer variables then X is constant and $Y = Z = W = 0$, if ϕ has only real variables then W is non-zero and $Y = Z = X = 0$. In other possible cases, it can be easily shown that one of Y, Z, W or X is non-zero. □

Theorem 2.1. *Let ϕ be a mixed E-convex function, then there exists a unique global minimum value in $\mathbb{R}$.*

Proof. Denote the minimum value of ϕ in $\Im|_{\mathbb{Z}}$ to be $\min\left\{\phi|_{\Im|_{\mathbb{Z}}}\right\}$. We know that every local minimum is also a global minimum for M-convex functions and vice versa. The restriction of ϕ to m-dimensional real space, $\phi|_{\mathbb{R}^m}$, has one global minimum value. Hence, in the entire domain of the function, $\min\left\{(\min\left\{\Phi_{1|_{\Im|_{\mathbb{Z}}}}\right\}, \min\left\{\Phi_{1|_{\mathbb{R}^m}}\right\})\right\}$ gives a unique global minimum value for mixed E-convex function ϕ. □

Note that the uniqueness of the global minimum value of a mixed E-convex function does not imply the uniqueness of the point which gives the global minimum value. The following theorem specifies the cases under which the global minimum point of a mixed E-convex function is unique.

Theorem 2.2. *The existence of a unique global minimum point of a mixed E-convex function $\phi : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$ depends on the functions $\overline{\phi}_1$ and $\overline{\phi}_2$ defined as above such that;*

Case 1: If the global minimum point of $\overline{\phi}$ (that is obtained from the global minimums of $\overline{\phi}_1$and $\overline{\phi}_2$) is the same as the global minimum of ϕ then the global minimum point of ϕ is unique,

Case 2: If the global minimum point of $\overline{\phi}$ is not in the domain of ϕ then the uniqueness of the global minimum point depends on the integer points in the neigborhood of the integer component of $\overline{\phi}|_{\Im|_{\mathbb{Z}} \times \mathbb{R}^m}$. In this case, the lowest value in the range gives the unique global minimum point if it is unique.

Proof. The uniqueness of the global minimum point of a mixed $E-$convex function $\phi : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$ follows from the global minimum point properties of the extension function $\overline{\phi}$. By the real function convexity property, there exists a unique global minimum point of $\overline{\phi}$. If the global minimum of $\overline{\phi}$ exists in the domain of ϕ then the global minimum point of ϕ is unique. If the global minimum point of $\overline{\phi}$ is not in the domain of ϕ then one can look at the integer points in the integer component neighborhood (componentwise for each integer component) of $\overline{\phi}|_{\Im|_{\mathbb{Z}} \times \mathbb{R}^m}$ to find the points that are possible global minimum points in the domain of ϕ. By comparison, one can observe the existence of a unique global minimum if there exists a unique point with the lowest value in the range of ϕ. The uniqueness of the minimum point value of ϕ is based on the closest integer values of ϕ_1 for all the components in the domain. This can be done by induction, that is by finding the minimum with respect to each component in the domain and checking the values based on the function that is given. □

Theorem 2.3. *If $\phi : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$ is a mixed E-convex function then the minimization of ϕ_1 in $\Im|_{\mathbb{Z}} \times \mathbb{R}^m$ depends on the local minimums of ϕ_1 and ϕ_2. In this case, the mixed variables of the function ϕ_1 define a new local minimum point set. The elements of this set are also global minimum points of ϕ_1.*

Proof. Suppose that ϕ is a mixed E-convex function. Considering the function ϕ_1, we have local minimum points that are also global minimum points for each fixed real variable. Extension of the domain to $\mathbb{R}^n$ with respect to the real variables will form another discrete set of minimum points that are not necessarily in $\Im|_{\mathbb{Z}}$. One can think of the new set of minimum points as the extension of the minimum set that exists on the restricted function ϕ_1. The extension of the domain of ϕ_1 to $\mathbb{R}^n$ gives a set of distinct functions in $\Im|_{\mathbb{Z}} \times \mathbb{R}^m$ and by finding the minimums of such distinct functions we can identify the new set of discrete minimums. The discrete set of minimums is isomorphic to $\Im|_{\aleph}$ with $|\aleph| = n+m$. Since every local minimum is a global minimum in $\Im|_{\aleph}$ for $|\aleph| = n+m$, the new set also forms a set of global minimums. □

Definition 2.3. A matrix L is called an M-matrix if it can be represented as $L = sI - B$ with a matrix B consisting of non-negative entries and a real number $s \geq \varkappa(B)$ where $\varkappa(B)$ denotes the spectrial radius of B, that is, the largest modulus of an eigenvalue of B.

Now consider the set $K = \left\{A : A = [a_{ij}]_{n\times n}\right\}$ of symmetric matrices in which the following conditions hold:

1. $a_{ij} \leq 0$ for $i \neq j$ for all i, j; $1 \leq i, j \leq n$.
2. $\sum\limits_{j=1}^{n} a_{ij} \geq 0$ for all i, $1 \leq i \leq n$.

Definition 2.4. A function $\phi : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$ is partial mixed E-convex if ϕ_1 is M-convex with respect to the integer variables and ϕ_2 is extended M-convex function with respect to the real variables.

In [5] it is shown that $L \in K$ is a positive semidefinite matrix by showing that any principal minor of L is non-negative. Considering the $(n+m)\times(n+m)$ M-matrix and extending m-dimensional integer space to real space, we can prove the following theorem;

Theorem 2.4. *A function ϕ is partial mixed E-convex if and only if the corresponding mixed Hessian matrix is positive definite.*

Proof. The proof follows from the fact that $\phi|_{\Im|_{\mathbb{Z}}\times\mathbb{Z}^m} : \Im|_{\mathbb{Z}} \times \mathbb{Z}^m \to \mathbb{R}$ is an M-convex function if and only if the corresponding $(n+m)$ dimensional discrete Hessian matrix is positive definite (Hirai and Murota [3]). Extending the m discrete variables of $\mathbb{Z}^m$ to real variables of $\mathbb{R}^m$, we still preserve the positive definiteness of the Hessian matrix which is the mixed Hessian matrix of $\phi : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$. □

Theorem 2.5. *A mixed function $\phi : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$ is a mixed E-convex function if and only if ϕ_1 is M convex and ϕ_2 is a real convex function.*

Proof. By definition if ϕ is a mixed $E-$convex function then ϕ_1 is M-convex and ϕ_2 is a real convex function. Suppose that ϕ_1 is M-convex and ϕ_2 is a real convex function then for each component of ϕ_1, there exists a real convex function attached to it. Since ϕ_1 itself is M-convex, this completes the proof. □

One important property of M convex functions is that every local minimum is also a global minimum and vice versa. A similar result can be given for mixed

E-convex functions as follows:

Theorem 2.6. *Let $\phi : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$ be a partial mixed E-convex function. Then the set of local minimums of ϕ form a set of global minimums and vice versa.*

Proof. Consider the integer convex portion ϕ_1. It is known that every local minimum of ϕ_1 is also a global minimum. Let the set of local minimums of ϕ_1 be Π_1 and Π_2 be the set of global minimums of ϕ_2. Adding a real variable to the domain of ϕ_1 forms the copies of real convex functions attached to every point of integer variable. It is not difficult to see that $\Pi_1 \times \Pi_2$ forms the set of local minimums. This is an integer set which is isomorphic to $\mathbb{Z}^{n+m}$ and in $\mathbb{Z}^{n+m}$ every local minimum is also a global minimum. This proves that the set of local minimums that is obtained for ϕ is also a set of global minimums and vice versa. □

3. Mixed E*-Convex Functions

$M^{\#}$-convex functions are conceptually equivalent to M-convex functions and the set of $M^{\#}$ convex functions is larger than the set of M-convex functions. By this eqiuvalance, the theorems stated for M-convex functions can be rephrased for $M^{\#}$-convex functions (Murota [5]). Using this useful information, after defining an $M^{\#}$-convex function, we will define mixed E*-convex functions and restate the theorems stated for mixed E-convex functions for mixed E*-convex functions.

Definition 3.1. Suppose that $\overline{\aleph} = \{x_0\} \cup \aleph$ where x_0 is not an element of $\aleph$. A function $\widetilde{\Omega} : \Im^{\overline{\aleph}}|_{\mathbb{Z}} \to \mathbb{R}$ is called an $M^{\#}$-convex function if $\widetilde{\Omega}(x_0, x) = \Omega(x)$ when $x_0 = -x(\aleph)$ where $\Omega : \Im|_{\mathbb{Z}} \to \mathbb{R}$ is an M-convex function.For consistency, we will write $\Im|_{\mathbb{Z}}$ instead of $\Im^{\overline{\aleph}}|_{\mathbb{Z}}$ throughout this paper.

Definition 3.2. A function $\widetilde{F} : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$ is defined to be mixed E^*-convex function if $\widetilde{F}$ is E-convex with respect to the integer variables and C^2 strictly convex with respect to the real variables.

Throughout this paper let $\widetilde{F} : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$, $\widetilde{F}|_{\Im|_{\mathbb{Z}}} = \widetilde{F}_1$, $\widetilde{F}|_{\mathbb{R}^m} = \widetilde{F}_2$, real variable extension of $\widetilde{F}_1$ to be $\overline{F}_1$, real variable extension of $\widetilde{F}$ to be $\overline{F} : \mathbb{R}^{n+m} \to \mathbb{R}$ unless stated otherwise.

Using the fact that the theorems stated for M-convex functions can also be restated for $M^{\#}$-convex functions, we will restate the theorems for mixed E^*-convex functions without giving any proofs since the proofs of these theorems follow similar to the proofs of the theorems stated for mixed E-convex functions.

Lemma 3.1. *The mixed Hessian matrix H defined for a mixed E^*-convex function is well defined.*

Many cases are possible, depending on the conditions placed on the upper right and lower left submatrices of H in a neighborhood of a point. In the simplest simple case we have the following lemma.

Lemma 3.2. *Let $\widetilde{F}$ be a mixed E^*-convex function that is linear for both the discrete and the real variables. Then the mixed Hessian matrix $\widetilde{H}$above is the zero matrix if the degree of the polynomial is one. If the degree of the polynomial is not one then the corresponding matrix has the form of* (2.6) *with at least one of X, W, Y or Z non-zero.*

Theorem 3.1. *Let $\widetilde{F}$ be a mixed E^*-convex function, then there exists a unique global minimum value in $\mathbb{R}$.*

Theorem 3.2. *The existence of a unique global minimum point of a mixed E^*-convex function $\widetilde{F} : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$ depends on the functions $\overline{F}_1$ and $\overline{F}_2$defined as above such that: If the global minimum point of $\overline{F}$ (that is obtained from the global minimums of $\overline{F}_1$ and $\overline{F}_2$) is the same as the global minimum of $\widetilde{F}$ then the global minimum point of $\widetilde{F}$ is unique; If the global minimum point of $\overline{F}$ is not in the domain of $\widetilde{F}$ then the uniqueness of the global minimum point depends on the integer points in the neigborhood of the integer component of $\overline{F}|_{\Im|_{\mathbb{Z}}\times\mathbb{R}^m}$. In this case, the lowest value in the range gives the unique global minimum if it is unique.*

Theorem 3.3. *If $\widetilde{F} : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$ is a mixed E^*-convex function then the minimization of $\tilde{F}_1$ in $\Im|_{\mathbb{Z}} \times \mathbb{R}^m$ depends on the local minimums of $\widetilde{F}_1$ and $\widetilde{F}_2$. In this case, the mixed variables of the function $\widetilde{F}_1$ define a new local minimum point set. The elements of this set are also global minimum points of $\widetilde{F}_1$.*

Definition 3.4. A function $\widetilde{F} : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$ is defined to be a partial mixed E^*-convex function if $\overline{F}_1$ is $M^{\#}$-convex with respect to the integer variables and F_2 is extended M convex function with respect to the real variables.

Theorem 3.4. *Let $\widetilde{F}$ be a mixed E^*-convex function with $\widetilde{F}_1$ to be an $M^{\#}$-convex function. A function $\widetilde{F}$ is mixed E^*-convex if and only if the corresponding mixed Hessian matrix is positive definite.*

Theorem 3.5. *A mixed function $\widetilde{F} : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$ is a mixed E^*-convex function if and only if $\tilde{F}_1$ is $M^{\#}$-convex and $\widetilde{F}_2$ is a real convex function.*

One important property of $M^{\#}$ convex functions is that every local minimum is also a global minimum and vice versa. A similar result can be given for mixed E^*-convex functions as follows:

Theorem 3.6. *Let* $\phi : \Im|_{\mathbb{Z}} \times \mathbb{R}^m \to \mathbb{R}$ *be a partial mixed* E^**-convex function. Then the set of local minimums of* ϕ *form a set of global minimums and vice versa.*

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