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A NOTE ON SEMILINEAR DIFFERENTIAL INCLUSIONS WITH TIME DELAYS AND IMPULSES UNDER LOWER SCORZA-DRAGONI PROPERTY

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Abstract: In this paper we estabilish sufficient conditions for the existence of mild solutions of impulsive semilinear functional differential inclusions with lower Scorza-Dragoni nonlinearities.

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1. Introduction

Impulsive differential equations or inclusions are used to describe various models of real processes and phenomena studied in physics, chemical technology, population dynamics, economics, etc. That is why they have been the object of a wide investigation. We refer for instance to the monographs [2], [3], [17] and to the new papers [6], [8], [15].

Moreover, many papers deal with the functional case (see e.g. [1], [4], [13]) and our paper takes its place in this field.

In particular, we provide the existence of mild solutions for an impulsive Cauchy problem governed by a semilinear functional differential inclusion where the linear part is time-depending and the nonlinearity is a lower Scorza-Dragoni multifunction. The proof of our main existence result is based on a selection

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theorem presented in [5].

2. Preliminaries

Let X, Y be Hausdorff topological spaces and, if necessary, Y be also linear. We will use the following notations: $\mathcal{P}(Y) = \{H \subset Y \,:\, H \neq \emptyset\}$; $\mathcal{P}_c(Y) = \{H \in \mathcal{P}(Y) \,:\, H \text{ convex}\}$; $\mathcal{P}_f(Y) = \{H \in \mathcal{P}(Y) \,:\, H \text{ closed}\}$; $\mathcal{P}_k(Y) = \{H \in \mathcal{P}(Y) \,:\, H \text{ compact}\}$; $\mathcal{P}_{fc}(Y) = \mathcal{P}_f(Y) \cap \mathcal{P}_c(Y)$; etc.

In the sequel it will be useful to consider the family $\mathcal{D}(Y)$, introduced by Michael in [14], defined as

$$\mathcal{D}(Y) = \{H \in \mathcal{P}_c(Y) \,:\, H \supset I(\overline{H})\}, \tag{1}$$

where $I(\overline{H}) = \{x \in \overline{H} \,:\, x \notin S\,,\ S \text{ supporting set for } \overline{H}\}$. We recall that a set S is a *supporting set* for $\overline{H}$ if it is a proper closed and convex subset of the (closed and convex) set $\overline{H}$ satisfying the property:

For every segment $[x_1, x_2] \subset \overline{H}$ such that $]x_1, x_2[\cap S \neq \emptyset$, then the whole segment $[x_1, x_2]$ is contained in S.

Let $F : X \to \mathcal{P}(Y)$ be a multifunction and $D \subset Y$ be a set. As the *preimage* of D we mean the set $F^{-1}(D) = \{x \in X \,:\, F(x) \subset D\}$.

The multifunction F is said to be:

Upper semicontinuous if $F^{-1}(V)$ is an open subset of X for every open $V \subset Y$;

Lower semicontinuous if $F^{-1}(W)$ is a closed subset of X for every closed $W \subset Y$.

For equivalent definitions and further details on this subject we refer to [11].

Let $[a, b]$ be an interval of the real line endowed with the usual Lebesgue measure λ defined on the Lebesgue σ-algebra Σ.

We say that a multifunction $F : [a, b] \times X \to \mathcal{P}(Y)$ verifies the *lower Scorza-Dragoni property* if:

(l-SD) For every $\epsilon > 0$ there exists a compact $K_\epsilon \subset [a, b]$ such that $\lambda([a, b] \setminus K_\epsilon) < \epsilon$ and $F_{|\,K_\epsilon \times X}$ is lower semicontinuous;

and that F is a *Michael map* if:

(M) for every closed set $Z \subset [a, b] \times X$ such that $F_{|Z}$ is lower semicontinuous, there exists a continuous selection of F on Z (i.e. there exists a continuous function $f : Z \to Y$ such that $f(t, x) \in F(t, x)$).

Let $(E, \|\cdot\|)$ be a real Banach space. We recall some notions (see, e.g. [7], [9], [11]).

A multifunction $F : [a,b] \to \mathcal{P}_k(E)$ is said to be:

Measurable if $F^{-1}(C) \in \Sigma$ for every closed $C \subset E$;

Strongly measurable if there exists a sequence $(F_n)_{n\in I\!N}$ of step multifunctions such that

$$\lim_{n\to+\infty} h(F_n(t), F(t)) = 0\ ,\ \text{a.e. } t \in [a,b],$$

where h is the Hausdorff metric on $\mathcal{P}_k(E)$.

Of course F is measurable if and only if $F_-(A) = \{t \in [a,b] \,:\, F(t) \cap A \neq \emptyset\} \in \Sigma$ for every open $A \subset E$. Moreover, if E is a separable Banach space, then all the above definitions are equivalent (cf. [11], Theorem 1.3.1).

By the symbol $L^1([a,b];E)$ we will denote the space of all Bochner summable functions and, for simplicity of notations, we will write $L^1_+([a,b])$ instead of $L^1([a,b]; I\!R^+)$.

We denote by $C_p([a,b];E)$ the space of all piece-wise continuous functions $c : [a,b] \to E$ with a finite number of points $t_* \in [a,b[$ such that

$$c(t_*^-) = \lim_{h\to 0^+} c(t_* - h) = c(t_*) \quad \text{and} \quad c(t_*^+) = \lim_{h\to 0^+} c(t_* + h) \text{ exists in } E\ .$$

Of course, the space $C_p([a,b];E)$ is a normed space endowed with the norm

$$\|c\|_{C_p} = \frac{1}{b-a}\int_a^b \|c(t)\|\, dt\ . \tag{2}$$

Given a multifunction $\mathcal{G} : [a,b] \to \mathcal{P}(E)$ we define

$$\mathcal{S}^1_{\mathcal{G}} = \{g \in L^1([a,b];E) \,:\, g(t) \in \mathcal{G}(t)\,,\ \text{a.e. } t \in [a,b]\}$$

and, for every $t \in [a,b]$, we set

$$\|\mathcal{G}(t)\| = \sup\{\|y\| \,:\, y \in \mathcal{G}(t)\}\ .$$

Finally, put $\Delta = \{(t,s) \in [a,b]\times[a,b] \,:\, a \le s \le t \le b\}$, we recall (see, e.g. [16]) that a two parameter family $\{T(t,s)\}_{(t,s)\in\Delta}$, $T(t,s) : E \to E$ bounded linear operator, is called an *evolution system* if the following conditions are satisfied:

1. $T(s,s) = I,\ a \le s \le b\ ;\quad T(t,r)T(r,s) = T(t,s),\ a \le s \le r \le t \le b;$
2. $(t,s) \mapsto T(t,s)$ is strongly continuous on Δ (see, e.g. [12]).

For every evolution system, we can consider the correspondent *evolution operator* $T : \Delta \to \mathcal{L}(E)$, where $\mathcal{L}(E)$ is the space of all bounded linear operators in E.

3. The Existence Results

Let us fix a positive number τ, an interval of the real line $[0, b]$ and a function $x \in C_p([-\tau, 0]; E)$. We consider the impulsive Cauchy problem with time delay

$$(P) \begin{cases} y'(t) \in A(t)y(t) + F(t, y_t), \quad \text{a.e. } \ t \in [0, b] \setminus \{t_1, \ldots, t_N\}, \\ y(t) = x(t), \quad t \in [-\tau, 0], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k}), \quad k = 1, \ldots, N, \end{cases}$$

where $\{A(t)\}_{t\in[0,b]}$ is a family of linear (not necessarily bounded) operators $A(t) : D(A) \to E$, $D(A)$ a dense subset of E not depending on t; $F : [0, b] \times C_p([-\tau, 0]; E) \to \mathcal{P}(E)$ is a multifunction; $y_t(\theta) = y(t+\theta)$, $\theta \in [-\tau, 0]$; $0 < t_1 < \cdots < t_N < b$; $I_k : C_p([-\tau, 0]; E) \to E$, $k = 1, \cdots, N$, are impulse functions.

Throughout the paper, on the linear part of the differential inclusion we will assume that:

(A) the family $\{A(t)\}_{t\in[0,b]}$ generates an evolution operator $T : \Delta \to \mathcal{L}(E)$.

We recall (see, e.g. [4]) that a function $y \in C_p([-\tau, b]; E)$ is said to be a *mild solution* of problem (P) if:

(i) $y(t) = T(t, 0)x(0) + \sum_{0<t_k<t} T(t, t_k)I_k(y_{t_k}) + \int_0^t T(t, s)f(s)\, ds \ , \ t \in [0, b],$

where $f \in \mathcal{S}^1_{F(\cdot, y_{(\cdot)})}$;

(ii) $y(t) = x(t) \ , \quad t \in [-\tau, 0]$;

(iii) $y(t_k^+) = y(t_k) + I_k(y_{t_k}) \, , \ k = 1, \cdots, N.$

Theorem 1. *Let E be a separable Banach space and assume that (A) holds. Suppose that $F : [0, b] \times C_p([-\tau, 0]; E) \to \mathcal{P}(E)$ satisfies (l-SD), (M) and hypotheses:*

(H1) There exists a function $\alpha \in L^1_+([0, b])$ such that for every $c \in C_p([-\tau, 0]; E)$

$$\|F(t, c)\| \le \alpha(t)(1 + \|c\|_{C_p}) \ , \ \text{a.e. } t \in [0, b];$$

(H2) There exists a function $\mu \in L^1_+([0, b])$ such that, for every bounded $D \subset C_p([-\tau, 0]; E)$,

$$\chi(F(t, D)) \le \mu(t) \sup_{-\tau\le\theta\le 0} \chi(D(\theta)), \quad \text{a.e.} \quad t \in [0, b],$$

where χ is the Hausdorff measure of noncompactness.

Then problem (P) has at least one mild solution on $[-\tau, b]$.

Proof. First of all, let us note that the Lebesgue measure on the Borel

σ-algebra of the interval $[0, b]$ is a Radon measure (cf. [9], Theorem A.2.67, and [10], Definition 7.6.8) and recall that $C_p([0, b]; E)$ is a normed space; further, multifunction F satisfies (l-SD) and (M). Therefore we can apply Theorem 3.1 in [5] (which holds good by using norm (2)) so we get that F has a Carathéodory selection, i.e. a function $f : [0, b] \times C_p([-\tau, 0]; E) \to E$ such that $f(t, \cdot)$ is continuous for every $t \in [0, b]$, $f(\cdot, c)$ is Borel-measurable for every $c \in C_p([-\tau, 0]; E)$, $f(t, c) \in F(t, c)$ for a.e. $t \in [0, b]$ and for every $c \in C_p([-\tau, 0]; E)$.

We consider now the multifunction $G : [0, b] \times C_p([-\tau, 0]; E) \to \mathcal{P}_{kc}(E)$ defined by means of the selection f as

$$G(t, c) = \{f(t, c)\}\ ,\ (t, c) \in [0, b] \times C_p([-\tau, 0]; E)$$

and we associate to G the impulsive Cauchy problem

$$\begin{cases} y'(t) \in A(t)y(t) + G(t, y_t)\ , \quad \text{a.e. } t \in [0, b] \setminus \{t_1, \dots, t_N\}, \\ y(t) = x(t)\ ,\ t \in [-\tau, 0], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k})\ ,\ k = 1, \dots, N\ . \end{cases} \tag{3}$$

We prove that G fulfills hypotheses (F1)-(F4) of Theorem 3.1 in [4].

First of all, fixed $c \in C_p([-\tau, 0]; E)$, the selector $f(\cdot, c)$ of $G(\cdot, c)$ is Borel-measurable and then measurable. Moreover, being E a separable Banach space, $f(\cdot, c)$ is also strongly measurable. Hence (F1) is satisfied.

Of course the multifunction $G(t, \cdot)$ is upper semicontinuous for each $t \in [0, b]$, so (F2) is verified too.

Hypothesis (F3) derives from (H1), in fact the following estimate holds

$$\|G(t, c)\| \le \|F(t, c)\| \le \alpha(t)\left(1 + \|c\|_{C_p}\right)\ ,\ \text{a.e. } t \in [0, b]\ .$$

Finally, from the monotonicity of the Hausdorff measure of noncompactness and by (H2), for every bounded set $D \subset C_p([-\tau, 0]; E)$ we obtain

$$\chi(G(t, D)) \le \chi(F(t, D)) \le \mu(t) \sup_{-\tau \le \theta \le 0} \chi(D(\theta))\ ,\ \text{a.e. } t \in [0, b]$$

and (F4) is satisfied.

We are now in position to apply Theorem 3.1 in [4] so that we obtain the existence of at least one mild solution of problem (3). It is easy to see that this function is a mild solution of (P). □

Remark 2. Let us observe that if $I_k = 0$ for every $k = 1, \cdots, N$, then Theorem 1 assures the existence of mild solutions on the whole interval $[-\tau, b]$ for the non-impulsive Cauchy problem

$$\begin{cases} y'(t) \in A(t)y(t) + F(t, y_t)\ , \quad \text{a.e. } t \in [0, b], \\ y(t) = x(t)\ ,\ t \in [-\tau, 0]\ . \end{cases}$$

Moreover, our theorem also guarantees the existence of mild solutions for the Cauchy problem with autonomuous linear part

$$\begin{cases} y'(t) \in Ay(t) + F(t, y_t)\ , \quad \text{a.e. } t \in [0,b] \setminus \{t_1, \ldots, t_N\}, \\ y(t) = x(t)\ ,\ t \in [-\tau, 0], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k})\ ,\ k = 1, \ldots, N, \end{cases} \tag{4}$$

where $A : D(A) \to E$ is the densely defined infinitesimal generator of a C_0-semigroup. In fact, put $A(t) = A$ for every $t \in [0,b]$ and recalled that a C_0-semigroup $\{U(t)\}_{t\in[0,b]}$ leads to an evolution system by means of the relation $T(t,s) = U(t-s)$ (cf., e.g. [16]), it is easy to verify that family $\{A(t)\}_{t\in[0,b]}$ satisfies (A). Then, by Theorem 1, we obtain the existence of a mild solution of (P); this solution can be rewritten as a mild solution of (4) by using again the relation $T(t,s) = U(t-s)$.

We wish to recall that if the multifunction F takes values in the family $\mathcal{D}(E)$ (see (1)), then (M) is easily satisfied (cf. [14], Theorem 3.1$'''$). Consequently we achieve the following:

Corollary 3. *Let E be a separable Banach space and assume that (A) holds. Suppose that $F : [0,b] \times C_p([-\tau,0];E) \to \mathcal{D}(E)$ satisfies (l-SD), (H1) and (H2). Then problem (P) has at least one mild solution on $[-\tau, b]$.*

Remark 4. We note that our existence results extend in a broad sense the first part of Theorem 3.1 in [4]. This derives from the chain inclusion (cf. [14])

$$\mathcal{P}_{fc}(E) \subset \mathcal{D}(E) \subset \mathcal{P}_c(E)$$

joined with the fact that the inclusion of $\mathcal{P}_{fc}(E)$ (all the more of $\mathcal{P}_{kc}(E)$) in $\mathcal{D}(E)$ is proper. This is obvious if E is a finite dimensional space since in this case $\mathcal{D}(E) = \mathcal{P}_c(E)$ (cf. [14] again). Whereas, if E has infinite dimension, the strict inclusion is proved by Example 3.2 in [5].

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